

ON A LOCAL REDUCTION OF A HIGHER ORDER
PAINLEVÉ EQUATION AND ITS UNDERLYING LAX PAIR
NEAR A SIMPLE TURNING POINT OF THE FIRST KIND

by

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Abstract. — We discuss a local reduction theorem for 0-parameter solutions of a higher order Painlevé equation and its underlying Lax pair near a simple turning point of the first kind when the size of the Lax pair is greater than 2. As a typical example of such higher order Painlevé equations the Noumi-Yamada systems are mainly considered.

Résumé (Sur une réduction locale au voisinage d'un point tournant simple de première espèce des équations de Painlevé d'ordre supérieur et de leur paire de Lax)

Nous considérons les solutions sans paramètre d'une équation de Painlevé d'ordre supérieur au voisinage d'un point tournant simple et sa paire de Lax associée. Nous présentons un théorème de réduction locale et nous développons comme cas typique l'exemple des systèmes de Noumi-Yamada.

1. Introduction

The local reduction theorem for 0-parameter solutions of the traditional (i.e., second order) Painlevé equations with a large parameter (cf. [3], see also [5]) is generalized to those of some higher order Painlevé equations in [6] (cf. [4] for its announcement). That is, it is shown in [6] that a 0-parameter solution of each member of the first and second Painlevé hierarchies $(P_J)_m$ ($J = \text{I, II-1 and II-2}$; $m = 1, 2, 3, \dots$) discussed in [2] can be locally reduced to a 0-parameter solution of the traditional first Painlevé equation

$$(P_1) \quad \frac{d^2 u}{dt^2} = \eta^2(6u^2 + t)$$

near a simple turning point of $(P_J)_m$ of the first kind in the sense of [2]. In [6], to construct a local transformation which reduces a 0-parameter solution of $(P_J)_m$ to

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that of (P_I) , we make essential use of the fact that the Lax pair $(L_J)_m$ associated with $(P_J)_m$ consists of 2×2 systems. The purpose of this paper is to discuss the local reduction theorem for 0-parameter solutions of higher order Painlevé equations near a simple turning point of the first kind in the case where the size of the underlying Lax pair is greater than 2.

In this paper, as an example of higher order Painlevé equations whose underlying Lax pair is of size greater than 2, we mainly discuss the Noumi-Yamada systems [7], i.e., higher order Painlevé equations with the affine Weyl group symmetry of type $A_l^{(1)}$ ($l = 2, 3, 4, \dots$). The Noumi-Yamada systems can be considered as higher order analogue of the traditional fourth and fifth Painlevé equations (P_{IV}) and (P_V) . As the size of the Lax pair associated with the Noumi-Yamada system of type $A_l^{(1)}$ is $l + 1$, the result of [6] is not applicable to this case; instead we construct the reduction of the underlying Lax pair of the Noumi-Yamada system to that of the traditional first Painlevé equation (P_I) . This means that the local reduction for a 0-parameter solution of the Noumi-Yamada system is also constructed implicitly. For the precise statement of our main theorem see Theorem 2.2 in Section 2.

The plan of the paper is as follows: After recalling the explicit form of the Noumi-Yamada systems and reviewing some basic properties of their Stokes geometry studied in [9], we state our main theorem in Section 2. To prove our main theorem, we construct two reductions of the underlying Lax pair of the Noumi-Yamada system to that of (P_I) by the medium of the local reduction of a pair of first order linear systems to its normal form at a (simple or double) turning point discussed in [8], and employ a kind of “matching” method for the two reductions thus constructed. In Section 3 we briefly explain the results of [8] necessary for the proof of our main theorem and study the structure of transformations which keep the normal form at a turning point invariant. Using these results and a matching method, we finally give a proof of our main theorem in Section 4.

2. Main result

To state our main theorem we need to prepare some notions and notations about the Noumi-Yamada system and its Stokes geometry. Let us first recall the explicit form of the Noumi-Yamada system and its underlying Lax pair.

The Noumi-Yamada system of type $A_l^{(1)}$ in case l is even (i.e., when $l = 2m$; $m = 1, 2, \dots$) is the following system of first order nonlinear differential equations:

$$(1) \quad \frac{du_j}{dt} = \eta \left[u_j(u_{j+1} - u_{j+2} + \dots - u_{j+2m}) + \alpha_j \right]$$

($j = 0, 1, \dots, 2m$), where α_j are complex parameters satisfying

$$(2) \quad \alpha_0 + \dots + \alpha_{2m} = \eta^{-1}$$

and the unknown functions u_j and the independent variable t are normalized so that

$$(3) \quad u_0 + \dots + u_{2m} = t$$

may be satisfied, while in case l is odd (i.e., when $l = 2m + 1; m = 1, 2, \dots$) it is given by

$$(4) \quad \frac{t}{2} \frac{du_j}{dt} = \eta \left[u_j \left(\sum_{1 \leq r \leq s \leq m} u_{j-1+2r} u_{j+2s} - \sum_{1 \leq r \leq s \leq m} u_{j+2r} u_{j+1+2s} \right) + \frac{t}{2} \alpha_j \right]$$

($j = 0, 1, \dots, 2m + 1$), where α_j, u_j and t satisfy the following:

$$(5) \quad \alpha_0 + \alpha_2 + \dots + \alpha_{2m} = \alpha_1 + \alpha_3 + \dots + \alpha_{2m+1} = \eta^{-1}/2,$$

$$(6) \quad u_0 + u_2 + \dots + u_{2m} = u_1 + u_3 + \dots + u_{2m+1} = t/2.$$

The Lax pair associated with the Noumi-Yamada system of type $A_l^{(1)}$ consists of the following first order $N \times N$ ($N = l + 1$) systems of linear differential equations:

$$(7) \quad \frac{\partial}{\partial x} \psi = \eta A \psi, \quad \frac{\partial}{\partial t} \psi = \eta B \psi,$$

where

$$(8) \quad A = -\frac{1}{x} \begin{pmatrix} \epsilon_1 & u_1 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \epsilon_{N-2} & u_{N-2} & 1 & \\ x & & & \epsilon_{N-1} & u_{N-1} & \\ xu_0 & x & & & & \epsilon_N \end{pmatrix}$$

and

$$(9) \quad B = \begin{pmatrix} q_1 & -1 & & & & \\ & q_2 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & q_{N-1} & -1 & \\ -x & & & & & q_N \end{pmatrix}.$$

That is, (1) (resp., (4)) describes the compatibility condition

$$(10) \quad \frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB - BA) = 0$$

of (7) for $l = 2m$, i.e., $N = 2m + 1$ (resp., for $l = 2m + 1$, i.e., $N = 2m + 2$). Here ϵ_j are parameters determined by the relations $\alpha_j = \epsilon_j - \epsilon_{j+1} + \eta^{-1} \delta_{j,0}$ and $\epsilon_1 + \dots + \epsilon_N = 0$ ($\delta_{j,k}$ stands for Kronecker's delta), and $q_j = q_j(t)$ are functions of t satisfying $q_{j+2} - q_j = u_j - u_{j+1}$ and $q_1 + \dots + q_N = -t/2$.

As (1) is equivalent to the traditional fourth Painlevé equation (P_{IV}) when $l = 2$ (i.e., $m = 1$), Equation (1) can be considered as a higher order fourth Painlevé equation; Equation (1) and its underlying Lax pair (7) for $l = 2m$ are respectively referred to as $(P_{IV})_m$ and $(L_{IV})_m$ in what follows. Similarly Equation (4) and its underlying Lax pair (7) for $l = 2m + 1$ are respectively referred to as $(P_V)_m$ and

$(L_V)_m$, as (4) is equivalent to the traditional fifth Painlevé equation (P_V) when $l = 3$ (i.e., $m = 1$).

Our problem is to analyze the Noumi-Yamada system and its underlying Lax pair near a simple turning point of the first kind. A turning point of the Noumi-Yamada system and its basic properties are studied in [9]. It is defined as a turning point of the linearized equation (“Fréchet derivative”) at a 0-parameter solution. Here a 0-parameter solution of the Noumi-Yamada system is a formal solution of the form

$$(11) \quad \widehat{u}_j = \widehat{u}_j(t, \eta) = \widehat{u}_{j,0}(t) + \eta^{-1}\widehat{u}_{j,1}(t) + \cdots$$

($0 \leq j \leq 2m$ for $(P_{IV})_m$ and $0 \leq j \leq 2m + 1$ for $(P_V)_m$), and the linearized equation at $\widehat{u} = \{\widehat{u}_j\}$ is an equation obtained by setting $u_j = \widehat{u}_j + \Delta u_j$ in $(P_{IV})_m$ or $(P_V)_m$ and by taking its linear part in $\{\Delta u_j\}$. Note that the linearized equation at a 0-parameter solution $\{\widehat{u}_j\}$ is a system of first order linear differential equations for $\Delta u = {}^t(\Delta u_0, \dots, \Delta u_l)$ ($l = 2m$ for $(P_{IV})_m$ and $l = 2m + 1$ for $(P_V)_m$) and can be expressed as

$$(12) \quad \frac{d}{dt}\Delta u = \eta C \Delta u, \quad C = C(t, \eta) = C_0(t) + \eta^{-1}C_1(t) + \cdots .$$

A turning point of the first kind of the Noumi-Yamada system is then, by definition, a point $t = \tau$ where two non-trivial solutions $\nu^\pm(t)$ of the characteristic equation $\det(\nu - C_0(t)) = 0$ of (12) merge and their values $\nu^\pm(\tau)$ are equal to 0. That is, if we let P denote a polynomial of ν defined by $\nu^{-1} \det(\nu - C_0(t))$ for $(P_{IV})_m$ and by $\nu^{-2} \det(\nu - C_0(t))$ for $(P_V)_m$ (cf. [9, Proposition 2.3]), a turning point of the first kind is a point $t = \tau$ where $\nu = 0$ is a double root of $P = 0$. Note that a turning point of the first kind is also a branch point of the Riemann surface \mathcal{R} associated with the 0-parameter solution. In what follows we assume that a turning point $t = \tau$ of the first kind is a square-root type branch point of \mathcal{R} and that τ is simple in the sense of [1]; to be more specific, using a local parameter $s = (t - \tau)^{1/2}$ of \mathcal{R} at $t = \tau$, we require that the polynomial $P = P(s, \nu)$ of ν should satisfy the following conditions at $(s, \nu) = (0, 0)$:

$$(13) \quad P(0, 0) = \frac{\partial P}{\partial \nu}(0, 0) = 0, \quad \frac{\partial P}{\partial s}(0, 0) \neq 0, \quad \frac{\partial^2 P}{\partial \nu^2}(0, 0) \neq 0.$$

Substituting a 0-parameter solution $\{\widehat{u}_j\}$ of the Noumi-Yamada system into the coefficients of the underlying Lax pair (7), we now obtain the Lax pair

$$(14) \quad \frac{\partial}{\partial x}\psi = \eta A\psi, \quad A = A(x, t, \eta) = A_0(x, t) + \eta^{-1}A_1(x, t) + \cdots ,$$

$$(15) \quad \frac{\partial}{\partial t}\psi = \eta B\psi, \quad B = B(x, t, \eta) = B_0(x, t) + \eta^{-1}B_1(x, t) + \cdots ,$$

the compatibility condition of which is satisfied as a formal power series of η^{-1} . Then, as is proved in [9, Theorem 2.1], a double turning point $x = b(t)$ of the first equation (14) of the Lax pair merges with a simple turning point $x = a(t)$ of (14) at a turning point $t = \tau$ of the first kind of the Noumi-Yamada system, provided that the following

genericity condition should hold at $x = a(t)$, which is also a turning point of the second equation (15) of the Lax pair:

(16) At $x = a(t)$ exactly two eigenvalues of $B_0(x, t)$ merge and the other eigenvalues are mutually distinct.

Note that the same pair of the eigenvalues of $A_0(x, t)$, denoted by $\lambda^\pm(x, t)$, merges both at $x = b(t)$ and at $x = a(t)$. Furthermore, letting $\nu^\pm(t)$ denote the two non-trivial solutions of the characteristic equation $\det(\nu - C_0(t)) = 0$ of (12) satisfying $\nu^+(\tau) = \nu^-(\tau) = 0$ and $\nu^-(t) = -\nu^+(t)$, we find that the following relation holds:

$$(17) \quad \frac{1}{2} \int_\tau^t (\nu^+(t) - \nu^-(t)) dt = \int_{a(t)}^{b(t)} (\lambda^+(x, t) - \lambda^-(x, t)) dx.$$

This relation (17) guarantees that, if $t = \sigma$ is a point on a Stokes curve of the Noumi-Yamada system emanating from τ , i.e., a curve in the t -plane (or, rather on the Riemann surface \mathcal{R}) given by

$$(18) \quad \text{Im} \int_\tau^t (\nu^+(t) - \nu^-(t)) dt = 0,$$

and further if $t = \sigma$ is sufficiently close to τ , then the two turning points $b(\sigma)$ and $a(\sigma)$ of (14) are connected by a Stokes curve (or, rather Stokes segment) of (14). The Stokes segment of (14), denoted by $\gamma = \gamma(\sigma)$, connecting $b(\sigma)$ and $a(\sigma)$ plays a crucially important role in the following argument; we try to construct a transformation which reduces the Lax pair (14) and (15) of the Noumi-Yamada system to that of the traditional first Painlevé equation (P_I) semi-globally near γ .

In view of (16), as the same pair $\lambda^\pm(x, t)$ of the eigenvalues of A_0 merges both at $x = b(t)$ and at $x = a(t)$, the Lax pair (14) and (15) can be simultaneously block-diagonalized in a neighborhood of $(x, t) = (a(\tau), \tau) (= (b(\tau), \tau))$. (For the block-diagonalization we refer the reader to, e.g., [8, Proposition 1]. See also [10], [11].) That is, (14) and (15) can be transformed into a system of the form

$$(19) \quad \frac{\partial}{\partial x} \tilde{\psi} = \eta \tilde{A}(x, t, \eta) \tilde{\psi}, \quad \tilde{A}(x, t, \eta) = \left(\begin{array}{c|c} A^{(1)} & 0 \\ \hline 0 & A^{(2)} \end{array} \right),$$

$$(20) \quad \frac{\partial}{\partial t} \tilde{\psi} = \eta \tilde{B}(x, t, \eta) \tilde{\psi}, \quad \tilde{B}(x, t, \eta) = \left(\begin{array}{c|c} B^{(1)} & 0 \\ \hline 0 & B^{(2)} \end{array} \right),$$

where $A^{(1)} = \sum_j \eta^{-j} A_j^{(1)}$ and $B^{(1)} = \sum_j \eta^{-j} B_j^{(1)}$ are (formal power series of η^{-1} with coefficients of) 2×2 matrices while $A^{(2)}$ and $B^{(2)}$ are $(l-1) \times (l-1)$ diagonal matrices with distinct diagonal components, by a transformation $\psi = (\sum_j \eta^{-j} P_j(x, t)) \tilde{\psi}$ in a neighborhood of $(x, t) = (a(\tau), \tau)$ (in particular, in a neighborhood of the Stokes segment γ). Here the eigenvalues of $A_0^{(1)}$ are given by the merging ones $\lambda^\pm(x, t)$ and hence the problem is reduced to that for the 2×2 blocks,

i.e., a pair of the following systems:

$$(21) \quad \frac{\partial}{\partial x} \varphi = \eta A^{(1)} \varphi, \quad A^{(1)} = A_0^{(1)}(x, t) + \eta^{-1} A_1^{(1)}(x, t) + \dots,$$

$$(22) \quad \frac{\partial}{\partial t} \varphi = \eta B^{(1)} \varphi, \quad B^{(1)} = B_0^{(1)}(x, t) + \eta^{-1} B_1^{(1)}(x, t) + \dots.$$

In what follows we assume that $x = b(t)$ is a “rank-zero type” double turning point of (21), i.e.,

$$(23) \quad \text{rank}(A_0^{(1)}(b(t), t) - \lambda_b(t)I_2) = 0,$$

where $\lambda_b(t)$ denotes the value at $x = b(t)$ of the two merging eigenvalues $\lambda^\pm(x, t)$ of $A_0^{(1)}$ and I_2 is the 2×2 identity matrix.

Remark 2.1. — *Although a rank-zero type double turning point is a degenerate one from the viewpoint of linear algebra and a double turning point satisfying*

$$(24) \quad \text{rank}(A_0^{(1)}(b(t), t) - \lambda_b(t)I_2) = 1$$

(“rank-one type”) should be more generic, every double turning point of (the first equation of) the Lax pair associated with a (traditional or higher order) Painlevé equation is of rank-zero type as far as we know. For example, in the cases of the traditional Painlevé equations and of the first and second Painlevé hierarchies discussed in [1] it is rigorously confirmed that all double turning points of the Lax pair are of rank-zero type. We surmise that any double turning point of the Lax pair associated with a higher order Painlevé equation is always of rank-zero type.

In the case of the first and second Painlevé hierarchies $(P_J)_m$ ($J = \text{I}, \text{II-1}, \text{II-2}$) discussed in [2] the underlying Lax pair consists of 2×2 systems and it is not necessary to use the block-diagonalization. In these cases, deriving a pair of Schrödinger (i.e., second order) equation $(SL_J)_m$ and its deformation equation $(D_J)_m$ from the Lax pair $(L_J)_m$ associated with $(P_J)_m$ and studying some analytic properties of these equations for one unknown function by making full use of their explicit forms, we construct in [6] a transformation $(\tilde{x}(x, t, \eta), \tilde{t}(t, \eta)) = (\sum_{j \geq 0} \eta^{-j} \tilde{x}_j(x, t), \sum_{j \geq 0} \eta^{-j} \tilde{t}_j(t))$ that brings $(SL_J)_m$ to (SL_I) , the Schrödinger equation underlying the traditional first Painlevé equation (P_I) , semi-globally near the Stokes segment γ and, furthermore, that reduces a formal series $b_j(t, \eta)$ ($j = 1, \dots, m$), whose elementary symmetric polynomials give 0-parameter solutions of $(P_J)_m$, to a 0-parameter solution $u_I(\tilde{t}, \eta) = \sum_j \eta^{-j} u_{I,j}(\tilde{t})$ of (P_I) in the sense that the following relation holds:

$$(25) \quad \tilde{x}(x, t, \eta) \Big|_{x=b_j(t, \eta)} = u_I(\tilde{t}(t, \eta), \eta).$$

(For the precise statement see [6, Proposition 3.2.1 and Theorem 3.2.1].) Applying the same technique to the 2×2 blocks (21) and (22) of the block-diagonalized Lax pair (19) and (20), we might obtain a similar conclusion also for the Noumi-Yamada systems. However, as the block-diagonalization has been employed, the explicit form

of (21) and (22) and that of the Schrödinger equation derived from them become too complicated to be analyzed by this technique. Instead, we discuss the semi-global transformation of (21) and (22) in the original matrix form, i.e., without rewriting them into a pair of single differential equations for one unknown function.

Now let us state our main theorem.

Theorem 2.2. — *Let τ be a simple turning point of the first kind of the Noumi-Yamada system $(P_J)_m$ ($J = \text{IV}, \text{V}; m = 1, 2, \dots$), and let $b(t)$ and $a(t)$ respectively be the double and simple turning points of the first equation (14) (i.e., equation in the x -direction) of the underlying Lax pair that merge at $t = \tau$. Suppose that the conditions (16) and (23) should be satisfied. We further let σ ($\neq \tau$) be a point that is sufficiently close to τ and that lies in a Stokes curve of $(P_J)_m$ emanating from τ , and let $\gamma = \gamma(\sigma)$ denote the Stokes segment of (14) which connects the two turning points $b(\sigma)$ and $a(\sigma)$. Then there exist a neighborhood Ω of γ , a neighborhood ω of σ , holomorphic functions $\tilde{x}_0(x, t)$ on $\Omega \times \omega$ and $\tilde{t}_0(t)$ on ω , and 2×2 matrices $P_j(x, t)$ ($j = 0, 1, 2, \dots$) whose entries are holomorphic functions on $\Omega \times \omega$ so that they satisfy the following relations:*

(i) *The function $\tilde{t}_0(t)$ satisfies*

$$(26) \quad \int_{\tau}^t (\nu^+(t) - \nu^-(t))dt = \left(2 \int_0^{\tilde{t}} \sqrt{12u_{\text{I},0}(\tilde{t})d\tilde{t}} \right) \Big|_{\tilde{t}=\tilde{t}_0(t)},$$

where ν^{\pm} denote the two non-trivial solutions of the characteristic equation $\det(\nu - C_0(t)) = 0$ of the Fréchet derivative of $(P_J)_m$ satisfying $\nu^+(\tau) = \nu^-(\tau) = 0$ and $\nu^-(t) = -\nu^+(t)$.

(ii) $\tilde{x}_0(b(t), t) = u_{\text{I},0}(\tilde{t}_0(t))$ and $\tilde{x}_0(a(t), t) = -2u_{\text{I},0}(\tilde{t}_0(t))$.

(iii) $d\tilde{t}_0/dt \neq 0$ on ω , $\partial\tilde{x}_0/\partial x \neq 0$ on $\Omega \times \omega$ and $\det P_0(x, t) \neq 0$ on $\Omega \times \omega$.

(iv) *By a change of variables $(x, t) \mapsto (\tilde{x}, \tilde{t}) = (\tilde{x}_0(x, t), \tilde{t}_0(t))$ and a transformation*

$$(27) \quad \varphi = \exp \left(\frac{\eta}{2} \int_{(a(\tau), \tau)}^{(x, t)} (\text{trace } A_0^{(1)} dx + \text{trace } B_0^{(1)} dt) \right) P(x, t, \eta) \tilde{\varphi}$$

with $P(x, t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, t)$, the 2×2 blocks (21) and (22) of the block-diagonalized Lax pair (19) and (20) of $(P_J)_m$ can be transformed to the underlying Lax pair of the traditional first Painlevé equation (P_{I}) , i.e.,

$$(L_{\text{I}}) \quad \frac{\partial}{\partial \tilde{x}} \tilde{\varphi} = \eta \tilde{A} \tilde{\varphi}, \quad \frac{\partial}{\partial \tilde{t}} \tilde{\varphi} = \eta \tilde{B} \tilde{\varphi},$$

where

$$(28) \quad \tilde{A} = \begin{pmatrix} \eta^{-1} du_{\text{I}}/d\tilde{t} & 4(\tilde{x} - u_{\text{I}}) \\ \tilde{x}^2 + u_{\text{I}}\tilde{x} + u_{\text{I}}^2 + \tilde{t}/2 & -\eta^{-1} du_{\text{I}}/d\tilde{t} \end{pmatrix},$$

$$(29) \quad \tilde{B} = \begin{pmatrix} 0 & 2 \\ \tilde{x}/2 + u_{\text{I}} & 0 \end{pmatrix},$$

and $u_{\text{I}} = u_{\text{I}}(\tilde{t}, \eta) = \sum_j \eta^{-j} u_{\text{I},j}(\tilde{t})$ (with $u_{\text{I},0} = \sqrt{-\tilde{t}/6}$) denotes a 0-parameter solution of (P_{I}) .

Remark 2.3. — We have not yet obtained an explicit formula like (25) that relates a 0-parameter solution of the Noumi-Yamada system to that of (P_{I}) . However, as the reduction of its underlying Lax pair is constructed, it can be considered that the reduction of a 0-parameter solution is also constructed in an implicit manner.

Remark 2.4. — Theorem 2.2 is also applicable to the traditional Painlevé equations (P_J) ($J = \text{II}, \dots, \text{VI}$) and the first and second Painlevé hierarchies $(P_J)_m$ ($J = \text{I}, \text{II-1}, \text{II-2}$) discussed in [2]. In these cases it is not necessary to assume the conditions (16) and (23) and the reasoning in Section 4 below gives a new proof for the known reduction theorem (except for the relation (25) between the two 0-parameter solutions).

3. Normal form of first order linear systems at a turning point

To construct a semi-global reduction of the Lax pair of $(P_J)_m$ to that of (P_{I}) , we use the local reduction of a pair of first order 2×2 systems of linear differential equations to its normal form at a turning point studied in [8]. In this section we review the results of [8] that are necessary for the proof of Theorem 2.2.

Let

$$(30) \quad \frac{\partial}{\partial x} \varphi = \eta A(x, t, \eta) \varphi, \quad A(x, t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} A_j(x, t),$$

$$(31) \quad \frac{\partial}{\partial t} \varphi = \eta B(x, t, \eta) \varphi, \quad B(x, t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} B_j(x, t)$$

be a pair of 2×2 systems, where $A_j(x, t)$ and $B_j(x, t)$ are 2×2 matrices with holomorphic entries. We assume that the compatibility condition

$$(32) \quad \frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta[A, B] = 0$$

of (30) and (31), where $[A, B] = AB - BA$ denotes the commutator of A and B , should be satisfied. By using a gauge transformation

$$(33) \quad \varphi = \exp \left(\frac{\eta}{2} \int_{(x_0, t_0)}^{(x, t)} (\text{trace } A_0 dx + \text{trace } B_0 dt) \right) \tilde{\varphi},$$

we may also assume without loss of generality that $\text{trace } A_0(x, t) = \text{trace } B_0(x, t) = 0$. (Note that it follows from the compatibility condition that $\omega = \text{trace } A_0 dx + \text{trace } B_0 dt$ is a closed 1-form in the (x, t) -space.)

We first discuss the normal form of the simultaneous system (30) and (31) at a rank-zero type double turning point.

Proposition 3.1. — *Let $x = b(t)$ be a rank-zero type (i.e., $\text{rank}(A_0(b(t), t) - \lambda_b(t)I_2) = 0$, where $\lambda_b(t)$ is the value at $x = b(t)$ of the merging eigenvalues of $A_0(x, t)$) double turning point of the first equation (30). Suppose that $B_0(x, t)$ has distinct eigenvalues $\pm\mu_b(t)$ ($\neq 0$) at $x = b(t)$. Then, if we define holomorphic functions $z = z(x, t)$ and $s = s(t)$ by*

$$(34) \quad z(x, t) = \left(2 \int_{b(t)}^x \sqrt{-\det A_0(x, t)} dx \right)^{1/2},$$

$$(35) \quad s(t) = \int^t \mu_b(t) dt + C_0$$

(where C_0 is an arbitrary constant independent of x and t), the simultaneous system (30) and (31) can be transformed into

$$(36) \quad \frac{\partial}{\partial z} \tilde{\varphi} = \eta \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} \tilde{\varphi}, \quad \frac{\partial}{\partial s} \tilde{\varphi} = \eta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\varphi}$$

by a change of variables $(x, t) \mapsto (z, s) = (z(x, t), s(t))$ and a formal transformation of the form

$$(37) \quad \varphi = P(x, t, \eta) \tilde{\varphi} = \left(\sum_{j=0}^{\infty} \eta^{-j} P_j(x, t) \right) \tilde{\varphi},$$

where $P_j(x, t)$ ($j = 0, 1, \dots$) are 2×2 matrices with holomorphic entries satisfying $\det P_0(x, t) \neq 0$.

Similarly the normal form of (30) and (31) at a simple turning point is described by the following

Proposition 3.2. — *Let $x = a(t)$ be a simple turning point of the first equation (30). Then, if we define a holomorphic function $z = z(x, t)$ by*

$$(38) \quad z(x, t) = \left(\frac{3}{2} \int_{a(t)}^x \sqrt{-\det A_0(x, t)} dx \right)^{3/2},$$

the simultaneous system (30) and (31) can be transformed into

$$(39) \quad \frac{\partial}{\partial z} \tilde{\varphi} = \eta \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \tilde{\varphi}, \quad \frac{\partial}{\partial t} \tilde{\varphi} = 0$$

by a change of variables $(x, t) \mapsto (z, t) = (z(x, t), t)$ and a formal transformation of the form (37).

For the proof of Propositions 3.1 and 3.2 see [8, Section 3].

To prove Theorem 2.2, we also use a transformation which keeps the above normal form at a turning point invariant. In the remaining part of this section we study the structure of such transformations.

Proposition 3.3. — *A transformation*

$$(40) \quad \varphi = P(z, s, \eta) \tilde{\varphi} = \left(\sum_{j=0}^{\infty} \eta^{-j} P_j(z, s) \right) \tilde{\varphi},$$

where P_j is a 2×2 matrix with holomorphic entries and $\det P_0(z, s) \neq 0$, keeps the normal form at a rank-zero type double turning point

$$(41) \quad \frac{\partial}{\partial z} \varphi = \eta \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} \varphi, \quad \frac{\partial}{\partial s} \varphi = \eta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \varphi$$

invariant if and only if P_j is of the following form:

$$(42) \quad P_j = \begin{pmatrix} \alpha_j & 0 \\ 0 & \beta_j \end{pmatrix},$$

where α_j and β_j ($j = 0, 1, \dots$) are constants independent of z and s with $\alpha_0 \beta_0 \neq 0$.

Proof. — Since we readily confirm that a transformation P of the form (42) keeps (41) invariant, it suffices to prove that a transformation (40) which keeps (41) invariant must be of the form (42).

Let us assume that (40) keeps (41) invariant. Then we have

$$(43) \quad \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} = P^{-1} \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} P - \eta^{-1} P^{-1} \frac{\partial P}{\partial z},$$

$$(44) \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = P^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P - \eta^{-1} P^{-1} \frac{\partial P}{\partial s},$$

that is,

$$(45) \quad \frac{\partial P}{\partial z} = \eta z \sum_{j=0}^{\infty} \eta^{-j} [J, P_j], \quad \frac{\partial P}{\partial s} = \eta \sum_{j=0}^{\infty} \eta^{-j} [J, P_j],$$

where

$$(46) \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The relations (45) first imply that the top order part (i.e., degree (-1) part in η^{-1}) of their right-hand sides should vanish, i.e., $[J, P_0] = 0$. Hence P_0 must be diagonal. Next, comparing the degree 0 part (in η^{-1}) of both sides of (45), we obtain

$$(47) \quad \frac{\partial P_0}{\partial z} = z[J, P_1], \quad \frac{\partial P_0}{\partial s} = [J, P_1].$$

Since P_0 is diagonal, the left-hand sides of (47) are diagonal, while the diagonal components of the right-hand sides vanish. Hence we find $\partial P_0 / \partial z = \partial P_0 / \partial s = 0$, that is, P_0 is of the form (42), and further we obtain $[J, P_1] = 0$. Then, by using an induction on j , we can prove that all P_j is of the form (42). This completes the proof of Proposition 3.3. \square

Proposition 3.4. — *A transformation*

$$(48) \quad \varphi = P(z, t, \eta)\tilde{\varphi} = \left(\sum_{j=0}^{\infty} \eta^{-j} P_j(z, t) \right) \tilde{\varphi},$$

where P_j is a 2×2 matrix with holomorphic entries and $\det P_0(z, t) \neq 0$, keeps the normal form at a simple turning point

$$(49) \quad \frac{\partial}{\partial z} \varphi = \eta \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \varphi, \quad \frac{\partial}{\partial t} \varphi = 0$$

invariant if and only if P_j is of the following form:

$$(50) \quad P_j = \alpha_j I_2 = \begin{pmatrix} \alpha_j & 0 \\ 0 & \alpha_j \end{pmatrix},$$

where α_j ($j = 0, 1, \dots$) are constants independent of z and t with $\alpha_0 \neq 0$.

Proof. — It suffices to prove that, if $P = \sum \eta^{-j} P_j(z, t)$ satisfies

$$(51) \quad \frac{\partial P}{\partial z} = \eta \sum_{j=0}^{\infty} \eta^{-j} \left[\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}, P_j \right], \quad \frac{\partial P}{\partial t} = 0,$$

then P_j is of the form (50).

We first note that

$$(52) \quad \left[\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = (c - bz) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (a - d) \begin{pmatrix} 0 & -1 \\ z & 0 \end{pmatrix}.$$

Since the degree (-1) part (in η^{-1}) of the right-hand sides of (51) vanishes, we then find that P_0 is of the form

$$(53) \quad P_0 = \alpha_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}$$

in view of (52). Next, if we write P_1 as

$$(54) \quad P_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

comparison of the degree 0 part of both sides of (51) entails that $\partial\alpha_0/\partial t = \partial\beta_0/\partial t = 0$ and

$$(55) \quad \begin{aligned} \frac{\partial P_0}{\partial z} &= \frac{\partial\alpha_0}{\partial z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\partial\beta_0}{\partial z} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}, P_1 \right] \\ &= (c_1 - b_1 z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (a_1 - d_1) \begin{pmatrix} 0 & -1 \\ z & 0 \end{pmatrix}. \end{aligned}$$

This implies that

$$(56) \quad \frac{\partial \alpha_0}{\partial z} = 0, \quad 2z \frac{\partial \beta_0}{\partial z} + \beta_0 = 0.$$

Hence $\beta_0 \equiv 0$ and α_0 is independent of z and t , i.e., P_0 is of the form (50). Furthermore we consequently obtain

$$(57) \quad \left[\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}, P_1 \right] = 0.$$

Thus the induction on j proceeds, completing the proof of Proposition 3.4. \square

Propositions 3.1, 3.2, 3.3 and 3.4 clarify the structure of the normal form at (rank-zero type double or simple) turning points and its invariant subgroup (i.e., stable subgroup) in the formal transformation group. On the other hand, it is well-known that at a regular point (i.e., at a point where eigenvalues of A_0 are distinct and so are eigenvalues of B_0) a pair of 2×2 systems of the form (30) and (31) can be simultaneously diagonalized as

$$(58) \quad \frac{\partial}{\partial x} \tilde{\varphi} = \eta \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \tilde{\varphi}, \quad \lambda^\pm = \lambda^\pm(x, t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} \lambda_j^\pm(x, t),$$

$$(59) \quad \frac{\partial}{\partial t} \tilde{\varphi} = \eta \begin{pmatrix} \mu^+ & 0 \\ 0 & \mu^- \end{pmatrix} \tilde{\varphi}, \quad \mu^\pm = \mu^\pm(x, t, \eta) = \sum_{j=0}^{\infty} \eta^{-j} \mu_j^\pm(x, t),$$

where $\lambda_0^+ + \lambda_0^- = \mu_0^+ + \mu_0^- = 0$ and $\lambda_0^+ \lambda_0^- \mu_0^+ \mu_0^- \neq 0$ hold (cf., e.g., [8, Section 3]). The structure of transformations which keep such a pair of diagonal systems invariant can be described as follows:

Proposition 3.5. — *A transformation*

$$(60) \quad \varphi = P(x, t, \eta) \tilde{\varphi} = \left(\sum_{j=0}^{\infty} \eta^{-j} P_j(x, t) \right) \tilde{\varphi},$$

where P_j is a 2×2 matrix with holomorphic entries and $\det P_0(x, t) \neq 0$, keeps a pair of diagonal systems

$$(61) \quad \frac{\partial}{\partial x} \varphi = \eta \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \varphi, \quad \frac{\partial}{\partial t} \varphi = \eta \begin{pmatrix} \mu^+ & 0 \\ 0 & \mu^- \end{pmatrix} \varphi$$

with $\lambda_0^+ + \lambda_0^- = \mu_0^+ + \mu_0^- = 0$ and $\lambda_0^+ \lambda_0^- \mu_0^+ \mu_0^- \neq 0$ invariant if and only if P_j is of the following form:

$$(62) \quad P_j = \begin{pmatrix} \alpha_j & 0 \\ 0 & \beta_j \end{pmatrix},$$

where α_j and β_j ($j = 0, 1, \dots$) are constants independent of x and t with $\alpha_0 \beta_0 \neq 0$.

As the proof of Proposition 3.5 is similar to that of Proposition 3.3, we omit it here.

4. Proof of Theorem 2.2

Thanks to the block-diagonalization, the problem is reduced to that for the 2×2 system (21) and (22). Furthermore, using a gauge transformation of the form (33), we may assume without loss of generality that $\text{trace } A_0^{(1)}(x, t)$ and $\text{trace } B_0^{(1)}(x, t)$ identically vanish. Thus, assuming $\text{trace } A_0^{(1)}(x, t) = \text{trace } B_0^{(1)}(x, t) = 0$, we discuss from now on the reduction of (21) and (22) to the underlying Lax pair (L_I) of the traditional first Painlevé equation (P_I) . For the sake of simplicity we abbreviate the coefficient matrices $A^{(1)}$ and $B^{(1)}$ of (21) and (22) as A and B in what follows.

The eigenvalues $\pm \sqrt{-\det A_0(x, t)}$ of $A_0(x, t)$ merge both at a rank-zero type double turning point $x = b(t)$ and at a simple turning point $x = a(t)$. Further, since the difference of two eigenvalues of A_0 is invariant under the gauge transformation (33), it follows from (17) that

$$(63) \quad \frac{1}{2} \int_{\tau}^t (\nu^+(t) - \nu^-(t)) dt = \int_{a(t)}^{b(t)} (\lambda^+(x, t) - \lambda^-(x, t)) dx = 2 \int_{a(t)}^{b(t)} \sqrt{-\det A_0(x, t)} dx.$$

On the other hand, the argument of [9, Section 3.2] verifies that the eigenvalues $\pm \mu_b(t)$ of $B_0(x, t)$ at $x = b(t)$ satisfy

$$(64) \quad \pm \mu_b(t) = \frac{1}{2} \nu^{\pm}(t),$$

and consequently they are distinct except at $t = \tau$. Hence Propositions 3.1 and 3.2 are applicable to the system (21) and (22); that is, letting (L) , (L_b) , and (L_a) respectively denote the system (21) and (22), the normal form (36) at a rank-zero type double turning point, and the normal form (39) at a simple turning point, we can transform (L) to (L_b) (resp., (L_a)) near $x = b(t)$ (resp., $x = a(t)$) by a change of variables $(z_b, s_b) = (z_b(x, t), s_b(t))$ (resp., $(z_a, s_a) = (z_a(x, t), s_a(t))$ with $s_a \equiv t$) and a formal transformation $P^b(x, t, \eta) = \sum_j \eta^{-j} P_j^b(x, t)$ (resp., $P^a(x, t, \eta) = \sum_j \eta^{-j} P_j^a(x, t)$) of the form (37). In a similar manner, by straightforward computations we readily confirm the following properties for the underlying Lax pair (L_I) of (P_I) :

$$(65) \quad \tilde{t} = 0 \text{ is a (unique) turning point of the first kind of } (P_I).$$

$$(66) \quad \det \tilde{A}_0(\tilde{x}, \tilde{t}) = -4(\tilde{x} - u_{I,0}(\tilde{t}))(\tilde{x} + 2u_{I,0}(\tilde{t})). \text{ In particular, (the first equation of) } (L_I) \text{ has a rank-zero type double turning point at } \tilde{x} = u_{I,0}(\tilde{t}) \text{ and a simple turning point at } \tilde{x} = -2u_{I,0}(\tilde{t}).$$

$$(67) \quad \text{The eigenvalues of } \tilde{B}_0 \text{ at } \tilde{x} = u_{I,0}(\tilde{t}) \text{ coincide with } \tilde{\nu}^{\pm}(\tilde{t})/2 = \pm \sqrt{12u_{I,0}(\tilde{t})}/2, \text{ a half of the characteristic roots of the Fréchet derivative of } (P_I) \text{ at a 0-parameter solution } \tilde{u} = u_I(\tilde{t}, \eta).$$

$$(68) \quad \frac{1}{2} \int_0^{\tilde{t}} (\tilde{\nu}^+(\tilde{t}) - \tilde{\nu}^-(\tilde{t})) d\tilde{t} \left(= \int_0^{\tilde{t}} \sqrt{12u_{I,0}(\tilde{t})} d\tilde{t} \right) = 2 \int_{-2u_{I,0}(\tilde{t})}^{u_{I,0}(\tilde{t})} \sqrt{-\det \tilde{A}_0(\tilde{x}, \tilde{t})} d\tilde{x}.$$

Hence we can apply Proposition 3.1 (resp., Proposition 3.2) also to (L_I) near $\tilde{x} = u_{I,0}(\tilde{t})$ (resp., $\tilde{x} = -2u_{I,0}(\tilde{t})$) to obtain a reduction of (L_I) to (L_b) (resp., (L_a)) through a change of variables $(\tilde{z}_b, \tilde{s}_b) = (\tilde{z}_b(\tilde{x}, \tilde{t}), \tilde{s}_b(\tilde{t}))$ (resp., $(\tilde{z}_a, \tilde{s}_a) = (\tilde{z}_a(\tilde{x}, \tilde{t}), \tilde{s}_a(\tilde{t}))$ with $\tilde{s}_a \equiv \tilde{t}$) and a formal transformation $\tilde{P}^b(\tilde{x}, \tilde{t}, \eta) = \sum_j \eta^{-j} \tilde{P}_j^b(\tilde{x}, \tilde{t})$ (resp., $\tilde{P}^a(\tilde{x}, \tilde{t}, \eta) = \sum_j \eta^{-j} \tilde{P}_j^a(\tilde{x}, \tilde{t})$) of the form (37).

By the medium of these transformations to the normal forms, the following two reductions of (L) to (L_I) are readily constructed; one is a reduction at $x = b(t)$, that is, a change of variables $(x, t) \mapsto (\tilde{x}, \tilde{t}) = (\tilde{x}_b(x, t), \tilde{t}_b(t))$ and a formal transformation $R_{\alpha, \beta}^b$ respectively defined by

$$(69) \quad (\tilde{z}_b(\tilde{x}, \tilde{t}), \tilde{s}_b(\tilde{t})) \Big|_{\substack{\tilde{x}=\tilde{x}_b(x,t) \\ \tilde{t}=\tilde{t}_b(t)}} = (z_b(x, t), s_b(t))$$

and

$$(70) \quad \varphi = R_{\alpha, \beta}^b(x, t, \eta) \tilde{\varphi} = P^b(x, t, \eta) P_{\alpha, \beta}(\eta) (\tilde{P}^b)^{-1}(\tilde{x}, \tilde{t}, \eta) \Big|_{\substack{\tilde{x}=\tilde{x}_b(x,t) \\ \tilde{t}=\tilde{t}_b(t)}} \tilde{\varphi},$$

where

$$(71) \quad P_{\alpha, \beta}(\eta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 + \eta^{-1} \alpha_1 + \cdots & 0 \\ 0 & \beta_0 + \eta^{-1} \beta_1 + \cdots \end{pmatrix}$$

(with α_j and β_j being constants independent of x and t), and another is a reduction at $x = a(t)$ defined by

$$(72) \quad (\tilde{z}_a(\tilde{x}, \tilde{t}), \tilde{t}) \Big|_{\substack{\tilde{x}=\tilde{x}_a(x,t) \\ \tilde{t}=\tilde{t}_a(t)}} = (z_a(x, t), \phi(t)),$$

$$(73) \quad \varphi = R^a(x, t, \eta) \tilde{\varphi} = P^a(x, t, \eta) (\tilde{P}^a)^{-1}(\tilde{x}, \tilde{t}, \eta) \Big|_{\substack{\tilde{x}=\tilde{x}_a(x,t) \\ \tilde{t}=\tilde{t}_a(t)}} \tilde{\varphi},$$

where $\phi(t)$ is a holomorphic function in a neighborhood of $t = \tau$ satisfying $(d\phi/dt)(\tau) \neq 0$. Here, in defining these two reductions, we have introduced a transformation $P_{\alpha, \beta}(\eta)$, which keeps the normal form (L_b) at $x = b(t)$ invariant in view of Proposition 3.3, and a change of variable $t \mapsto \phi(t)$ in the t -space, which clearly keeps the normal form (L_a) at $x = a(t)$ invariant, so that we may employ a kind of “matching” method: As a matter of fact, we prove in what follows that these two reductions give the same one through a suitable choice of α , β and $\phi(t)$.

Let us first consider the change of variables. Proposition 3.1 and (69) together with (64) and (67) tell us that the change of variables $(\tilde{x}_b(x, t), \tilde{t}_b(t))$ at $x = b(t)$ is determined by the relations

$$(74) \quad \int_{u_{I,0}(\tilde{t})}^{\tilde{x}} \sqrt{-\det \tilde{A}_0(\tilde{x}, \tilde{t})} d\tilde{x} \Big|_{\substack{\tilde{x}=\tilde{x}_b(x,t) \\ \tilde{t}=\tilde{t}_b(t)}} = \int_{b(t)}^x \sqrt{-\det A_0(x, t)} dx,$$

$$(75) \quad \int_0^{\tilde{t}} \sqrt{12u_{I,0}(\tilde{t})} d\tilde{t} \Big|_{\tilde{t}=\tilde{t}_b(t)} = \frac{1}{2} \int_{\tau}^t (\nu^+(t) - \nu^-(t)) dt + C_0,$$

where C_0 is a constant independent of x and t , while Proposition 3.2 and (72) entail that $(\tilde{x}_a(x, t), \tilde{t}_a(t)) = (\tilde{x}_a(x, t), \phi(t))$ is determined by

$$(76) \quad \int_{-2u_{I,0}(\tilde{t})}^{\tilde{x}} \sqrt{-\det \tilde{A}_0(\tilde{x}, \tilde{t})} d\tilde{x} \Big|_{\substack{\tilde{x}=\tilde{x}_a(x,t) \\ \tilde{t}=\phi(t)}} = \int_{a(t)}^x \sqrt{-\det A_0(x, t)} dx.$$

Now we choose C_0 to be 0 and define $\tilde{t}_b(t)$ by the relation (75). That is, we define $\tilde{t}_b(t)$ to be a constant multiple of

$$(77) \quad \left(\int_{\tau}^t (\nu^+(t) - \nu^-(t)) dt \right)^{4/5}.$$

Note that, since the assumption (13) implies that $\nu^{\pm}(t)$ is of exactly order $(t - \tau)^{1/4}$ at $t = \tau$, $\tilde{t}_b(t)$ is holomorphic in a neighborhood of τ . It then follows from (63), (68) and (75) (with $C_0 = 0$) that

$$(78) \quad \int_{-2u_{I,0}(\tilde{t})}^{u_{I,0}(\tilde{t})} \sqrt{-\det \tilde{A}_0(\tilde{x}, \tilde{t})} d\tilde{x} \Big|_{\tilde{t}=\tilde{t}_b(t)} = \int_{a(t)}^{b(t)} \sqrt{-\det A_0(x, t)} dx.$$

This relation (78) guarantees that $\tilde{x}_b(x, t)$ determined by (74) also satisfies the equation (76) for $\tilde{x}_a(x, t)$ with $\phi(t)$ being replaced by $\tilde{t}_b(t)$. We thus conclude that the two change of variables $(\tilde{x}_b(x, t), \tilde{t}_b(t))$ and $(\tilde{x}_a(x, t), \phi(t))$ coincide by setting $\phi(t) = \tilde{t}_b(t)$. (The holomorphy of $\tilde{x}_b(x, t)$ in a neighborhood of the Stokes segment $\gamma = \gamma(\sigma)$ is verified by the same reasoning as that of [6, Section 3.2].)

Next let us discuss matching between the two formal transformations $R_{\alpha,\beta}^b$ and R^a . We prove that, if we choose a suitable (α, β) , the transformation

$$(79) \quad \tilde{\varphi} = (R^a)^{-1} \varphi = (R^a)^{-1} R_{\alpha,\beta}^b \tilde{\varphi}$$

is an identity operator in the following manner: First we note that $R_{\alpha,\beta}^b$ (resp., R^a) is holomorphically extended along γ except at the terminal point $x = a(t)$ (resp., $x = b(t)$) since each coefficient of $R_{\alpha,\beta}^b$ and R^a respectively satisfies a linear ordinary differential equation with singularities only at $x = b(t)$ and $x = a(t)$. We now pick up a regular point $\tilde{x} = \tilde{c}$ of (L_I) between the two turning points $u_{I,0}(\tilde{t})$ and $-2u_{I,0}(\tilde{t})$ and consider an auxiliary transformation \tilde{P}^c which reduces (L_I) to a pair of 2×2 diagonal systems of the form (58) and (59) at $\tilde{x} = \tilde{c}$. Since both $R_{\alpha,\beta}^b$ and R^a transform (L) to (L_I) , the transformation (79) keeps (L_I) invariant and consequently

$$(80) \quad (\tilde{P}^c)^{-1} (R^a)^{-1} R_{\alpha,\beta}^b \tilde{P}^c$$

keeps the pair of diagonal systems invariant. It then follows from Proposition 3.5 that for any (α, β) the degree j part (with respect to η^{-1}) of (80) must be of the form (62), that is,

$$(81) \quad (\tilde{P}^c(\tilde{x}, \tilde{t}, \eta))^{-1} \tilde{P}^a(\tilde{x}, \tilde{t}, \eta) (P^a(x, t, \eta))^{-1} P^b(x, t, \eta) \times \\ \times P_{\alpha,\beta}(\eta) (\tilde{P}^b)^{-1}(\tilde{x}, \tilde{t}, \eta) \tilde{P}^c(\tilde{x}, \tilde{t}, \eta) \Big|_{\substack{\tilde{x}=\tilde{x}_b(x,t) \\ \tilde{t}=\tilde{t}_b(t)}} = P_{\hat{\alpha},\hat{\beta}}(\eta)$$

holds with some $\hat{\alpha} = \hat{\alpha}_0 + \eta^{-1}\hat{\alpha}_1 + \dots$ and $\hat{\beta} = \hat{\beta}_0 + \eta^{-1}\hat{\beta}_1 + \dots$ each coefficient of which is independent of x and t . Letting

(82)

$$Q_1(x, t, \eta) = \begin{pmatrix} a_1(x, t, \eta) & b_1(x, t, \eta) \\ c_1(x, t, \eta) & d_1(x, t, \eta) \end{pmatrix}, \quad Q_2(x, t, \eta) = \begin{pmatrix} a_2(x, t, \eta) & b_2(x, t, \eta) \\ c_2(x, t, \eta) & d_2(x, t, \eta) \end{pmatrix}$$

respectively denote

$$(83) \quad Q_1 = (\tilde{P}^c(\tilde{x}, \tilde{t}, \eta))^{-1} \tilde{P}^a(\tilde{x}, \tilde{t}, \eta) (P^a(x, t, \eta))^{-1} P^b(x, t, \eta) \Big|_{\substack{\tilde{x}=\tilde{x}_b(x, t), \\ \tilde{t}=\tilde{t}_b(t)}}$$

$$(84) \quad Q_2 = (\tilde{P}^b)^{-1}(\tilde{x}, \tilde{t}, \eta) \tilde{P}^c(\tilde{x}, \tilde{t}, \eta) \Big|_{\substack{\tilde{x}=\tilde{x}_b(x, t), \\ \tilde{t}=\tilde{t}_b(t)}}$$

we thus find that for any $(\alpha, \beta) = (\alpha(\eta), \beta(\eta))$ (with $\alpha_0\beta_0 \neq 0$) there exists $(\hat{\alpha}, \hat{\beta}) = (\hat{\alpha}(\eta), \hat{\beta}(\eta))$ (with $\hat{\alpha}_0\hat{\beta}_0 \neq 0$) for which the following relation holds (as formal power series of η^{-1}):

(85)

$$\begin{pmatrix} a_1(x, t, \eta) & b_1(x, t, \eta) \\ c_1(x, t, \eta) & d_1(x, t, \eta) \end{pmatrix} \begin{pmatrix} \alpha(\eta) & 0 \\ 0 & \beta(\eta) \end{pmatrix} \begin{pmatrix} a_2(x, t, \eta) & b_2(x, t, \eta) \\ c_2(x, t, \eta) & d_2(x, t, \eta) \end{pmatrix} = \begin{pmatrix} \hat{\alpha}(\eta) & 0 \\ 0 & \hat{\beta}(\eta) \end{pmatrix}.$$

Lemma 4.1. — *If*

(86)

$$Q_1(x, t, \eta) = \begin{pmatrix} a_1(x, t, \eta) & b_1(x, t, \eta) \\ c_1(x, t, \eta) & d_1(x, t, \eta) \end{pmatrix} \quad \text{and} \quad Q_2(x, t, \eta) = \begin{pmatrix} a_2(x, t, \eta) & b_2(x, t, \eta) \\ c_2(x, t, \eta) & d_2(x, t, \eta) \end{pmatrix}$$

with $\det Q_1 \cdot \det Q_2 \neq 0$ satisfy (85) for any $(\alpha(\eta), \beta(\eta))$ with some $(\hat{\alpha}(\eta), \hat{\beta}(\eta))$, then either (87) or (88) below holds.

$$(87) \quad a_1a_2 \text{ and } d_1d_2 \text{ are invertible and independent of } x \text{ and } t, \text{ and } b_j = c_j = 0 \text{ (} j = 1, 2 \text{)}.$$

$$(88) \quad b_1c_2 \text{ and } c_1b_2 \text{ are invertible and independent of } x \text{ and } t, \text{ and } a_j = d_j = 0 \text{ (} j = 1, 2 \text{)}.$$

Proof. — The relation (85) implies

$$(89) \quad \begin{aligned} \alpha a_1a_2 + \beta b_1c_2 &= \hat{\alpha}, & \alpha c_1b_2 + \beta d_1d_2 &= \hat{\beta}, \\ \alpha a_1b_2 + \beta b_1d_2 &= \alpha c_1a_2 + \beta d_1c_2 = 0. \end{aligned}$$

In particular, since α and β can be chosen arbitrarily, we have

$$(90) \quad a_1b_2 = b_1d_2 = c_1a_2 = d_1c_2 = 0.$$

Hence, noting that $\det Q_1 \cdot \det Q_2 \neq 0$, we obtain

$$(91) \quad b_j = c_j = 0 \text{ (} j = 1, 2 \text{)} \quad \text{or} \quad a_j = d_j = 0 \text{ (} j = 1, 2 \text{)}.$$

In case $b_j = c_j = 0$ ($j = 1, 2$), (89) also entails that a_1a_2 and d_1d_2 are independent of x and t . Thus (87) holds. Similarly (89) entails (88) in case $a_j = d_j = 0$ ($j = 1, 2$). \square

By Lemma 4.1 we obtain

$$(92) \quad a_1 a_2 \alpha(\eta) = \hat{\alpha}(\eta) \quad \text{and} \quad d_1 d_2 \beta(\eta) = \hat{\beta}(\eta),$$

or

$$(93) \quad b_1 c_2 \beta(\eta) = \hat{\alpha}(\eta) \quad \text{and} \quad c_1 b_2 \alpha(\eta) = \hat{\beta}(\eta).$$

Hence in both cases a suitable choice of $(\alpha(\eta), \beta(\eta))$ can attain $\hat{\alpha}(\eta) = \hat{\beta}(\eta) = 1$, i.e.,

$$(94) \quad (\tilde{P}^c)^{-1} (R^a)^{-1} R_{\alpha, \beta}^b \tilde{P}^c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, if we choose $(\alpha(\eta), \beta(\eta))$ suitably, $(R^a)^{-1} R_{\alpha, \beta}^b$ becomes an identity operator. This completes the proof of Theorem 2.2.

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