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ISOMONODROMY FOR COMPLEX LINEAR q-DIFFERENCE EQUATIONS

by

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Anyone who considers transcendental
means of producing Galois groups
is, of course, in a state of sin. (*)

Abstract. — The words "monodromy" and "isomonodromy" are used in the theory of difference and q-difference equations by Baranovsky-Ginzburg, Jimbo-Sakai, Borodin, Krichever,... although it is not clear that phenomena of branching during analytic continuation are involved there. In order to clarify what is at stake, we survey results obtained during the last few years, mostly by J.-P. Ramis, J. Sauloy and C. Zhang. Links to Galois theory (as developped by P. Etingof, M. van der Put & M. Singer, Y. André, L. Di Vizio...) are briefly mentioned. A tentative definition of isomonodromy deformations is given along with some elementary results.

***Résumé* (Isomonodromie des équations aux q-différences complexes).** — Les mots « monodromie« et « isomonodromie » ont été employés en théorie des équations aux différences et aux q-différences par Baranovsky-Ginzburg, Jimbo-Sakai, Borodin, Krichever,... bien que, dans un tel contexte, n'apparaissent pas clairement des phénomènes de ramification par prolongement analytique. Afin de clarifier ce qui est en jeu, nous décrivons des résultats obtenus ces dernières années, principalement par J.-P. Ramis, J. Sauloy et C. Zhang. Les liens avec la théorie de Galois (telle qu'elle a été développée par P. Etingof, M. van der Put & M. Singer, Y. André, L. Di Vizio...) sont brièvement mentionnés. Une définition expérimentale de déformation isomonodromique est proposée, ainsi que quelques résultats élémentaires.

0. Introduction

0.1. Roots. — In recent years, the words "monodromy", "isomonodromy" have been used in various places in the context of difference and q-difference equations, *e.g.*, see V. Baranovsky & V. Ginzburg ([**6**]), M. Jimbo & H. Sakai, drawing on previous results

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(*) Adapted from von Neumann who said "*arithmetical* means of producing *random digits*", see [**41**].

of Jimbo and Miwa ([**21**]), A. Borodin ([**10**]) and, more recently[(1)], I. Krichever ([**22**]). However, in these contexts, it is not clear that problems of multivalued solutions and branching at singularities are really involved, as it is the case in the classical setting of linear differential equations in the complex plane. The goal of this survey, is to summarize what can be said about an underlying geometry or topology of solutions encoded in a monodromy group or a Galois group, even if the solutions are taken to be uniform. We shall stick to q-differences, since the theory looks much better behaved there than for differences. Moreover, we shall almost only refer to work conducted under the impulse of Jean-Pierre Ramis, mostly by J.-P. Ramis, C. Zhang and the author. Note that this is meant to be a survey paper: essentially no proofs are given. On the other hand, for a survey with a broader scope, [**14**] is recommended.

While the prehistory of q-difference equations may be thought to have started with Euler, the archetypal example certainly is Heine's *basic hypergeometric series*, here written for a "base" $q \in \mathbf{C}$ such that $|q| > 1$ (see [**14**]):

$$\Phi(a, b, c; q, z) = \sum_{n\geq 0} \frac{(a;p)_n (b;p)_n}{(c;p)_n (p;p)_n} z^n, \qquad \text{where } p = q^{-1} \text{ and } (x;p)_n = \prod_{i=0}^{n-1} (1 - xp^i).$$

It is a q-analogue of the Gauss hypergeometric series

$$F(\alpha, \beta, \gamma; z) = \sum_{n\geq 0} \frac{(\alpha)_n (\beta)_n}{(1)_n (\gamma)_n} z^n, \qquad \text{where } (\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i).$$

The most obvious analogy is that, if one takes $a = p^\alpha$, $b = p^\beta$, $c = p^\gamma$ and lets q go to 1, then, the coefficients of the series defining $\Phi(a, b, c; q, z)$ tend to the coefficients of the series defining $F(\alpha, \beta, \gamma; z)$.

A deeper analogy is related to functional equations. The function $\Phi = \Phi(a, b, c; q, z)$ is solution of a second order linear q-difference equation with rational coefficients, that is, it satisfies a $\mathbf{C}(z)$-linear relation on $\Phi(z)$, $\Phi(qz)$ and $\Phi(q^2 z)$. This relation can be written in terms of the operator σ_q defined by $\sigma_q \phi(z) = \phi(qz)$, thus giving rise to the relation

$$\sigma_q^2 \Phi - \lambda \sigma_q \Phi + \mu \Phi = 0 \quad \text{with} \begin{cases} \lambda = \dfrac{(a+b)z - (1 + c/q)}{abz - c/q} \\ \mu = \dfrac{z - 1}{abz - c/q} \end{cases}. \tag{0.0.1}$$

[(1)] The ArXiv preprint by Krichever appeared one month after the Painlevé conference for which this talk was prepared.

It can also be given in terms of the operator δ_q defined by $\delta_q\phi(z) = \dfrac{\phi(qz)-\phi(z)}{q-1}$. It, then, takes the form

$$\delta_q^2\Phi - \tilde{\lambda}(q)\delta_q\Phi + \tilde{\mu}(q)\Phi = 0 \quad \text{with} \begin{cases} \tilde{\lambda}(q) = \dfrac{\lambda-2}{q-1} \\ \tilde{\mu}(q) = \dfrac{\mu-\lambda+1}{(q-1)^2} \end{cases}. \tag{0.0.2}$$

If one brutally (or heuristically) replaces the operator δ_q by the Euler differential operator $\delta = z\, d/dz$, and the coefficients by their limit as q goes to 1, one finds the corresponding *hypergeometric differential equation* satisfied by $F = F(\alpha,\beta,\gamma;z)$:

$$\delta^2 F - \tilde{\lambda}\delta F + \tilde{\mu} F = 0 \quad \text{with} \begin{cases} \tilde{\lambda} = \dfrac{(\alpha+\beta)z+(1-\gamma)}{1-z} \\ \tilde{\mu} = \dfrac{\alpha\beta z}{1-z} \end{cases}. \tag{0.0.3}$$

Since this equation was the first instance of the so-called Riemann-Hilbert correspondence, one would expect this limiting process to be reflected on the monodromy: the general theory shall be mentioned in 1.3, this particular example being dealt with, in full detail, in [**35**]. We shall rather use the operator σ_q and also rather use systems than equations. For instance, putting $X = \begin{pmatrix} f \\ \sigma_q f \end{pmatrix}$, we get the system

$$\sigma_q X = AX \quad \text{with } A = \begin{pmatrix} 0 & 1 \\ -\mu & \lambda \end{pmatrix} \in GL_2(\mathbf{C}(z)). \tag{0.0.4}$$

The modern history begins with the famous paper by Birkhoff about the so-called generalized Riemann problem, [**8**]. There, he tackled what we perhaps would call nowadays the Riemann-Hilbert problem of classifying differential equations by their singularities and (loosely said) global geometric behaviour. For regular singular differential equations, the former are encoded in the local Jordan structure (generically, the eigenvalues or *exponents*) and the latter means the monodromy representation or, in less intrinsic terms, the knowledge of sufficiently many connection matrices. Birkhoff showed that, to a large extent, the problem can be posed and solved in parallel for differential, difference and q-difference equations. For definiteness, from now on, we consider (as Birkhoff did) q-difference systems meromorphic over the Riemann sphere. To be more precise, we first introduce some notations.

Throughout the text, q is a fixed complex number with modulus $|q| > 1$[(2)]. We also fix a $\tau \in \mathcal{H}$ (the Poincaré half plane) such that $q = e^{-2\imath\pi\tau}$. The field K of

(2) The opposite convention (that is, $0 < |q| < 1$) holds equally often in the litterature, for instance in [**18**]; some formulas or definitions (e.g., classical "basic" functions or the Newton polygon) do depend on the chosen convention. The fundamental fact, if one wants to do some analysis, is that $|q| \neq 1$ (at least in the present state of our technology).

coefficients is one of the following: the field $\mathbf{C}(z)$ of rational functions (*global* case), the field $\mathbf{C}(\{z\})$ of convergent Laurent series meromorphic at 0 (*analytic* or *convergent* local case) and the field $\mathbf{C}((z))$ of formal Laurent series meromorphic at 0 (*formal* local case); we understand meromorphic Laurent series to have finitely many negative exponents. Any of these fields can be endowed with an automorphism σ_q defined by $(\sigma_q f)(z) = f(qz)$. A linear q-difference equation of order n may be written

$$\sigma_q^n(f) + a_1\sigma_q^{n-1}(f) + \cdots + a_n f = 0\ ,\ a_1, \ldots, a_n \in K\ ,\ a_n \neq 0. \tag{0.0.5}$$

By vectorializing, *i.e.*, setting

$$X = X_f \underset{\text{def}}{=} \begin{pmatrix} f \\ \sigma_q f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix}$$

$$\text{and} \qquad A = A_{\underline{a}} \underset{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{pmatrix}, \tag{0.0.6}$$

the equation (0.0.5) may be turned into a system

$$\sigma_q X = AX\ ,\ A \in GL_n(K). \tag{0.0.7}$$

For any such equation or system with coefficients in the field K, one will look for solutions in some K-algebra of functions $\mathcal{A}$ endowed with a dilatation operator σ_q extending the one of K. One possible choice for $\mathcal{A}$ is the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions over $\mathbf{C}^*$, with the natural operation defined by $(\sigma_q f)(z) = f(qz)$. The subalgebra of *q-constants*

$$\mathcal{A}^{\sigma_q} \underset{def}{=} \{f \in \mathcal{A} \ /\ \sigma_q f = f\}$$

is then the field $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ of q-invariant meromorphic functions. Letting $z = e^{2\imath\pi x}$, we see that $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ is isomorphic to the field of meromorphic functions over $\mathbf{C}$ with periods 1 and τ, thus, to a field of elliptic functions. More geometrically, $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ can be identified in a natural way to the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. The latter is an elliptic curve, since the exponential map $x \mapsto e^{2\imath\pi x}$ makes $\mathbf{C}$ a covering of $\mathbf{C}^*$ and induces an isomorphism $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \to \mathbf{E}_q$.

If we find a fundamental solution of (0.0.7) in $\mathcal{A} = \mathcal{M}(\mathbf{C}^*)$, that is, a matrix $\mathcal{X} \in GL_n(\mathcal{A})$ such that $\sigma_q\mathcal{X} = A\mathcal{X}$, then the vector solutions $X \in \mathcal{A}^n$ are exactly the vectors $\mathcal{X}C$ with $C \in (\mathcal{A}^{\sigma_q})^n$, that is, they form a vector space of rank n over the field of constants $\mathcal{M}(\mathbf{E}_q)$. As a matter of fact, contrary to the case of differential equations, a fundamental solution in $\mathcal{A} = \mathcal{M}(\mathbf{C}^*)$ always exists: one does not have to rely on multivalued functions (but, see remark 0.1 below).

Birkhoff considered systems (0.0.7) for $K = \mathbf{C}(z)$. These should be classified with respect to rational equivalence:

$$(0.0.8) \qquad B \sim A \quad \Longleftrightarrow \quad B = F[A] \underset{def}{=} (\sigma_q F) A F^{-1} \text{ for an } F \in GL_n(\mathbf{C}(z)).$$

Note that the gauge transformation $X \mapsto Y = FX$ changes solutions of the system $\sigma_q X = AX$ into solutions of the system $\sigma_q Y = BY$. By the way, Birkhoff proved that any such system is equivalent to the system obtained through (0.0.6) from some equation (0.0.5), a result known today as the *cyclic vector lemma*.

To begin with, the matrix F has coefficients in the field $K = \mathbf{C}(z)$ (global classification); but intermediate results involve local classification, for which we allow local gauge transforms $F \in GL_n(\mathbf{C}(\{z\}))$ or $F \in GL_n(\mathbf{C}((z)))$. In this setting, the only possible *local* information seems to be located at 0 and ∞, since they are the only points fixed by the automorphism $z \mapsto qz$. Birkhoff (relying on previous results by Adams and Carmichael) then defined what it means for a system to be singular regular at these points and built multivalued local solutions at 0 and ∞. The *a priori* local solutions $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ thus obtained are actually meromorphic all over $\mathbf{C}^*$, because the functional equation $\sigma_q X = AX$ (A rational) expands any given disk of convergence by the factor $|q| > 1$.

Then, solutions $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ being given, he defines their connection matrix P through the relation: $\mathcal{X}^{(0)} = \mathcal{X}^{(\infty)} P$. Since $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ are fundamental solutions of the same q-difference system, the matrix P is q-invariant, thus elliptic: it can therefore be encoded by finitely many numerical invariants. These, of course, should be joined with the local invariants at 0 and ∞ (the exponents).

In order to compare the class of q-difference systems (up to rational equivalence) to the class of such sets of invariants (up to natural symetries), Birkhoff counted the number of free parameters on both sides and found them equal. Then, he formulated the inverse problem in the generic case (the local matrices $A(0)$ and $A(\infty)$ are semi-simple): does every such family of numerical invariants come from a regular singular system ? He solved this "generalized Riemann problem" affirmatively. Here, the main tool was the "preliminary theorem", better known as "Birkhoff factorization of matrices". Nowadays, it is rather formulated as the Birkhoff-Grothendieck theorem about the classification of holomorphic vector bundles over the Riemann sphere (see [**5**] or [**25**]), making it quite clear that it has a topological meaning. Besides, this theorem was used in this form by Röhrl in [**31**] to solve the Riemann-Hilbert problem for differential equations (see also [**32**]).

Remark 0.1. — The possibility of a uniform fundamental solution was seemingly ignored by the ancient authors, including Birkhoff. Hence, they had multivalued solutions and their constants were not quite elliptic functions. For instance, in [**8**], the connection matrix P has a constant automorphy factor when going from z to $ze^{2\imath\pi}$. However, it is still rigid enough to be encoded by finitely many numerical invariants.

0.2. Organisation of the paper. — We survey here some recent progress in q-difference equations. Birkhoff's results were modernized in [**35**]. Solutions of Fuchsian systems are built from theta functions and power series, thereby yielding an elliptic connection matrix. The resulting classification scheme is described in 1.1 and 1.2. The most striking feature of this approach is *confluence* to classical monodromy, explained in 1.3. This amounts to a topological interpretation of all known classical q-analogies and can be found in [**35**]. At the end of section 1, we add some remarks about the transition from classification to Galois theory for q-difference equations.

We, then, tackle Galois theory by transcendental means. For Fuchsian equations, we define and compute the local Galois group with the help of a classification by flat holomorphic vector bundles over an elliptic curve $\mathbf{E}_q$ and deduce the local monodromy; this is done in section 2. In section 3, to get the global Galois group of Fuchsian equations along the same lines, we generalize Birkhoff's scheme with a more intrinsic construction of the connection matrix. Then, we propose a tentative computation of global monodromy in the Abelian regular case with the help of the geometric class field theory of the elliptic curve $\mathbf{E}_q$. The results of sections 2 and 3 are taken from [**36**].

In section 4, we address the corresponding problems for general q-difference systems. The local theory, involving a Stokes phenomenon, is summarized in 4.1. The main results about classification come from [**29**](3), also based on [**38**] and [**37**]. Their application to Galois theory in 4.2 will be published in [**34**], as well as the (so far, more fragmentary) global theory of 4.3.

Last, we try to address the motivating subject of this survey (and the subject of the Conference) in an elementary discussion of isomonodromy and integrability for q-difference equations, in section 5.

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(3) Until this is published, see [**30**] and the subsequent notes.

1. Birkhoff revisited

Following a suggestion by Birkhoff in [**8**], J.-P. Ramis, in [**26**], emphasized and initiated the use of theta functions and of the geometry of the elliptic curve in the study of q-difference equations. One salient feature of this angle of attack is that the necessity of multivalued functions totally disappears. Instead of branching at 0 and ∞, the elementary functions of the theory come with discrete logarithmic spirals of zeroes and poles and the constants (q-invariant functions) of the theory, e.g., the coefficients of the Birkhoff connection matrix, are elliptic functions. Theta functions can then be taken as a natural scale for asymptotics and were used that way in [**27**][(4)].

1.1. Local resolution. — We shall give here some more useful notation and terminology. The local study (at 0) of equation (0.0.5) involves some kind of *Newton polygon at* 0, which was first introduced by Adams in [**1**]. This is the convex hull of the set $\{(i,j)\ /\ i \in \{0,\dots,n\}\ ,\ j \geq v_0(a_i)\}$, where $v_0(a)$ denotes the order at 0 of the series $a \in K$ (z-adic valuation), and we have written $a_0 = 1$ for simplicity. The boundary of the Newton polygon is made of two vertical half-lines and $k \geq 1$ lower sides of rational *slopes* $\mu_1 > \cdots > \mu_k$ (here numbered from right to left). The lengths of horizontal projections (absolute abcissae) of these sides are positive integers $r_1, \dots, r_k \in \mathbf{N}^*$, the *multiplicities* of the slopes. Giving the slopes and their multiplicities allows one to recover the Newton polygon, and we will identify these two kinds of data.

Definition 1.1. — An equation is said to be *pure of slope* μ if its Newton polygon has only one slope $\mu_1 = \mu$, and it is said to be *Fuchsian*, or *regular singular*, if it is pure of slope 0.

We shall give the definition of *regularity* (having an ordinary point at 0) only for systems, see the next definition. There are criteria of Fuchsianity resting on orders of growth (or decay) of the solutions near 0. Similar definitions and criteria can be formulated at ∞ (this is also true of all that follows). Birkhoff considered systems with matrices A such that $A(0) \in GL_n(\mathbf{C})$, whence the following definitions:

Definition 1.2. — The system (0.0.7) will be said to be *regular singular* or *Fuchsian* at 0 if A is locally equivalent to $A' \in GL_n(K)$ such that $A'(0) \in GL_n(\mathbf{C})$. It will be said to be *regular* or to have an *ordinary point* at 0 if A is locally equivalent to $A' \in GL_n(K)$ such that $A'(0) = I_n$.

Here, "locally" means: through a gauge transformation with coefficients in $\mathbf{C}(\{z\})$ if $K = \mathbf{C}(z)$ or $\mathbf{C}(\{z\})$, with coefficients in $\mathbf{C}((z))$ if $K = \mathbf{C}((z))$. One can then prove that a system (0.0.7) is Fuchsian if and only if it is equivalent to a system (0.0.6) obtained by starting from a Fuchsian equation; also, a rational system that is Fuchsian

(4) Results about asymptotics can also be found in [**7**].

at 0 and at ∞ is rationally equivalent to one such that $A'(0), A'(\infty) \in GL_n(\mathbf{C})$. One easily checks that, for a system with coefficients in $\mathbf{C}(\{z\})$ (*a fortiori* in $\mathbf{C}(z)$), regularity (having an ordinary point at 0) is equivalent to the condition that the system has a fundamental solution $\mathcal{X} \in GL_n(\mathbf{C}(\{z\}))$. This is similar to the case of regular (or ordinary) points for differential equations.

These definitions are local at 0 (or ∞). Intermediary singularities (those in $\mathbf{C}^*$) are defined in a different way. For the sake of clarity, we introduce now the relevant terminology, although it won't be used in the local study.

Definition 1.3. — A *singularity* of the meromorphic matrix $A(z)$ is either a pole of $A(z)$ or of $A^{-1}(z)$ in $\mathbf{C}^*$. Assuming $A(z) \in \mathbf{C}(\{z\})$, the system (0.0.7) is said to be *singular* at $z_0 \in \mathbf{C}^*$ if z_0 is a singularity of $A(z)$. If $A(z)$ and $A^{-1}(z)$ are holomorphic over $\mathbf{C}^*$ (with values in $GL_n(\mathbf{C})$), the matrix $A(z)$ and the system (0.0.7) are said to be *regular over* $\mathbf{C}^*$.

As is the case for differential equations, a basic lemma says that any Fuchsian system, is locally equivalent to a system with constant coefficients. We state it in the convergent case.

Lemma 1.4. — *Let $A \in GL_n(\mathbf{C}(\{z\}))$ be Fuchsian at 0. Then, there exists $F^{(0)} \in GL_n(\mathbf{C}(\{z\}))$ and $A^{(0)} \in GL_n(\mathbf{C})$ such that $F^{(0)}[A^{(0)}] = A$.*

The latter equality reads $F^{(0)}(qz) = A(z)F^{(0)}(z)\left(A^{(0)}\right)^{-1}$. This implies that, if A is rational, then $F^{(0)}$ is actually meromorphic all over $\mathbf{C}$: it has, *a priori*, as many half discrete logarithmic spirals $z_i q^{\mathbf{N}^*}$ of singularities as $A(z)$ has singularities $z_i \in \mathbf{C}^*$.

We now show how to get a fundamental solution for (0.0.7) in the Fuchsian case at 0. We write $A = F^{(0)}[A^{(0)}]$ as in the lemma. Then, $\mathcal{X}^{(0)} = F^{(0)} e_{A^{(0)}}$ is such a fundamental solution if $e_{A^{(0)}}$ is so for the system with matrix $A^{(0)}$. For the latter, we use the multiplicative Dunford decomposition:

$$A^{(0)} = A_s^{(0)} A_u^{(0)},$$

where $A_s^{(0)}$ is semi-simple, $A_u^{(0)}$ is unipotent and they commute. One builds $e_{A_s^{(0)}}$ from elementary functions such that $\sigma_q e_c = c e_c$ (this accounts for the eigenvalues c of $A^{(0)}$, also called *exponents*) and $e_{A_u^{(0)}}$ from an elementary function such that $\sigma_q l = l + 1$ (in case $A_u^{(0)}$ is nontrivial). We do not give the details. Last, we put

$$e_{A^{(0)}} = e_{A_s^{(0)}} e_{A_u^{(0)}}.$$

Classically (Adams, Carmichael, Birkhoff ...), one took $e_c = z^{\log_q(c)}$ and $l = \log_q(z)$. These functions have no singularities over $\mathbf{C}^*$, but they are multivalued and so will be $\mathcal{X}^{(0)}$ with those choices.

Following [**26**], one can instead build *single valued* fundamental solutions in $\mathcal{M}(\mathbf{C}^*)$. This uses Jacobi's theta function $\theta_q \in \mathcal{O}(\mathbf{C}^*)$, which is defined as:

$$\theta_q(z) = \sum_{n \in \mathbf{Z}} (-1)^n q^{-n(n-1)/2} z^n. \tag{1.4.1}$$

It satisfies the q-difference equation:

$$\sigma_q \theta_q = -qz\theta_q. \tag{1.4.2}$$

For instance, one can then take $e_c = e_{q,c}$ and $l = l_q$, where::

$$e_{q,c}(z) = \theta_q(z)/\theta_q(c^{-1}z) \text{ and} \tag{1.4.3}$$

$$l_q(z) = z\theta_q'(z)/\theta_q(z). \tag{1.4.4}$$

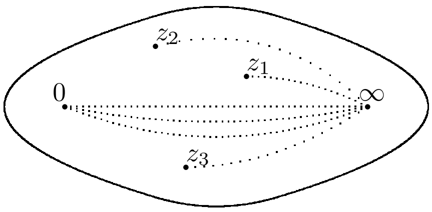
Since θ_q has simple zeroes on the discrete logarithmic spiral $q^{\mathbf{Z}}$, instead of ramification at 0 and ∞, the solution $\mathcal{X}^{(0)}$ inherits discrete logarithmic q-spirals (or half-spirals) of poles.

Remark 1.5. — As noted before, the fundamental solution $\mathcal{X}^{(0)}$ for the Fuchsian system (0.0.7) has, in some sense, moderate growth near 0. However, as a rule, it has an essential singularity at 0. As an example, let us follow the "non Fuchsian" function θ_q *versus* the "Fuchsian" function $e_{q,c}$ along the discrete half-spiral $q^{-\mathbf{N}}z_0$, for some $z_0 \notin q^{\mathbf{Z}} \cup cq^{\mathbf{Z}}$. We assume $c \notin q^{\mathbf{Z}}$, so that $e_{q,c}$ indeed has an essential singularity at 0 (coming from the denominator $\theta_q(c^{-1}z)$). We find, for $n \in \mathbf{N}$:

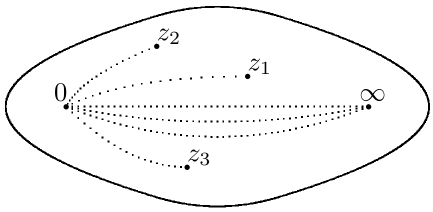
$$\begin{aligned} \theta_q(q^{-n}z_0) &= (-1)^n q^{n(n-1)/2} z_0^{-n} \theta_q(z_0), \\ e_{q,c}(q^{-n}z_0) &= c^{-n} e_{q,c}(z_0). \end{aligned}$$

Thus, along a generic half q-spiral, $e_{q,c}$ has polynomial growth while θ_q has not.

If A is as well Fuchsian at ∞, we end up with two fundamental solutions $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$, each with discrete logarithmic q-spirals of singularities (accounting for the local Jordan structures at 0 and ∞, that is, those of $A^{(0)}$ and $A^{(\infty)}$) and also *half* discrete logarithmic q-spirals of singularities (accounting for the singularities of $A(z)$ and inherited through the gauge transformations $F^{(0)}$ and $F^{(\infty)}$).



Singularities of $\mathcal{X}^{(0)}$



Singularities of $\mathcal{X}^{(\infty)}$

1.2. Connection matrix. — To the rational system $A(z)$ Fuchsian at 0 and ∞, Birkhoff then associated the *connection matrix*:

$$P = \left(\mathcal{X}^{(\infty)}\right)^{-1} \mathcal{X}^{(0)}.$$

The latter is σ_q-invariant, hence (almost) elliptic. Indeed, if the elementary functions were chosen multivalued, one may have $P(ze^{2\imath\pi}) \neq P(z)$. However, if one takes *uniform* elementary functions, as we just did, then the Birkhoff connection matrix P is truly elliptic: $P \in GL_n(\mathcal{M}(\mathbf{E}_q))$. This, of course, comes from the identification of $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ with $\mathcal{M}(\mathbf{E}_q)$.

Theorem 1.6. — *(Birkhoff, [**8**].) From the connection matrix P together with the local data $A^{(0)}$ and $A^{(\infty)}$, one can recover the rational system A.*

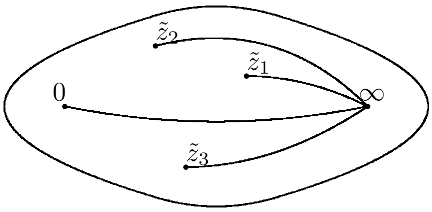
This can be made more precise: the theorem in section 2.2, p. 1041 of [**35**] provides an explicit one to one correspondance between sets of equivalence classes (it will be made more functorial in the next section). However, the heart of the theorem is the above existence assertion, which relies directly on Birkhoff decomposition of matrices.

> ***Problem:*** *to interpret the connection matrix in Galois or in monodromy terms. By the way: the same problem holds for the local data encoded in $A^{(0)}$ and $A^{(\infty)}$.*

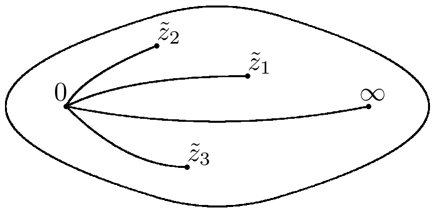
Since all functions are uniform, this will not come from ambiguity in analytic continuation. In order to get into the problem, we shall need some Tannakian formalism which we give in section 2.

1.3. Confluence. — In order to give a geometric interpretation to the connection matrix, one can draw on the classical q-analogy alluded to in the introduction: $\delta_q = \frac{\sigma_q - 1}{q-1} \to \delta = z\frac{d}{dz}$ when $q \to 1$. The phenomenon of *confluence* is described more precisely in [**35**]. The choice of this term may be justified by the fact that the poles of the meromorphic solutions of q-difference equations will be seen to *come together* and condensate into cuts for the multivalued solutions of differential equations.

Suppose $(A_q - I_n)/(q-1) \to \tilde{B}$. Then, when $q \to 1$, under adequate hypotheses, a (single valued) fundamental solution $\mathcal{X}_q^{(0)}$ of $\sigma_q X = A_q X$ over $\mathbf{C}^*$ converges to a (multivalued) fundamental solution $\tilde{\mathcal{X}}^{(0)}$ of $\delta X = \tilde{B}X$. Discrete spirals of poles of $\mathcal{X}_q^{(0)}$ condensate into cuts of $\tilde{\mathcal{X}}^{(0)}$, which are continuous spirals $\tilde{z}_j q^{\mathbf{R}}$, where $\tilde{z}_0 = 1$ and $\tilde{z}_1, \ldots, \tilde{z}_m$ are the poles of $\tilde{B}$ on $\mathbf{C}^*$. The same holds at ∞.

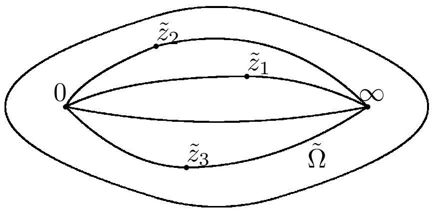


Definition domain of $\tilde{\mathcal{X}}^{(0)}$



Definition domain of $\tilde{\mathcal{X}}^{(\infty)}$

The connection matrix $P_q = \left(\mathcal{X}_q^{(\infty)}\right)^{-1} \mathcal{X}_q^{(0)}$ converges to $\tilde{P} = \left(\tilde{\mathcal{X}}^{(\infty)}\right)^{-1} \tilde{\mathcal{X}}^{(0)}$, which is locally constant on the open set: $\tilde{\Omega} = \mathbf{C}^* \setminus \bigcup \tilde{z}_j q^{\mathbf{R}}$, with values $\tilde{P}_1, \ldots, \tilde{P}_m$.



Definition domain of $\tilde{P}$

Theorem 1.7. — *The monodromy matrix around $\tilde{z}_j$ is $\tilde{P}_j^{-1}\tilde{P}_{j-1}$.*

The example of the q-hypergeometric series degenerating to the classical hypergeometric series is treated at length in [**35**].

1.4. Renewal. — As for Galois theory, the idea to extend Picard-Vessiot theory and differential algebra to difference or q-difference equations has been around for quite a while, see for instance [**17**] or [**12**]. However, from the very beginning, stands a difficulty about the field of constants of such a theory. While, in differential Picard-Vessiot theory, constants are functions with a zero derivative, hence elements of $\mathbf{C}$ (at least in the complex analytic theory), in our setting, constants are σ_q-invariants, hence, as we saw, elliptic functions and it is not clear that algebraic groups over the field of constants could be descended to the field $\mathbf{C}$. It is indeed necessary to have, for all exponents $c \in \mathbf{C}^*$, functions such that $\sigma_q e_c = c e_c$. If we require these functions to be uniform meromorphic, we get a family of elliptic functions $\frac{e_c e_d}{e_{cd}}$ which can be shown to generate a transcendental extension of $\mathbf{C}$.

Bypassing this difficulty, P. Etingof gave in [**16**] the first precise Galoisian interpretation of the connection matrix. He considered only *regular* systems, for which there is no need for the e_c or for special functions of any kind, everything being solvable with power series (see definition 1.2). Thanks to this, the field of constants is then

$\mathbf{C}$ and Etingof was able to use the *values* of the Birkhoff connection matrix P to generate the Picard-Vessiot Galois group: more precisely, he shows the $P(a)^{-1}P(b)$, for regular points $a, b \in \mathbf{C}^*$ to be a set of generators.

These results were extended to general linear q-difference equations by M. van der Put and M. Singer in [**25**]. They use as a substitute for special functions a family of symbolic solutions e_c that are (algebraically) constrained to the relations $e_{cd} = e_c e_d$ and build a universal Picard-Vessiot extension over $\mathbf{C}$. They also recover the Galois group by Tannakian construction in the spirit of [**13**]. A minor complication of the theory is that the universal Picard-Vessiot extension is not an integral ring (and cannot be embedded into a field). A more serious backdraw is that the use of symbolic solutions wipes out the function theoretic point of view.

In [**2**], Y. André took up the program of Birkhoff to unify differential, difference and q-difference equations and developped the notion of non commutative differentials and connections. Among other things, this allowed him to define the Galois groups for mixed families, involving *simultaneously* any of the three types of such functional equations, and to prove a specialization theorem for these families. This is purely algebraic, but the confluence result we have met in subsection 1.3 can be seen as an analytical incarnation of (a particular case of) this theorem. The algebraic viewpoint has also led L. Di Vizio to results of an arithmetical nature, although they rather deal with the *generic* Galois group ([**15**]).

2. Local monodromy for Fuchsian equations

2.1. Some Tannakian formalism. — The following considerations will be needed for the construction of the local as well as the global monodromy and Galois groups.

In the study of differential equations, solutions of the system $\delta X = AX$ (δ a derivation) are seen as null-vectors of the operator $\Delta_A : X \mapsto \delta X - AX$ and the latter is abstracted as a connection on a differential module. In analogy with this, we see the solutions of the system (0.0.7) as invariant vectors of the semi-linear operator $\Phi_A : X \mapsto A^{-1}\sigma_q X$. Abstracting the latter, we call *q-difference module* a pair $M = (V, \Phi)$ where V is a finite dimensional vector space over K and Φ a σ_q-linear automorphism; this means a group automorphism of V satisfying the following rule (which replaces here Leibnitz rule):

$$\forall a \in K \ , \ \forall x \in V \ , \ \Phi(ax) = \sigma_q(a)\Phi(x).$$

Actually, choosing a base allows one to identify any q-difference module with (K^n, Φ_A) for some $A \in GL_n(K)$. A morphism from the q-difference module $M = (V, \Phi)$ to the q-difference module $N = (W, \Psi)$ is a K-linear map f from V to W such that $\Psi \circ f = f \circ \Phi$; thus, an isomorphism from (K^n, Φ_A) to (K^n, Φ_B) is the same as a gauge transformation (0.0.8) from A to B. Things can be made even more intrinsic (and maybe further clarified) by noting that the category $DiffMod(K, \sigma_q)$ of q-difference modules over (K, σ_q) is equivalent to the category of finite length left $\mathcal{D}_{q,K}$-modules, where we write $\mathcal{D}_{q,K} = K\left\langle \sigma, \sigma^{-1} \right\rangle$ for the Öre algebra of non commutative Laurent polynomials characterized by the relation $\sigma.f = \sigma_q(f).\sigma$ (the operation of Φ being identified with left multiplication by σ). One can then prove that the Newton polygon is an intrinsic object in the following sense: it depends only on the (isomorphism class of) the module obtained from an equation.

At any rate, classical linear operations (tensor product, internal Hom, dual ...) may be defined in $DiffMod(K, \sigma_q)$ turning it into a rigid **C**-linear Abelian tensor category: this is because for any of our three base fields, the subfield of constants is **C**. The category $DiffMod(K, \sigma_q)$ is moreover Tannakian, and we shall describe fibre functors and the corresponding Tannakian Galois groups.

Our main object of interest is $\mathcal{E} = DiffMod(\mathbf{C}(z), \sigma_q)$. As a step towards this global study, we shall consider the localized category $\mathcal{E}^{(0)} = DiffMod(\mathbf{C}(\{z\}), \sigma_q)$ and the corresponding category $\mathcal{E}^{(\infty)}$ involving $w = 1/z$. In particular, we shall deal with the full subcategory $\mathcal{E}_f^{(0)}$ of $\mathcal{E}^{(0)}$ (resp. $\mathcal{E}_f^{(\infty)}$ of $\mathcal{E}^{(\infty)}$, resp. $\mathcal{E}_f$ of $\mathcal{E}$) having as objects modules that are Fuchsian at 0 (resp. at ∞, resp. at both 0 and ∞). These are all Tannakian subcategories of each other.

Since we know that the categories at stake are Tannakian, it is tempting to use the construction of $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ to define fibre functors $\omega^{(0)}$ and $\omega^{(\infty)}$, then to interpret the construction of P as providing a tensor isomorphism $\omega^{(0)} \to \omega^{(\infty)}$, whence a path in the Galois groupoid. However, *the formation of $\mathcal{X}^{(0)}$, $\mathcal{X}^{(\infty)}$ and P is not tensor compatible*, because $e_c e_d \neq e_{cd}$. Moreover, as noted in section 1.4, this is unavoidable while using meromorphic functions.

The local classification of global systems (0.0.7) with $A \in GL_n(\mathbf{C}(z))$ deals with gauge transformations with coefficients in $\mathbf{C}(\{z\})$ (local analytic classification) or in $\mathbf{C}((z))$ (formal analytic classification). One finds that, for Fuchsian systems, the local equivalence classes of such systems have the same explicit description in both settings (formal or analytic), see [**25**] or [**38**]. As a consequence of the explicit description, one does not change the local category if we allow objects over the coefficient field $\mathbf{C}(\{z\})$ or even $\mathbf{C}((z))$, instead of $\mathbf{C}(z)$. Therefore, the object of this section is the category $\mathcal{E}_f^{(0)}$. Note that results rather similar to those described here were obtained by V. Baranovsky and V. Ginzburg in [**6**]. A precise comparison is made in [**36**].

2.2. Equivalence with flat holomorphic vector bundles over $\mathbf{E}_q$. — Write

$\mathcal{P}^{(0)}$ the full subcategory of $\mathcal{E}_f^{(0)}$ made of *flat* objects, that is, modules $(\mathbf{C}(\{z\})^n, \Phi_A)$ with $A \in GL_n(\mathbf{C})$. From lemma 1.4, the canonical inclusion of $\mathcal{P}^{(0)}$ in $\mathcal{E}_f^{(0)}$ is an equivalence of categories.

To the flat object $(\mathbf{C}(\{z\})^n, \Phi_A)$, we associate a flat holomorphic vector bundle over the elliptic curve $\mathbf{E}_q$ in the following way. First, we consider the trivial holomorphic vector bundle $\mathbf{C}^* \times \mathbf{C}^n$ over $\mathbf{C}^*$. We write $\sim_A$ for the equivalence relation on the total space $\mathbf{C}^* \times \mathbf{C}^n$ generated by the relations $\forall (z,X) \in \mathbf{C}^* \times \mathbf{C}^n$, $(z,X) \sim_A (qz, AX)$. The quotient set $\mathcal{F}_A = (\mathbf{C}^* \times \mathbf{C}^n)/\sim_A$ is then naturally endowed with a structure of holomorphic vector bundle over $\mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{E}_q$, and this bundle is flat. The following is proven in [**36**]:

Theorem 2.1. — *We thereby obtain an equivalence of the category $\mathcal{E}_f^{(0)}$ with the category $Fib_p(\mathbf{E}_q)$ of flat holomorphic vector bundles over $\mathbf{E}_q$.*

The latter is to be understood as a full subcategory of the category of all holomorphic vector bundles over $\mathbf{E}_q$. In particular, we shall have no use for connections on these bundles and we do not consider them as part of the structure.

It is well known (Weil correspondence, see [**42**],[**19**] or [**20**]) that flat holomorphic vector bundles over a Riemann surface X are "the same" as finite dimensional complex linear representations of its fundamental group $\pi_1(X)$ (we assume X to be connected and omit the base point). However, it should be noticed that isomorphic vector bundles do not correspond to isomorphic representations in the usual sense. To make this more precise, we introduce a category whose objects are the finite dimensional complex linear representations of $\Gamma = \pi_1(X)$. Here, we see the group Γ as acting on the universal covering $\tilde{X}$ of X. Then we take as morphisms from the representation $\rho : \Gamma \to GL(V)$ to the representation $\rho' : \Gamma \to GL(V)'$ all *equivariant* mappings: these are the holomorphic maps $\phi : \tilde{X} \to L(V,V')$ (the space of $\mathbf{C}$-linear maps from V to V') such that $\forall z \in \tilde{X}, \forall \gamma \in \Gamma$, $\phi(\gamma z) \circ \rho(\gamma) = \rho'(\gamma) \circ \phi(z)$. This category admits the (usual) category of finite dimensional complex linear representations of Γ as an essential but not full subcategory.

The above correspondence stems from the fact that any *continuous* vector bundle over X trivializes over the covering $\tilde{X}$ of X and that $X = \tilde{X}/\Gamma$. In the case that X is the elliptic curve $\mathbf{E}_q$, there is a tower of coverings:

$$\mathbf{C} \to \mathbf{C}^* \to \mathbf{E}_q,$$

realizing $\mathbf{E}_q$ as a quotient of the intermediate covering $\mathbf{C}^*$ by the group $q^{\mathbf{Z}}$ of its automorphisms, and *holomorphic* vector bundles already trivialize over the open Riemann surface $\mathbf{C}^*$. This provides another description of $Fib_p(\mathbf{E}_q)$, similar to the previous one, but where we take $\tilde{X} = \mathbf{C}^*$ and $\Gamma = \mathbf{Z}$ (acting through powers of q).

Proposition 2.2. — *The category $Fib_p(\mathbf{E}_q)$ is equivalent to the category $\mathcal{R}$ having as objects the finite dimensional complex representations of $\mathbf{Z}$ and, as morphisms from $\rho : \mathbf{Z} \to GL(V)$ to $\rho' : \mathbf{Z} \to GL(V')$ respectively characterized by $A = \rho(1)$ and $A' = \rho'(1)$, all holomorphic maps $F : \mathbf{C}^* \to L(V,V')$ such that $\forall z \in \mathbf{C}^*$, $F(qz) \circ A = A' \circ F(z)$.*

The equivalence of $\mathcal{E}_f^{(0)}$, $Fib_p(\mathbf{E}_q)$ and $\mathcal{R}$ is to be understood as a Tannakian equivalence, that is, compatible with linear operations (tensor product, internal Hom, dual).

2.3. Galois group and monodromy group. — To compute the Galois group of $\mathcal{E}_f^{(0)}$ one has to describe fibre functors, which can be done in two ways:

1. Algebraic: since holomorphic vector bundles over the Riemann surface $\mathbf{E}_q$ are the same thing as algebraic vector bundles over the algebraic curve $\mathbf{E}_q$, the functor $A \rightsquigarrow \mathcal{F}_A$ from $\mathcal{E}_f^{(0)}$ to $Fib_p(\mathbf{E}_q)$ can itself be seen as a fibre functor over the base $\mathbf{E}_q$. As such, according to [**13**], it gives rise to a Galois groupoid over $\mathbf{E}_q$ in the algebro-geometric sense.
2. Analytic: using the equivalence of $\mathcal{E}_f^{(0)}$ with a category of equivariant bundles over $\mathbf{C}^*$, we can actually define a family fibre functors indexed by $\mathbf{C}^*$ and thereby define a Galois groupoid with base $\mathbf{C}^*$.

We follow here the latter path, which gives a richer information.

First recall the Tannakian category $Rep_{\mathbf{C}}(\mathbf{Z})$ of finite dimensional complex representations of $\mathbf{Z}$. It has a natural fibre functor $\omega^{(0)} : Rep_{\mathbf{C}}(\mathbf{Z}) \to Vect_{\mathbf{C}}^f$, *i.e.* the forgetful functor, which sends each representation to its underlying finite dimensional complex vector space and each morphism of representations to the underlying linear map. One proves $Rep_{\mathbf{C}}(\mathbf{Z})$ to be neutral tannakian over $\mathbf{C}$, thus, equivalent to the category of representations of a unique complex proalgebraic group:

$$\mathbf{Z}^{alg} = Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C},$$

the *proalgebraic hull* of $\mathbf{Z}$.

Our category $\mathcal{R}$ has the same objects as $Rep_{\mathbf{C}}(\mathbf{Z})$ but more morphisms. Therefore, there is a (not fully) faithful essentially surjective functor $Rep_{\mathbf{C}}(\mathbf{Z}) \to \mathcal{R}$ and it is compatible with linear operations.

For each $a \in \mathbf{C}^*$ we define a fibre functor $\omega_a^{(0)}$ on $\mathcal{R}$ over $\mathbf{C}$: on objects, it has the same effect as $\omega^{(0)}$ (it sends a representation to its underlying space), but on morphisms, it is richer: it sends a morphism F from $\rho : \mathbf{Z} \to GL(V)$ to $\rho' : \mathbf{Z} \to GL(V')$ to the linear map $F(a)$. Indeed, recall from proposition 2.2 that F is a holomorphic map from $\mathbf{C}^*$ to $L(V, V')$, so that $F(a) \in L(V, V')$. All these $\omega_a^{(0)}$ have the same restriction $\omega^{(0)}$ to $Rep_{\mathbf{C}}(\mathbf{Z})$. General nonsense and a small computation then yield:

Theorem 2.3. — *The local Galois groupoid, with base* $\mathbf{C}^*$*, is given by:*

$$Iso^{\otimes}(\omega_a^{(0)}, \omega_b^{(0)}) = \{(\gamma, \lambda) \in \mathbf{Z}^{alg} \ / \ \gamma(q)a = b\}.$$

Corollary 2.4. — *The local Galois group (relative to any base point in* $\mathbf{C}^*$*) is:*

$$G_f^{(0)} = \{(\gamma, \lambda) \in \mathbf{Z}^{alg} \ / \ \gamma(q) = 1\}.$$

We see that $G_f^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)}$, where the semi-simple component $G_{f,s}^{(0)}$ is $\{\gamma \in Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \ / \ \gamma(q) = 1\}$ and the unipotent component is $G_{f,u}^{(0)} = \mathbf{C}$.

The unipotent component is plain enough. We can understand better the semi-simple component by extracting some sort of compact form of it, the subgroup $M_{f,s}^{(0)}$

of continuous elements of $G^{(0)}_{f,s}$. We notice that elements of $M^{(0)}_{f,s}$ must send $\mathbf{C}^*$ to $U = \{u \in \mathbf{C} \ / \ |u| = 1\}$, since they factor through the compact group $\mathbf{C}^*/q^{\mathbf{Z}}$. Using the splitting $\mathbf{C}^* = U \times q^{\mathbf{R}}$, where $q^{\mathbf{R}}$ is defined as $e^{2\imath\pi\tau\mathbf{R}}$, we can introduce two particular elements of $M^{(0)}_{f,s}$:

$$\begin{cases} \gamma_1 : ue^{-2\imath\pi\tau y} \mapsto u \\ \gamma_2 : ue^{-2\imath\pi\tau y} \mapsto e^{2\imath\pi y} \end{cases}, \text{ where } |u| = 1 \text{ and } y \in \mathbf{R}.$$

Proposition 2.5. — *The group $M^{(0)}_{f,s}$ is the free Abelian group generated by γ_1 and γ_2. It is Zariski-dense in $G^{(0)}_{f,s}$.*

We see γ_1, γ_2 as generating the (semi-simple component of the) fundamental group of an infinitesimal elliptic curve and consider $M^{(0)}_{f,s}$ as the (semi-simple component of the) the local monodromy. Thus, what we got amounts to a Schlesinger-type density theorem.

We also note without further detail that the behaviour of $M^{(0)}_{f,s}$ (generators and relations) under confluence ($q \to 1$) can be precisely described and linked to the differential Galois group and the monodromy group of the differential equation at the limit [**36**]. This is a particular case of the specialisation theorem in [**2**], with a bit more transcendental information.

3. Global monodromy for Fuchsian equations

Here, we deal with the full subcategory $\mathcal{E}_f$ of $\mathcal{E} = DiffMod(\mathbf{C}(z), \sigma_q)$ whose objects are the modules that are Fuchsian at 0 and at ∞. Such a module can be incarnated by a matrix $A \in GL_n(\mathbf{C}(z))$ such that $A(0)$, $A(\infty) \in GL_n(\mathbf{C})$. These are precisely the systems considered by Birkhoff. We take up his basic idea: to localize at 0 and at ∞ and to keep track of the global information ithrough the connection matrix, which glues solutions at 0 and at ∞.

3.1. Equivalence with connection triples. — To a Fuchsian system A, we can associate (non canonically) two systems with constant coefficients $A^{(0)}, A^{(\infty)} \in GL_n(\mathbf{C})$ which are respectively locally equivalent to A at 0 and at ∞ through gauge transformations $F^{(0)} \in GL_n(\mathcal{M}(\mathbf{C}))$ and $F^{(\infty)} \in GL_n(\mathcal{M}(\mathbf{S} \setminus \{\infty\}))$: one has $F^{(0)}[A^{(0)}] = A$ and $F^{(\infty)}[A^{(\infty)}] = A$. Then $F = \left(F^{(\infty)}\right)^{-1} F^{(0)} \in GL_n(\mathcal{M}(\mathbf{C}^*))$ is such that $(\sigma_q F)A^{(0)} = A^{(\infty)} F$.

More geometrically, we define two flat holomorphic vector bundles $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(\infty)}$ over $\mathbf{E}_q$ and a *meromorphic* isomorphism $\phi : \mathcal{F}^{(0)} \to \mathcal{F}^{(\infty)}$. This is, of course, an intrinsic version of Birkhoff matrix: the latter can be recovered by trivializing the pullbacks of $\mathcal{F}^{(0)}$ and of $\mathcal{F}^{(\infty)}$ over $\mathbf{C}^*$ through the choice of meromorphic frames, for instance our fundamental solutions $e_{q,A^{(0)}}$ and $e_{q,A^{(\infty)}}$. However, *the construction of ϕ is tensor compatible*, in contrast to the construction of Birkhoff's connection matrix.

Theorem 3.1. — *We thus obtain an equivalence of Tannakian categories over* $\mathbf{C}$ *from* $\mathcal{E}_f$ *to the category of such triples* $(\mathcal{F}^{(0)}, \phi, \mathcal{F}^{(\infty)})$.

The above description of how to get a triple from a system is not canonical, but there is a way to make it functorial ([**36**]).

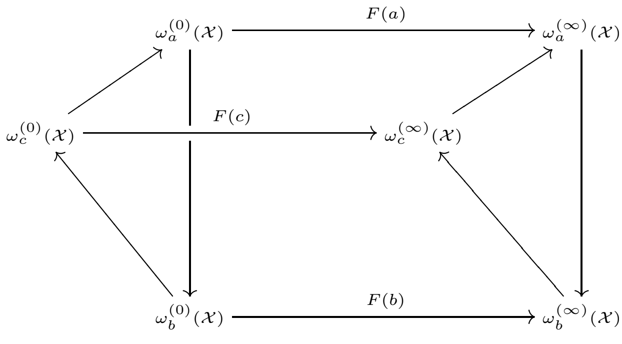
Now, from section 2, we get, for each $a \in \mathbf{C}^*$, two fibre functors $\omega_a^{(0)}$ and $\omega_a^{(\infty)}$. Recall that they are defined by pulling back the bundles $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(\infty)}$ through the covering $\mathbf{C}^* \to \mathbf{E}_q$, which trivializes them. Then, the central component ϕ of the triple $\mathcal{X} = (\mathcal{F}^{(0)}, \phi, \mathcal{F}^{(\infty)})$ is incarnated by a matrix F meromorphic over $\mathbf{C}^*$, actually, the same F intoduced at the beginning of this section. Evaluation at a yields a linear map $F(a) : \omega_a^{(0)}(\mathcal{X}) \to \omega_a^{(\infty)}(\mathcal{X})$.

Proposition 3.2. — *In this way, we obtain a tensor isomorphism of fibre functors:*

$$\left[(\mathcal{F}^{(0)}, \phi, \mathcal{F}^{(\infty)}) \rightsquigarrow F(a)\right] \in Iso^{\otimes}(\omega_a^{(0)}, \omega_a^{(\infty)})$$

Remark 3.3. — We are not being quite rigorous here, as the linear map $F(a)$ needs not to be defined for all $\mathcal{X}$ and all a: in principle, the meromorphic matrix F has as singularities the q-spirals generated by the singularities of $A(z)$ (the rational system it comes from). So the tensor isomorphism just constructed is really defined over some Tannakian subcategory, for instance, the subcategory generated by a particular system of interest. The corresponding machinery is detailed in [**36**].

The following diagram may help to visualize the corresponding Galois groupoid. The base of the groupoid is the disjoint union $\mathbf{C}^* \vee \mathbf{C}^*$ which we can see as two parallel vertical punctured planes. The point a of each plane corresponds to the fibre functor $\omega_a^{(0)}$ (resp. $\omega_a^{(\infty)}$). Pathes between a and b within the first plane come for the elements of $Iso^{\otimes}(\omega_a^{(0)}, \omega_b^{(0)})$ described in section 2, and the corresponding pathes in the second plane are similar. Then, for each $a \in \mathbf{C}^*$, we just defined a tensor isomorphism from $\omega_a^{(0)}$ to $\omega_a^{(\infty)}$, to be visualized as a horizontal path joining the two components.



Theorem 3.4. — *Together with* $G_f^{(0)}$ *and* $G_f^{(\infty)}$, *these elements Zariski-generate the global Galois groupoid.*

Recall from definition 1.2 and the remarks in section 1.4 that, in the case of a regular system, no q-character or q-logarithm is involved in the local resolution. Then, the local bundles are trivial as are the local Galois groupoids: one should see the above diagram as having each of its $\mathbf{C}^*$-components shrunk to a point.

As a corollary, we can state Etingof result ([**16**]):

Corollary 3.5. — *The Galois group of a regular system with connection matrix P is generated by the values $P(a)^{-1}P(b)$.*

Confluence. — The precise relationship of the intrinsic (and tensor compatible) description above with the one by Birkhoff is the following. With the notations introduced at the beginning of this section, the local solutions of Birkhoff are $\mathcal{X}^{(0)} = F^{(0)} e_{q,A^{(0)}}$ and $\mathcal{X}^{(\infty)} = F^{(\infty)} e_{q,A^{(\infty)}}$, so that the connection matrix reads:

$$\begin{aligned} P &= \left(\mathcal{X}^{(\infty)}\right)^{-1} \mathcal{X}^{(0)} \\ &= \left(e_{q,A^{(\infty)}}\right)^{-1} F e_{q,A^{(0)}}. \end{aligned}$$

Thus, the local and global contributions of the monodromy are mixed in P, while they are separated in the bundle encoding: this explains why the latter behaves better with respect to linear constructions.

The proof of the confluence result stated in 1.3 actually deals separately with $e_{q,A^{(0)}}$ and $e_{q,A^{(\infty)}}$ on the one hand, $F^{(0)}$ and $F^{(\infty)}$ on the other, so that the “central component” $F = \left(F^{(\infty)}\right)^{-1} F^{(0)}$ itself has a limit. From this result, we deduce the confluence of the Galois group (*cf* [**36**]), which is again a special case of [**2**] with a more function theoretic flavour.

3.2. Monodromy for Abelian regular equations. — The previous description of the global (or connection) component of the Galois group or groupoid involves uncountably many generators, indexed by $a \in \mathbf{E}_q \setminus \{ \text{ singularities } \}$. This cannot be considered as satisfactory in the spirit of a Riemann correspondance, for we should like the monodromy to involve a tractable set of generators and relations.

To tackle the problem, we restrict ourselves to the case of regular equations, where there is no interference of the local monodromy: we saw that, in that case, the Galois group is generated by the values $P(a)^{-1}P(b)$, computed from the connection matrix P. Since P is elliptic, we may see it as meromorphic over $\mathbf{E}_q$. Also, up to conjugacy, we may assume that the Galois group is generated by the values $P(a)$ of P. We thus have a linear algebraic group rationally parameterized by an elliptic curve, with known prescribed singularities.

Now we further assume that our equation is Abelian, that is, its Galois group is Abelian. Geometric class field theory, which describes commutative linear algebraic groups parameterized by algebraic curves, with known prescribed singularities, is suited for our needs. The theory is due to Lang and Rosenlicht and expounded in [**39**]. Applied to our setting, it gives the following statement:

Theorem 3.6. — *The Abelian regular systems with singularities in a prescribed finite subset $S \subset \mathbf{E}_q$ are classified by the representations of the following algebraic group:*

$$\pi^1_{ab,S,reg} = \frac{L_{S,s}}{L'_{S,s}} \times L_{S,u}, \quad \textit{where: } \begin{cases} L_{S,s} = \frac{\mathbf{G_m}^S}{\Delta} \simeq \mathbf{G_m}^{|S|-1}, \\ L_{S,u} \simeq \prod_{p\in S} (1 + t_p\mathbf{C}[[t_p]]) \end{cases},$$

where $\mathbf{G_m}$ is the multiplicative group, where Δ is the diagonal in $\mathbf{G_m}^S$ and where

$$\begin{cases} Rel_{\mathbf{E}_q}(S) = \{(n_p)_{p\in S} \in \mathbf{Z}^S \ / \ \sum\limits_{p\in S} n_p p = 0_{\mathbf{E}_q}\} \\ L'_{S,s} = \textit{ image in } L_{S,s} \textit{ of } \{(x_p)_{p\in S} \ / \ \forall (n_p)_{p\in S} \in Rel_{\mathbf{E}_q}(S) \ , \ \prod\limits_{p\in S} x_p^{n_p} = 1\} \end{cases}.$$

Example 3.7. — *We just illustrate the case of a rank 1 system (taken from [**36**]). We consider the equation $\sigma_q y = ay$, where:*

$$a(z) = a_0 \prod_{i=1}^r \frac{1 - u_i^{-1} z}{1 - v_i^{-1} z} = a_\infty \prod_{i=1}^r \frac{1 - u_i w}{1 - v_i w}$$

One has used $w = \frac{1}{z}$; the above requires that $a_\infty \prod u_i = a_0 \prod v_i$. Then the connection function*:*

$$p(z) = \frac{e_{q,a_0}(z)}{e_{q,a_\infty^{-1}}(w)} \prod_{i=1}^r \frac{u_i \Theta_q(z/u_i)}{v_i \Theta_q(z/v_i)}$$

is elliptic. In the regular case, one has $a_0 = a_\infty = 1$, $\prod u_i = \prod v_i$ and the connection function is:

$$p(z) = \prod_{i=1}^r \frac{u_i \Theta_q(z/u_i)}{v_i \Theta_q(z/v_i)}.$$

The connection component is the subgroup of $\mathbf{C}^*$ generated by the values $\frac{p(b)}{p(a)}$, where a, b run through $\mathbf{C}^* - \{u_1, \ldots, u_r, v_1, \ldots, v_r\}$. One can of course fix a. This group is clearly connected, so it has to be $\mathbf{C}^*$ (the general case) or trivial. The latter occurs if $p(z)$ is constant, that is, if the given equation is (equivalent to) the trivial equation $\sigma_q f = f$.

4. Monodromy for irregular equations

In this section, we address the local study of the category $\mathcal{E} = DiffMod(\mathbf{C}(z), \sigma_q)$. For reasons similar to those given at the beginning of 2, this amounts to dealing with the localized category $\mathcal{E}^{(0)} = DiffMod(\mathbf{C}(\{z\}), \sigma_q)$. However, we shall find out that here, the formal and analytic equivalence are by no means the same: in the general case of irregular equations, there is a Stokes phenomenon. We shall call "formalisation" the base change functor $-\otimes \mathbf{C}((z))$, going from $\mathcal{E}^{(0)}$ to $\hat{\mathcal{E}}^{(0)} = DiffMod(\mathbf{C}((z)), \sigma_q)$.

4.1. Local analytic classification

The canonical filtration by the slopes. — Recall from sections 1.1 and 1.2 that, to each q-difference equation, system or module, is attached a Newton polygon at 0. Recall also from the classical theory of complex linear differential equations how the Newton polygon of a differential equation, system or module can be used to produce solutions, using exponentials of rational functions and formal power series; equivalently, analytic linear differential operators admit a formal factorisation with "pure" factors; equivalently again, differential modules can be split as a direct sum of "pure" modules (one slope only) *in the formal category.* However, the formal series which appear in the resolution or in the factorisation are, as a rule, divergent and differential modules cannot be simplified within the analytic category.

Similar methods are available for q-difference equations, with (for instance) theta functions in guise of exponential factors. Adams and Carmichael found that one can therefore build a fundamental basis of formal solutions, but Adams, in [**1**], found the following striking fact:

***Theorem 4.1* (*Adams' Lemma,* [1]).** — *Solutions built from the first (leftmost) slope of the Newton polygon are convergent.*

For a proof, see [**38**]. This implies that any analytic q-difference operator can be factored *within the analytic category.* As for q-difference modules, the following is proved in *loc. cit.*:

Theorem 4.2. — *Any q-difference module admits a unique filtration* $(M^{\geq \mu})_{\mu \in \mathbf{Q}}$ *with* $M^{(\mu)} = \frac{M^{\geq \mu}}{M^{>\mu}}$ *pure of slope* μ*. After formalisation, this filtration splits.*

Actually, the *solutions* successively built from Adam's lemma may involve ramification of the complex variable if some slope is non integral. However, the *factorisation, filtration and splitting* mentioned in the theorem are valid without any restrictive assumption (e.g., integrality) on the slopes.

Definition 4.3. — We call *tamely irregular* a direct sum of pure modules or, what amounts to the same, a module M that is isomorphic to the graded module $\bigoplus M^{(\mu)}$.

The corresponding full subcategory of $\mathcal{E}^{(0)}$ will be denoted by $\mathcal{E}^{(0)}_{mi}$. The importance of these constructions in classification matters (and for Galois theory) stems from the following facts:

Theorem 4.4. — *The filtration is functorial and* $gr : M \rightsquigarrow \bigoplus M^{(\mu)}$ *is a faithful exact* $\mathbf{C}$*-linear* $\otimes$*-compatible functor. After formalisation, gr becomes isomorphic to the identity functor.*

We should notice that, in this way, our filtration resemble those axiomatized in [**33**] but are very different from those defined for differential equations or in arithmetic, like in [**3**].

Although the above constructions are defined quite generally, concrete applications, like normal forms or explicit Galoisian operators, are much easier to describe when the slopes are integral[(5)]. So, from now on, we consider only modules with integral slopes. We let $\mathcal{E}_1^{(0)}$ (resp. $\mathcal{E}_{mi,1}^{(0)}$) be the full subcategory of $\mathcal{E}^{(0)}$ made of modules (resp. direct sums of pure modules) with integral slopes. They are Tannakian. The following is an easy consequence of the previous discussion.

Theorem 4.5. — *The Galois group $G_1^{(0)}$ of $\mathcal{E}_1^{(0)}$ is the semi-direct product of the Galois group $G_{mi,1}^{(0)} = \mathbf{C}^* \times G_f^{(0)}$ of $\mathcal{E}_{mi,1}^{(0)}$ by a prounipotent group that we call St.*

We shall now want to build explicit Galoisian *Stokes operators* generating the so-called *Stokes group St*, thereby obtaining the so-called *wild monodromy* (see [**28**]).

Classification by the Stokes sheaf. — The following constructions were mainly motivated by classification concerns and they are not all directly related to the Galois and monodromy group. However, it seems difficult to completely separate the two trends of thought.

As follows from theorem 4.4, the formal class of a q-difference module M in $\mathcal{E}^{(0)}$ is entirely determined by the associated graded module, an object of $\mathcal{E}_{mi}^{(0)}$. In the case of $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_{mi,1}^{(0)}$ (integral slopes), the formal classification and graded modules are moreover easily described in terms of the local Fuchsian classification (which is the same for the formal and analytic categories): indeed, a pure module of integral slope is the tensor product of a rank 1 module by a Fuchsian module, so that we just have to add to the Fuchsian invariants a grading by the slopes. On the side of solutions, this means allowing integral powers of theta functions.

As it is customary for differential equations, we shall study analytical classes within a formal class or, what amounts to the same, isomorphism classes of modules with a prescribed graded module. Thus, we fix a formal class, by giving a direct sum $M_0 = P_1 \oplus \cdots \oplus P_k$ of pure modules with slopes $\mu_1 > \cdots > \mu_k$ and ranks $r_1, \ldots, r_k$. Each of the pure components of M_0 can be put into the form $P_i = (\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{-\mu_i} A_i})$. Therefore, one has $M_0 = (\mathbf{C}(\{z\})^n, \Phi_{A_0})$ for some block-diagonal matrix A_0:

$$A_0 = \begin{pmatrix} z^{-\mu_1} A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{-\mu_k} A_k \end{pmatrix}, \tag{4.5.1}$$

where the μ_i and r_i are as above and $A_i \in GL_{r_i}(\mathbf{C})$ $(i = 1, \ldots, k)$.

(5) After submission of this paper, the preprint [**24**] by M. van der Put and M. Reversat appeared. It contains, among other results, a complete classification of pure modules without restrictive assumptions on the slopes. It can certainly be used to extend the results described here about classification and Galois theory.

The objects we want to classify are, precisely, modules in $\mathcal{E}_1^{(0)}$ *endowed with a formalisation*; by this, we mean pairs (M, g) made up of such a module and an isomorphism $g : gr(M) \to M_0$. We consider two such objects (M, g) and (M', g') to be equivalent if there is an isomorphism $u : M \to M'$ compatible with the formalisations, that is, $g' \circ gr(u) = g$. We shall write $\mathcal{F}(M_0)$ for the set of such *isoformal analytic classes*. A pair (M, g) is thus described by giving the module M the shape $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, with:

$$A = A_U \underset{def}{=} \begin{pmatrix} z^{-\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{-\mu_k}A_k \end{pmatrix}, \tag{4.5.2}$$

where

$$U = (U_{i,j})_{1\le i<j\le k} \in \prod_{1\le i<j\le k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\})).$$

In the above representation, a morphism $M \to M'$ is described by a matrix $F \in GL_n(\mathbf{C}(\{z\}))$ such that $F[A_U] = A_{U'}$. Since the filtration is functorial, F must be upper triangular by blocks. To express the requirement that it is compatible with the formalisation, we introduce the algebraic subgroup $\mathfrak{G}$ of GL_n made up of matrices of the form:

$$F = \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix}. \tag{4.5.3}$$

Then, the requirement is that $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$.

Theorem 4.4 entails that, for any A_U as above, there is a unique $F \in \mathfrak{G}(\mathbf{C}((z)))$ such that $F[A_0] = A_U$. We write it $\hat{F}(U)$. We also see that $\hat{F}(U,V) = \hat{F}(V)\hat{F}(U)^{-1}$ is the unique $F \in \mathfrak{G}(\mathbf{C}((z)))$ such that $F[A_U] = A_V$ and that the corresponding pairs are equivalent if and only if $\hat{F}(U,V) \in \mathfrak{G}(\mathbf{C}(\{z\}))$. We conclude that sending A_U to $\hat{F}(U)$ induces a one-to-one correspondance between $\mathcal{F}(M_0)$ and the left quotient $\mathfrak{G}(\mathbf{C}(\{z\}))\backslash\mathfrak{G}^{A_0}(\mathbf{C}((z)))$, where $\mathfrak{G}^{A_0}(\mathbf{C}((z))) = \{F \in \mathfrak{G}(\mathbf{C}((z)))\ /\ F[A_0] \in GL_n(\mathbf{C}(\{z\}))\}$.

The following result of [**29**] originates in the so-called *Birkhoff-Guenther normal form* of [**9**]:

Theorem 4.6. — *The set $\mathcal{F}(M_0)$ of isoformal analytic classes can be parameterized by polynomial matrices with prescribed degrees:*

$$\forall i,j \text{ s.t. } 1 \le i < j \le k\ ,\ \textit{coeffs}(U_{i,j}) \in \sum_{\mu_j \le d < \mu_i} \mathbf{C}z^d.$$

Therefore, it is an affine algebraic variety of dimension $\sum_{i<j} r_i r_j(\mu_i - \mu_j)$.

In *loc. cit.*, a theory of asymptotic expansions adapted to the type of divergence of q-calculus was developped . This allows one to define a sheaf $\Lambda_I(M_0)$ of automorphisms of M_0 infinitely tangent to identity. Note that this is a sheaf over $\mathbf{E}_q$, in the same way as the classical analogue is a sheaf over the circle of directions. The following is a q-analogue of one of the Malgrange-Sibuya theorems.

Theorem 4.7. — *There is a natural correspondance:* $\mathcal{F}(M_0) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0))$.

There are two proofs. One is sheaf theoretic, using Newlander-Nirenberg theorem; this actually yields a more general result of Malgrange-Sibuya type. The other proof is analytic, using a discrete summation process for q-Gevrey divergent series; this yields an explicit construction of cocycles. Together, these results give a rather complete solution to the local analytic classification problem.

One remarkable feature of the discrete summation process just mentioned is that the *directions* of the classical theory (summation directions, Stokes lines, anti-Stokes lines ...) are here replaced by points on $\mathbf{E}_q$. Indeed, the kernels of the q-Laplace operators are made up from theta functions, the zeroes of which have to be taken avoiding particular q-spirals in $\mathbf{C}^*$, and the latter are the same as points on $\mathbf{E}_q$.

4.2. Local monodromy

Algebraic summation. — It is not presently clear that these transcendental methods allow a Galoisian interpretation. Therefore, we shall consider an alternative more algebraic solution that was expounded in [**37**]. We generalize the directions of discrete summation (points on $\mathbf{E}_q$) in the following way. We keep the previous notations.

Definition 4.8. — An *allowed summation divisor* for A_0, is a family $(D_{i,j})_{1\leq i<j\leq k}$ of effective divisors over $\mathbf{E}_q$, such that:

$$\deg D_{i,j} = \mu_i - \mu_j \text{ and } D_{i,j} = D_{i,l} + D_{l,j};$$

we moreover require that the evaluation $ev_{\mathbf{E}_q}(D_{i,j})$ does not belong to the image $\overline{(-1)^{\mu_i-\mu_j}\frac{Sp(A_i)}{Sp(A_j)}}$ of $(-1)^{\mu_i-\mu_j}\frac{Sp(A_i)}{Sp(A_j)} \subset \mathbf{C}^*$ in $\mathbf{E}_q$.

The evaluation of a divisor $\sum n_i[a_i]$ on $\mathbf{E}_q$ is $\sum n_i a_i \in \mathbf{E}_q$ (computed using the group law). The matrix A_i was defined in equation (4.5.1) and we write $Sp(A_i)$ for its spectrum.

Theorem 4.9. — *For each A_U, there is a unique $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ such that $F[A_0] = A_U$ and $div_{\mathbf{E}_q}(F_{i,j}) \geq -D_{i,j}$. It is asymptotic to the formal gauge transformation $\hat{F}(U)$ (defined just before theorem 4.6).*

The notation $\text{div}_{\mathbf{E}_q}(f)$ makes sense for any meromorphic germ f in a punctured neighborhood of 0 in $\mathbf{C}^*$ such that the divisor of its zeroes and poles in $\mathbf{C}^*$ is q-invariant near 0, which obviously holds for the coefficients of $F_{i,j}$. The condition

$\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geq -D_{i,j}$ then means that the multiplicities of the poles of these coefficients are not greater than the corresponding coefficients in $D_{i,j}$.

We write $F_D(U)$ for the F of the theorem and we see it as the result of the summation of $\hat{F}(U)$ along the direction D. The condition on the $ev_{\mathbf{E}_q}(D_{i,j})$ in the definition is to be compared to prohibited Stokes directions.

Corollary 4.10. — *The sheaf of solutions of A_U is locally isomorphic to the sheaf of solutions of of A_0. Hence, it is a vector bundle over $\mathbf{E}_q$.*

The triangular structure of $F_D(U) \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ has a nice consequence. To each block corresponding to a pure module P_i, one can naturally associate a vector bundle that is the tensor product of a line bundle (corresponding to the theta factor) by a flat vector bundle (corresponding to the Fuchsian part): we call *pure* such a vector bundle. The following allows a new and simpler proof of theorem 4.7, although this does not apply to the more general Malgrange-Sibuya type theorem that the transcendental methods provide.

Theorem 4.11. — *The Stokes sheaf $\Lambda_I(M_0)$ has a devissage by pure vector bundles.*

Galoisian Stokes operators. — We have two ways to define fibre functors on $\mathcal{E}_1^{(0)}$. To describe them, we shall first introduce functors to the category of vector bundles over $\mathbf{E}_q$. As noted before, they allow us to define Galois groupoids with base the algebraic curve $\mathbf{E}_q$; moreover, they convey a nice geometrical picture. However, we consider more significant the fibre functors to the category of vector spaces obtained by pulling back these vector bundles to $\mathbf{C}^*$ (where they trivialize), thus getting groupoids with base $\mathbf{C}^*$. Indeed, the algebraic groupoid with base $\mathbf{E}_q$ is obtained from the transcendental one with base $\mathbf{C}^*$ by some kind of folding through the exponential map, and some information is thereby lost.

In section 2,we saw how to associate a vector bundle to each object in $\mathcal{E}_f^{(0)}$. The process can be extended to $\mathcal{E}_{mi,1}^{(0)}$ by sending the rank 1 module $(\mathbf{C}(\{z\}), z)$ to a line bundle of degree 1, then a pure module to a pure bundle in a tensor compatible way. We therefore get a first fibre functor $\hat{\omega}^{(0)}$ by sending M to the vector bundle associated to $gr(M)$. That we do get a fibre functor follows from theorem 4.4.

Second, we can proceed as we did for Fuchsian modules by restricting to an essential subcategory where the construction is well behaved. From 4.6, we know that the full subcategory $\mathcal{P}_1^{(0)}$ of objects in Birkhoff-Guenther normal form is essential, and it is easily seen to be a Tannakian subcategory. Then we can use the same construction as on $\mathcal{P}^{(0)}$:

$$\frac{\mathbf{C}^* \times \mathbf{C}^n}{(z,X) \sim_A (qz, A(z)X)} \to \mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{E}_q.$$

We thus obtain another fibre functor $\omega^{(0)}$ on $\mathcal{E}_1^{(0)}$. Its restriction to $\mathcal{E}_{mi,1}^{(0)}$ is $\hat{\omega}^{(0)}$ and both restrict to the previous construction in the case of a Fuchsian module.

Now, for each allowed summation divisor D and each module M described by a matrix A_U, $F_D(U)$ is a meromorphic isomorphism from $\hat{\omega}^{(0)}(M)$ to $\omega^{(0)}(M)$. While $M \rightsquigarrow F_D(U)$ is tensor-compatible, it is not, in general functorial. However, if one uses a divisor D with support one point $a \in \mathbf{E}_q$, one does get a tensor isomorphism ϕ_a from $\hat{\omega}^{(0)}$ to $\omega^{(0)}$. One can prove, moreover, that the $\phi_{a,a'} = \phi_a^{-1} \circ \phi_{a'}$ provide generators of $Aut^{\otimes}(\hat{\omega}^{(0)})$, the required Galoisian Stokes operators. All these constructions and results will be detailed in [**34**].

Vector bundles and classification. — For a Fuchsian module, the associated flat bundle loses no information and the corresponding functor is an equivalence of categories. This remains true for tamely irregular modules (direct sums of pure modules). However, it is false when one gets to general irregular modules.

The functor $\omega^{(0)}$ from $\mathcal{E}_1^{(0)}$ to the category of holomorphic vector bundles over $\mathbf{E}_q$ is essentially surjective and faithful, but not fully faithful. Non isomorphic modules may give rise to isomorphic bundles: the isomorphisms will be lifted to isomorphisms of trivial bundles over $\mathbf{C}^*$, but these cannot be defined at 0, where, as a rule, they have wild behaviour.

However, if one gives a bundle together with a devissage by pure bundles and if one considers morphisms of bundles compatible with this enriched structure, then one can choose trivialisations on $\mathbf{C}^*$ such that the spaces of morphisms are filtered by subspaces with prescribed q-Gevrey growth conditions, and these allow one to recover the morphisms of modules. The corresponding functor is then fully faithful, but its essential image is unclear.

In the same circle of ideas, one should note that the filtration of q-difference modules is by no way related to the Harder-Narasimhan filtration on the associated bundle. Indeed, the slopes of a module have no intrinsic meaning on the side of the bundle.

4.3. Global monodromy. — The previous construction of the vector bundle $\omega^{(0)}(M)$ rests on the existence of a Birkhoff-Guenther normal form for M (theorem 4.6), which yields a matrix that is regular over $\mathbf{C}^*$ (see definition 1.3). This was obtained only for modules with integral slopes. There is, however, a construction of $\omega^{(0)}(M)$ that is valid without such restrictive assumption. Moreover, it is easily generalized to cover non local aspects of the monodromy.

Geometric constructions. — We start from a rather general q-difference system:

$$\sigma_q X = AX \ , \ A(z) \in GL_n(\mathcal{M}(\mathbf{C}^*)). \tag{4.11.1}$$

Recall that we defined a singularity of $A(z)$ to be either a pole of $A(z)$ or of $A^{-1}(z)$ in $\mathbf{C}^*$. The set of singularities of A shall be denoted $Sing(A)$.

We write $\pi : \mathbf{C}^* \to \mathbf{E}_q$ for the canonical projection and consider a connected open subset U of $\mathbf{C}^*$, such that:

$$\pi(U) = \mathbf{E}_q. \tag{4.11.2}$$

We note that, while the restriction $\pi : U \to \mathbf{E}_q$ is a local isomorphism, it is no longer a covering.

Let $\sim_A$ be the equivalence relation on $U \times \mathbf{C}^n$ generated by the relations:

$$\forall z \in U \text{ such that } qz \in U \setminus Sing(A)\ ,\ \forall X \in \mathbf{C}^n\ ,\ (z,X) \sim_A (qz, A(z)X),$$

(this means the smallest equivalence relation that contains all relations of the previous form).

Proposition 4.12. — *We moreover assume that*

$$U \cap q^{-1}U \cap Sing(A) = \varnothing. \tag{4.12.1}$$

Then, $\frac{U\times\mathbf{C}^n}{\sim_A}$ *is a holomorphic vector bundle over* $\mathbf{E}_q$*. The sheaf of sections of this bundle is given by the following; the* $\mathcal{O}_{\mathbf{E}_q}(V)$*-module of sections over an open subset* V *of* $\mathbf{E}_q$ *is:*

$$\mathcal{F}_U(V) = \{X \in \mathcal{M}_{\mathbf{C}^*}(\pi^{-1}(V))^n \cap \mathcal{O}_{\mathbf{C}^*}(U)\ /\ \sigma_q X = AX\}. \tag{4.12.2}$$

Actually, the solution X is *a priori* defined (and holomorphic) on U, but the functional equation $\sigma_q X = AX$ may be used to get a meromorphic extension to $\pi^{-1}(V)$, since we took A to be meromorphic all over $\mathbf{C}^*$. Since a holomorphic vector bundle over a compact Riemann surface is meromorphically trivial, we get:

Corollary 4.13. — *The system (4.11.1) has a fundamental basis of (uniform, meromorphic) solutions.*

To my knowledge, this basic statement was first proven by Praagman in [**23**] (also see [**25**]).

We shall keep in mind the most important example of an open annulus $U = \mathcal{C}(r,R) \underset{def}{=} \{z \in \mathbf{C}\ /\ r < |z| < R\}$, where $r \geq 0$ and $R > |q|r$. In particular, in the case where $U = \overset{\circ}{D}(0,R) \underset{def}{=} \mathcal{C}(0,R)$ and R is sufficiently small so that (4.12.1) is satisfied, we recover the previous local constructions.

Sheaf theoretic constructions. — We now relax the assumption (4.12.1) on the open domain U. Then, equality (4.12.2) still defines a sheaf $\mathcal{F}_U$ over $\mathbf{E}_q$, actually, an $\mathcal{O}_{\mathbf{E}_q}$-module. The existence of a fundamental basis of solutions of (4.11.1) was proved using a "good" domain U, but is valid anyhow. So, we call $\mathcal{X}_0 \in GL_n(\mathcal{M}(\mathbf{C}^*))$ a fundamental matrix solution, *i.e.* $\sigma_q \mathcal{X}_0 = A\mathcal{X}_0$.

Lemma 4.14. — *With the usual identification of* $\mathcal{O}_{\mathbf{E}_q}(V)$ *with the subring* $\mathcal{O}_{\mathbf{C}^*}(\pi^{-1}(V))^{\sigma_q}$ *of* σ_q*-invariant elements of* $\mathcal{O}_{\mathbf{C}^*}(\pi^{-1}(V))$*, one has:*

$$\mathcal{X}_0^{-1}\mathcal{F}_U(V) = \{\Phi \in \mathcal{M}_{\mathbf{E}_q}(V)^n\ /\ \mathcal{X}_0\ \Phi \textit{ is holomorphic on } U \cap \pi^{-1}(V)\}.$$

This allows a description rather similar as the one with "diviseurs matriciels" in [**42**], and one proves:

Theorem 4.15. — *The* $\mathcal{O}_{\mathbf{E}_q}$*-module* $\mathcal{F}_U$ *is locally free of rank* n*.*

Thus, one may associate to any q-difference module various holomorphic vector bundles, depending on the choice of "big domains" U. Since holomorphic vector bundles over open Riemann surfaces are trivial, the concrete form of the last theorem is that $A(z)$ is equivalent to some $A^{(U)} \in GL_n(\mathcal{O}(\mathbf{C}^*))$, via a gauge transformation $F^{(U)}$ that is regular holomorphic on U and meromorphic elsewsere. Moreover, for any two such open domains U, U', there is a corresponding *meromorphic* gauge transformation $F^{(U,U')}$ carrying $A^{(U)}$ to $A^{(U')}$, a kind of generalized connection matrix.

Localizing at intermediate singularities. — The constructions above are all functorial and tensor compatible, hence they provide us with many fibre functors. For these to be of any use, we must be able to define tensor isomorphisms between these fibre functors (pathes in the Galois groupoid).

For simplicity, we shall assume that the singularities $z_1, \ldots, z_k$ of A on $\mathbf{C}^*$ are all non equivalent modulo $q^{\mathbf{Z}}$ (this is easy to realize) and that their moduli are far enough from each other, that one can choose radii $R_1 < \cdots < R_k$ such that: $|z_i| < R_i < |qz_i|$ for $i = 1, \ldots, k$. Said otherwise, the annuli $U_i = \mathcal{C}(R_i, R_{i+1})$ separate the singularities and their boundaries actually split the "singular pairs" (z_i, qz_i). For convenience, we take $R_0 = 0$ and $R_{k+1} = \infty$, so that U_0 and U_k are respectively neighborhoods of 0 and ∞ in $\mathbf{C}^*$, and one can take as $A^{(U_0)}$ and $A^{(U_k)}$ the Birkhoff-Guenther normal forms (if slopes are integral). In case the singularities are not so nicely scattered, a similar construction is possible with topological annuli (domains bounded by two topological circles).

Then, for $i = 1, \ldots, k$, the gauge transform $F^{(U_{i-1}, U_i)}$ is singular only at z_i and the product of these k matrices is the connection matrix. So, in some sense, we have a localisation at intermediate singularities, as in 3.2 for the Abelian case. It is also possible to extract some invariants, either by algebra (invariant factors) or by analysis (residues). However, so far, we have been unable to extract enough information to get monodromy as a representation. This is probably for lack of some kind of normal forms for the matrices $A^{(U)}$. This is to be contrasted with the work of I. Krichever ([**22**]), which does contain explicit invariants (although with much more restrictive assumptions).

5. Rudiments of isomonodromy for Fuchsian equations

5.1. Isomonodromy and integrability. — In [**21**], M. Jimbo and H. Sakai derive a q-analogue of Painlevé PVI equation from a condition of isomonodromy for a family of rank 2 rational linear q-difference systems. The salient features of their work are the following:

1. Although they consider a family $(A_t(z))_t$ indexed by a complex parameter t, the kind of invariance they require for monodromy is q-constancy: the systems A_t and A_{qt} are assumed to have the same monodromy.

2. They consider the monodromy as totally encoded in the connection matrix. Thus, after having put their family in a somewhat rigid form, they only assume that A_t and A_{qt} have the same connection matrix.
3. Although they do not care for local monodromies, it so happens that these are indeed q-constant. At ∞, one simply has $A_{qt}(\infty) = A_t(\infty)$. At 0, however, one has $A_{qt}(0) = qA_t(0)$. But, after section 2, this also means that the monodromies are the same.
4. From the fact that A_t and A_{qt} have the same connection matrix, they deduce the existence of a holomorphic family $B_t(z)$ of gauge transformations $B_t : A_t \to A_{qt}$.

The last point deserves some attention. First of all, from Birkhoff theory (as reinterpreted in the previous sections), we know that the equal monodromy condition on A_t and A_{qt} does imply that there is a gauge transformation B_t as above; what is stronger is to obtain it as a continuous family.

Second, writing $A(t,z)$ and $B(t,z)$ instead of $A_t(z)$ and $B_t(z)$, one finds the relation:

$$B(t,qz)A(t,z) = A(qt,z)B(t,z). \tag{5.0.1}$$

This is reminiscent at the same time of a Lax pair and of an integrability condition. We shall take it as a definition: the family $(A_t(z))_t$ is said to be integrable if there is a (holomorphic, meromorphic or rational) family $(B_t(z))_t$ such that (5.0.1) holds. Note that, if $q \to 1$, a small formal computation shows that (5.0.1) indeed degenerates into a classical integrability condition. But, we have other ways of checking that our definition makes good sense. One may want to define a solution to the system (A, B) as a function $X(t,z)$ such that

$$\begin{cases} X(t,qz) = A(t,z)X(t,z) \\ X(qt,z) = B(t,z)X(t,z) \end{cases}.$$

Then, (5.0.1) expresses the formal compatibility condition for this system. More geometrically, one may want to consider $X(t,z)$ as a section of some vector bundle over $\mathbf{E}_q \times \mathbf{E}_q$. Then, we should quotient $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^n$ (or some adequate subset) by the equivalence relation generated by the relations: $(t,z,X) \sim (t,qz,A(t,z)X)$ and $(t,z,X) \sim (qt,z,B(t,z)X)$. Thus, we should like to make $\mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^n$ an equivariant (trivial) holomorphic vector bundle under the action of the group $q^{\mathbf{Z}} \times q^{\mathbf{Z}}$. Then, (5.0.1) is a necessary consequence of the commutativity of this group. Last, note that there exists a notion of *discrete connection* (the word connection is here in relation to differential geometry) and (5.0.1) expresses that we have an integrable discrete connection in the sense of [**40**].

That integrability implies isomonodromy is almost tautological, since the first expresses that A_t and A_{qt} are isomorphic through a *holomorphic family* of gauge transformations B_t, and the mere existence of the *individual* transforms B_t implies that

A_t and A_{qt} have the same monodromy. To get a converse theorem, we first remember that, if A_t and A_{qt} have the same monodromy, then they must be isomorphic through some gauge transformation B_t (this is just Riemann-Hilbert equivalence, as we adapted it from Birkhoff); then, we should prove that this can be done familywise. We shall do so generically.

5.2. Pointwise versus familywise isomorphism. — Before giving an idea of the proof, it is worth making some further remarks about the isomonodromy condition of Jimbo and Sakai. The fact that the same dilatation factor q acts on the variable z and the parameter t can be seen as essential or as accidental. In some sense, the converse theorem can be formulated as follows: if two families $(A_t(z))_t$ and $(A'_t(z))_t$ are pointwise isomorphic, then, they are familywise isomorphic. If this is proven, then take $A'_t = A_{qt}$ and get the conclusion. Of course, this should work exactly the same if one takes $A'_t = A_{q't}$ (for some $q' \in \mathbf{C}^*$) or even A_{t+1} (mixing q-difference equation with constancy in tems of difference equations) or $A'_t = A_{t_0}$ (for some fixed t_0). Actually, with a very little bit of algebraic tools, one can even mix q-differences with derivation by putting $A'_t = A_{t+\epsilon}$, where $\epsilon^2 = 0$ as for dual numbers.

On the other hand(6), we noticed that, in [**21**], the local family $A_t(0)$ is not q-constant, but its monodromy is. Recall that local monodromy sees the eigenvalues of $A_t(0)$ only modulo $q^{\mathbf{Z}}$, that is, it sees their images in $\mathbf{E}_q$. This expresses the fact that the corresponding moduli space is a space of vector bundles over $\mathbf{E}_q$, hence has the jacobian variety $\mathbf{E}_q$ as component. Thus, we have maps from the parameter space in t to $\mathbf{E}_q$. Declaring, for instance, q'-constancy in t yields maps from $\mathbf{E}_{q'}$ to $\mathbf{E}_q$, and these will be trivial if q' is unrelated to q. In *loc. cit.*,the relation $A_{q't}(0) = qA_t(0)$ would not be possible with a rational family for a generic q'.

Local study. — Let U be an open domain in $\mathbf{C}$. We consider the following ring $\mathcal{R}$ of so-called Hartogs series: these are functions $a(t,z) = \sum_{k\geq k_0} a_k(t)z^k$, with the a_k holomorphic over U and the above series having a uniform positive radius of convergence r_K in z for any compact subset K of U. We shall define a family of q-difference systems to be an invertible matrix $A(t,z) \in GL_n(\mathcal{R})$. We say that $A(t,z)$ and $B(t,z)$ are equivalent if there exists $F(t,z) \in GL_n(\mathcal{R})$ such that $F(t,qz)A(t,z) = B(t,z)F(t,z)$. We speak of Birkhoff equivalence if, moreover $\forall t \in U$, $F(t,0) = I_n$.

To motivate the use of Birkhoff equivalence and the following lemma, we should comment on the basic lemma 1.4. Starting from a Fuchsian system, the first step is to reduce it, through a rational gauge transformation obtained by a pivot type algorithm, to a *non resonant* system $A(z)$. This means that $A(0) \in GL_n(\mathbf{C})$ and, for any two distinct eigenvalues c, d of $A(0)$, we have $c/d \notin q^{\mathbf{Z}}$. Then one proves that

(6) The following argument was communicated to me by Daniel Bertrand at the end of the talk at Angers.

$A(z)$ is Birkhoff equivalent to $A(0)$: there is a unique formal $F = I_n + zF_1 + \cdots$ such that $F[A(0)] = A$ and this F converges.

To study families of systems, some kind of rigidity must be assumed so that the invariants depend smoothly on the parameter. For instance, the results of subsection 1.3 were obtained under assumption of non resonance. Here, we do the same.

Lemma 5.1. — *Assume that for all $t \in U$, the system $A(t,z)$ is non resonant. Then the family $A(t,z)$ is Birkhoff equivalent to the family $A(t,0)$.*

Now, to get a local result is rather easy: if two families have, pointwise, the same local monodromies, then, they are equivalent. The only technical ingredient is the use of the theorem of invariant factors over the field $\mathcal{M}(U)$, to get (generically) a normal form for all $A(t,0)$ and all $B(t,0)$ at the same time.

Global study. — We know that, if two families $A(t,z)$ and $B(t,z)$ of *rational* systems have, pointwise, the same monodromies, they are pointwise equivalent. We want to prove that this stays true familywise. From the local study, we may assume that $A(t,0) = B(t,0)$ and $A(t,\infty) = B(t,\infty)$. Then, we want to lift the pointwise equivalence of their connection matrix into a familywise equivalence. Again, arguments from linear algebra entail this generically.

Theorem 5.2. — *If the family $(A_t(z))_t$ is integrable, then it is isomonodromic. The converse is true except for a discrete subset of parameters.*

The result thus obtained is rather incomplete in the following sense: it holds only generically and we have to eliminate a discrete subset of parameters, which, of course, is unavoidable; but we have not characterized this singular subset, although it probably can be done.

Remark 5.3. — As kindly pointed out by the referee, there is a striking analogy of this section with the paper [**4**], the main result of which can indeed be understood as an isomonodromy statement.

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