

SPECIAL POLYNOMIALS ASSOCIATED WITH RATIONAL AND ALGEBRAIC SOLUTIONS OF THE PAINLEVÉ EQUATIONS

by

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Abstract. — Rational solutions of the second, third and fourth Painlevé equations (P_{II} – P_{IV}) can be expressed in terms of logarithmic derivatives of special polynomials that are defined through coupled second order, bilinear differential-difference equations which are equivalent to the Toda equation.

In this paper the structure of the roots of these special polynomials, and the special polynomials associated with algebraic solutions of the third and fifth Painlevé equations, is studied and it is shown that these have an intriguing, highly symmetric and regular structure. Further, using the Hamiltonian theory for P_{II} – P_{IV} , it is shown that all these special polynomials, which are defined by differential-difference equations, also satisfy fourth order, bilinear ordinary differential equations.

Résumé (Polynômes spéciaux associés aux solutions rationnelles ou algébriques des équations de Painlevé)

On peut exprimer les solutions rationnelles des équations P_{II} , P_{III} et P_{IV} en fonction des dérivées logarithmiques de polynômes spéciaux définis par des équations différences-différentielles bilinéaires d'ordre deux couplées et équivalentes à l'équation de Toda.

Dans cet article nous étudions la configuration des racines de ces polynômes spéciaux et des polynômes spéciaux associés aux solutions algébriques des équations de Painlevé P_{III} et P_V . Nous mettons en évidence une structure étonnante, fortement symétrique et régulière. En outre, appliquant la théorie hamiltonienne à P_{II} , P_{III} et P_{IV} , nous montrons que tous ces polynômes spéciaux, définis par des équations différences-différentielles, satisfont aussi à des équations différentielles ordinaires bilinéaires d'ordre 4.

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1. Introduction

In this paper our interest is in rational solutions of the second, third and fourth Painlevé equations (P_{II}–P_{IV})

$$(1.1) \quad w'' = 2w^3 + zw + \alpha,$$

$$(1.2) \quad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w},$$

$$(1.3) \quad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},$$

where $' \equiv d/dz$ and α, β, γ and δ are arbitrary constants and algebraic solutions of P_{III} and the fifth Painlevé equation (P_V)

$$(1.4) \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1} \right) (w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left(\alpha w + \frac{\beta}{w} \right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}.$$

The six Painlevé equations (P_I–P_{VI}), were discovered by Painlevé, Gambier and their colleagues whilst studying which second order ordinary differential equations of the form

$$(1.5) \quad w'' = F(z, w, w'),$$

where F is rational in w' and w and analytic in z , have the property that the solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property* (cf. [34]). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions. Indeed Iwasaki, Kimura, Shimomura and Yoshida [35] characterize the Painlevé equations as “the most important nonlinear ordinary differential equations” and state that “many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions” (see also [14, 75]). The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution (cf. [34, 75]).

Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications including statistical mechanics, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics. Further the Painlevé equations have attracted much interest since they also arise as reductions of the soliton equations which are solvable by inverse scattering (cf. [1], and references therein, for further details).

Vorob'ev [79] and Yablonskii [80] expressed the rational solutions of P_{II} (1.1) in terms of the logarithmic derivative of certain special polynomials which are now

known as the *Yablonskii–Vorob’ev polynomials* (see §2 below). Okamoto [60] derived analogous special polynomials related to some of the rational solutions of P_{IV} , these polynomials are now known as the *Okamoto polynomials* (see §4.2 below), which have been generalised by Noumi and Yamada [58] so that all rational solutions of P_{IV} can be expressed in terms of the logarithmic derivative of special polynomials (see §4.3 below). Umemura [77] derived associated analogous special polynomials with certain rational and algebraic solutions of P_{III} , P_V and P_{VI} which have similar properties to the Yablonskii–Vorob’ev polynomials and the Okamoto polynomials (see also [56, 81]). Subsequently there have been several studies of special polynomials associated with the rational solutions of P_{II} [26, 38, 40, 68], the rational and algebraic solutions of P_{III} [39, 59], the rational solutions of P_{IV} [26, 41, 58], the rational solutions of P_V [51, 57] and the algebraic solutions of P_{VI} [45, 44, 50, 69, 70]. Many of these papers are concerned with the combinatorial structure and determinant representation of the polynomials, often related to the Hamiltonian structure and affine Weyl symmetries of the Painlevé equations. Typically these polynomials arise as the “ τ -functions” for special solutions of the Painlevé equations and are generated through nonlinear, three-term recurrence relations which are Toda-type equations that arise from the associated Bäcklund transformations of the Painlevé equations. Additionally the coefficients of these special polynomials have some interesting, indeed somewhat mysterious, combinatorial properties (cf. [56, 75, 77]).

Clarkson and Mansfield [22] investigated the locations of the zeroes of the Yablonskii–Vorob’ev polynomials in the complex plane and showed that these zeroes have a very regular, approximately triangular structure (see also [15]). An earlier study of the distribution of the zeroes of the Yablonskii–Vorob’ev polynomials is given by Kametaka, Noda, Fukui, and Hirano [42] — see also [35, p. 255, p. 339]. The structure of the zeroes of the polynomials associated with rational and algebraic solutions of P_{III} is studied in [17], which essentially also have an approximately triangular structure, and with rational solutions of P_{IV} in [16], which have an approximate rectangular and combinations of approximate rectangular and triangular structures. The term “approximate” is used since the patterns are not exact triangles and rectangles since the zeroes lie on arcs rather than straight lines.

In this paper we review the studies of special polynomials associated with rational solutions of P_{II} , P_{III} and P_{IV} in §§2–4, respectively, and special polynomials associated with algebraic solutions of P_{III} and P_V in §5 and §6, respectively. Further we discuss the rational solutions of the Hamiltonian systems associated with P_{II} , P_{III} and P_{IV} , respectively. In particular, it is shown that the associated special polynomials, which are defined by differential-difference equations, also satisfy fourth order, bilinear ordinary differential equations. This is analogous to classical orthogonal polynomials, such as Hermite, Laguerre and Jacobi polynomials, which satisfy linear ordinary

differential, difference and differential-difference equations (cf. [3, 7, 71]), and so provides further evidence that the Painlevé equations are nonlinear special functions. In §7 we discuss the interlacing of the roots of these special polynomials in the complex plane. In §8 we discuss our results and pose some open questions.

2. Special Polynomials Associated with Rational Solutions of P_{II}

Rational solutions of P_{II} , for $\alpha = n \in \mathbb{Z}$, can be expressed in terms of the logarithmic derivative of special polynomials which are defined through a second order, bilinear differential-difference equation, see equation (2.2) below. These special polynomials were introduced by Vorob'ev [79] and Yablonskii [80], now known as the *Yablonskii–Vorob'ev polynomials*, which are given in the following theorem (see also [26, 68, 75, 78]).

Theorem 2.1. — *Rational solutions of P_{II} exist if and only if $\alpha = n \in \mathbb{Z}$, which are unique, and have the form*

$$(2.1) \quad w_n = w(z; n) = \frac{d}{dz} \left\{ \ln \left[\frac{Q_{n-1}(z)}{Q_n(z)} \right] \right\},$$

for $n \geq 1$, where the polynomials $Q_n(z)$ satisfy the differential-difference equation

$$(2.2) \quad Q_{n+1}Q_{n-1} = zQ_n^2 - 4 \left[Q_n Q_n'' - (Q_n')^2 \right],$$

with $Q_0(z) = 1$ and $Q_1(z) = z$. The other rational solutions of P_{II} are given by $w_0 = 0$ and $w_{-n} = -w_n$.

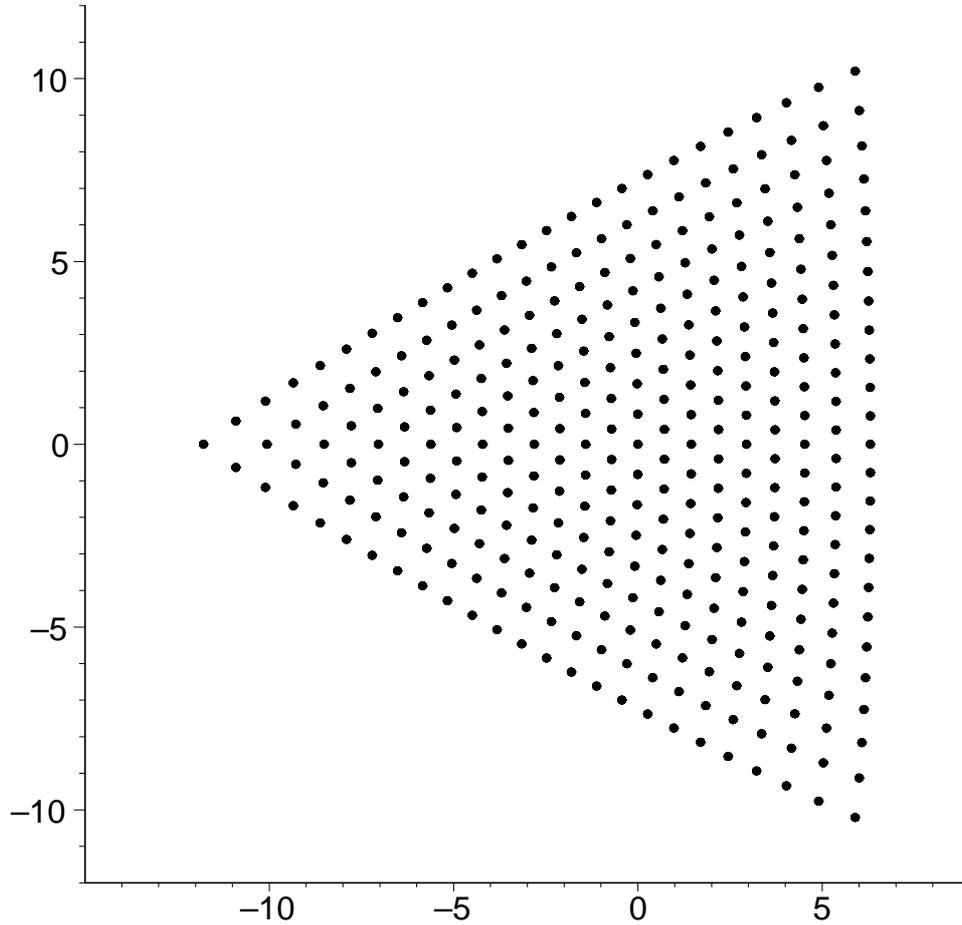
The Yablonskii–Vorob'ev polynomials $Q_n(z)$ are monic polynomials of degree $\frac{1}{2}n(n+1)$ with integer coefficients. It is clear from the recurrence relation (2.2) that the $Q_n(z)$ are rational functions, though it is not obvious that in fact they are polynomials since one is dividing by $Q_{n-1}(z)$ at every iteration. Hence it is somewhat remarkable that the Yablonskii–Vorob'ev polynomials are polynomials. A list of the first few Yablonskii–Vorob'ev polynomials and plots of the locations of their zeros in the complex plane are given in [22]. A plot of the roots of $Q_{25}(z)$ in the complex plane is given in Figure 2. The interlacing of the roots of these special polynomials in the complex plane is discussed in §7.

It is well-known that P_{II} can be written as the Hamiltonian system [60]

$$(2.3) \quad q' = \frac{\partial H_{II}}{\partial p} = p - q^2 - \frac{1}{2}z, \quad p' = -\frac{\partial H_{II}}{\partial q} = 2qp + \alpha + \frac{1}{2},$$

where the (non-autonomous) Hamiltonian $H_{II}(q, p, z; \alpha)$ is given by

$$(2.4) \quad H_{II}(q, p, z; \alpha) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q.$$

FIGURE 2.1. Roots of the Yablonskii–Vorob’ev polynomial $Q_{25}(z)$

Eliminating p in (2.3) then $q = w$ satisfies P_{II} , whilst eliminating q yields

$$(2.5) \quad pp'' = \frac{1}{2} \left(\frac{dp}{dz} \right)^2 = \frac{1}{2}(p')^2 + 2p^3 - zp^2 - \frac{1}{2}(\alpha + \frac{1}{2})^2,$$

which is known as P_{34} , since it is equivalent to equation XXXIV of Chapter 14 in [34]. The Hamiltonian function $\sigma(z; \alpha) = H_{\text{II}}(q, p, z; \alpha)$, where p and q satisfy (2.3), satisfies the second order, second degree equation [36, 60]

$$(2.6) \quad (\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \frac{1}{4}(\alpha + \frac{1}{2})^2.$$

Equation (2.6), which was first derived by Chazy [12] and rederived by Bureau [10, 9, 11], is equation SD-I.d in the classification of second order, second degree equations by Cosgrove and Scoufis [23] and arises in various applications including random matrix theory (cf. [24, 73]). Conversely if $\sigma(z; \alpha)$ is a solution of (2.6), then

$$(2.7) \quad q(z; \alpha) = \frac{4\sigma''(z; \alpha) + 2\alpha + 1}{8\sigma'(z; \alpha)}, \quad p(z; \alpha) = -2\sigma'(z; \alpha).$$

are solutions of (2.3) [60]. The relationship between the Hamiltonian function and associated τ -functions is, up to a multiplicative constant, given by [60]

$$\sigma_n = \sigma(z; n) = \frac{d}{dz} \ln \tau_n,$$

where τ_n satisfies the Toda equation

$$(2.8) \quad \tau_n \tau_n'' - (\tau_n')^2 = C \tau_{n+1} \tau_{n-1},$$

with C a constant. Solutions of (2.2) and (2.8) are related by $\tau_n = Q_n \exp(-z^3/24)$, with $C = -\frac{1}{4}$, and so rational solutions of (2.6) have the form

$$(2.9) \quad \sigma_n = -\frac{1}{8}z^2 + \frac{d}{dz} \ln Q_n.$$

Using this Hamiltonian formalism for P_{II} , it can be shown that the Yablonskii–Vorob’ev polynomials $Q_n(z)$ satisfy a fourth order bilinear ordinary differential equation and a fourth order, second degree, hexa-linear (i.e. homogeneous of degree six) difference equation (see also [15]). Differentiating (2.6) with respect to z yields

$$(2.10) \quad \sigma''' + 6(\sigma')^2 + 2z\sigma' - \sigma = 0,$$

and then substituting (2.9) into (2.10) yields the fourth order, bilinear equation

$$(2.11) \quad Q_n Q_n'''' - 4Q_n' Q_n''' + 3(Q_n'')^2 - z [Q_n Q_n'' - (Q_n')^2] - Q_n Q_n' = 0.$$

We remark that substituting (2.9) into (2.6) yields the third order, second degree, quad-linear (i.e. homogeneous of degree four) equation

$$(2.12) \quad \begin{aligned} & Q_n^2 (Q_n''')^2 + Q_n''' [4(Q_n')^3 - 6Q_n Q_n' Q_n'' - \frac{1}{2}Q_n^3] + 4Q_n (Q_n'')^3 \\ & - (Q_n'')^2 [3(Q_n')^2 + zQ_n^2] + \frac{1}{2}Q_n Q_n' Q_n'' (4zQ_n' - Q_n) \\ & - (Q_n')^3 (zQ_n' - Q_n) + \frac{1}{2}zQ_n^3 Q_n' - \frac{1}{4}n(n+1)Q_n^4 = 0. \end{aligned}$$

Additionally Q_n satisfies the fourth order, second degree, hexa-linear difference equation

$$(2.13) \quad \begin{aligned} & 16(2n+1)^4 Q_n^6 - 8(2n+1)^2 (Q_{n+2} Q_n^3 Q_{n-1}^2 + 2Q_{n+1}^3 Q_{n-1}^3 + Q_{n-2} Q_n^3 Q_{n+1}^2 \\ & - 4z Q_{n+1}^2 Q_n^2 Q_{n-1}^2) + (Q_{n+2} Q_{n-1}^2 - Q_{n+1}^2 Q_{n-2})^2 = 0 \end{aligned}$$

(see [15] for details). Hence the Yablonskii–Vorob’ev polynomials Q_n satisfy nonlinear ordinary differential equations (2.11) and (2.12), the difference equation (2.13) as well

as the differential-difference equation (2.2); see [15] for further differential-difference equations satisfied by the Yablonskii–Vorob’ev polynomials.

It seems reasonable to expect that the ordinary differential equations (2.11) and (2.12) will be useful for proving properties of the Yablonskii–Vorob’ev polynomials since there are more techniques for studying solutions of ordinary differential equations than for difference equations or differential-difference equations. For example, suppose we seek a polynomial solution of (2.12) with $\alpha = n$ in the form

$$Q_n(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_1z + a_0,$$

where it has been assumed, without loss of generality, that the coefficient of z^r is unity since (2.11) is homogeneous. Then it is easy to show that necessarily $r = \frac{1}{2}n(n+1)$, which is a simple proof of the degree of $Q_n(z)$. Similarly it is straightforward to show using (2.11) that $a_{r-3j-1} = 0$ and $a_{r-3j-2} = 0$ and to derive recurrence relations for the coefficients a_{r-3j} . Kaneko and Ochiai [43] derive formulae for the coefficients of the lowest degree term of the Yablonskii–Vorob’ev polynomials; the other coefficients remain to be determined, which is an interesting problem.

3. Special Polynomials Associated with Rational Solutions of P_{III}

3.1. Rational solutions and Bäcklund transformations of P_{III} . — In this section we consider the generic case of P_{III} when $\gamma\delta \neq 0$, then we set $\gamma = 1$ and $\delta = -1$, without loss of generality (by rescaling w and z if necessary), and so consider

$$(3.1) \quad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + w^3 - \frac{1}{w}.$$

The location of rational solutions for the generic case of P_{III} given by (3.1) is stated in the following theorem due to Gromak, Laine and Shimomura [32, p. 174] (see also [52, 54]).

Theorem 3.1. — *Equation (3.1), i.e. P_{III} with $\gamma = -\delta = 1$, has rational solutions if and only if $\alpha + \varepsilon\beta = 4n$, with $n \in \mathbb{Z}$ and $\varepsilon = \pm 1$. Generically, except when α and β are both integers, these rational solutions have the form $w = P_{n^2}(z)/Q_{n^2}(z)$, where $P_{n^2}(z)$ and $Q_{n^2}(z)$ are polynomials of degree n^2 with no common roots.*

We remark that the rational solutions of the generic case of P_{III} (3.1) lie on the lines $\alpha + \varepsilon\beta = 4n$ in the α - β plane, rather than isolated points as is the case for P_{IV} .

The Bäcklund transformations of P_{III} are described in the following theorem due to Gromak [28, 29] (see also [52, 54] and the references therein).

Theorem 3.2. — Suppose $w = w(z; \alpha, \beta, 1, -1)$ is a solution of P_{III} , then $w_j = w_j(z; \alpha_j, \beta_j, 1, -1)$, $j = 1, 2, \dots, 6$, are also solutions of P_{III} where

$$(3.2a) \quad w_1 = \frac{zw' + zw^2 - \beta w - w + z}{w(zw' + zw^2 + \alpha w + w + z)}, \quad \alpha_1 = \alpha + 2, \quad \beta_1 = \beta + 2,$$

$$(3.2b) \quad w_2 = -\frac{zw' - zw^2 - \beta w - w + z}{w(zw' - zw^2 - \alpha w + w + z)}, \quad \alpha_2 = \alpha - 2, \quad \beta_2 = \beta + 2,$$

$$(3.2c) \quad w_3 = -\frac{zw' + zw^2 + \beta w - w - z}{w(zw' + zw^2 + \alpha w + w - z)}, \quad \alpha_3 = \alpha + 2, \quad \beta_3 = \beta - 2,$$

$$(3.2d) \quad w_4 = \frac{zw' - zw^2 + \beta w - w - z}{w(zw' - zw^2 - \alpha w + w - z)}, \quad \alpha_4 = \alpha - 2, \quad \beta_4 = \beta - 2.$$

$$(3.2e) \quad w_5 = -w, \quad \alpha_5 = -\alpha, \quad \beta_5 = -\beta$$

$$(3.2f) \quad w_6 = 1/w, \quad \alpha_6 = -\beta, \quad \beta_6 = -\alpha.$$

3.2. Associated special polynomials. — Umemura [77], see also [17, 39, 81], derived special polynomials associated with rational solutions of P_{III} , which are defined in Theorem 3.3; though these are actually polynomials in $1/z$ rather than polynomials in z . Further Umemura states that these “polynomials” are the analogues of the Yablonskii–Vorob’ev polynomials associated with rational solutions of P_{II} and the Okamoto polynomials associated with rational solutions of P_{IV} .

Theorem 3.3. — Suppose that $T_n(z; \mu)$ satisfies the recursion relation

$$(3.3) \quad zT_{n+1}T_{n-1} = -z \left[T_n \frac{d^2 T_n}{dz^2} - \left(\frac{dT_n}{dz} \right)^2 \right] - T_n \frac{dT_n}{dz} + (z + \mu)T_n^2,$$

with $T_{-1}(z; \mu) = 1$ and $T_0(z; \mu) = 1$. Then

$$(3.4) \quad w_n(z; \mu) \equiv w(z; \alpha_n, \beta_n, 1, -1) = \frac{T_n(z; \mu - 1) T_{n-1}(z; \mu)}{T_n(z; \mu) T_{n-1}(z; \mu - 1)},$$

satisfies P_{III} , with $\alpha_n = 2n + 2\mu - 1$ and $\beta_n = 2n - 2\mu + 1$.

The “polynomials” $T_n(z; \mu)$ are rather unsatisfactory since they are polynomials in $\xi = 1/z$ rather than polynomials in z , which would be more natural. However it is straightforward to determine a sequence of functions $S_n(z; \mu)$ which are generated through an equation that are polynomials in z . These are given in the following theorem, proved in [17, 37], which generalizes the work of Kajiwara and Masuda [39].

Theorem 3.4. — Suppose that $S_n(z; \mu)$ satisfies the recursion relation

$$(3.5) \quad S_{n+1}S_{n-1} = -z \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right)^2 \right] - S_n \frac{dS_n}{dz} + (z + \mu)S_n^2,$$

with $S_{-1}(z; \mu) = S_0(z; \mu) = 1$. Then

(3.6)

$$w_n = w(z; \alpha_n, \beta_n, 1, -1) = 1 + \frac{d}{dz} \left\{ \ln \left[\frac{S_{n-1}(z; \mu - 1)}{S_n(z; \mu)} \right] \right\} \equiv \frac{S_n(z; \mu - 1) S_{n-1}(z; \mu)}{S_n(z; \mu) S_{n-1}(z; \mu - 1)},$$

satisfies P_{III} with $\alpha_n = 2n + 2\mu - 1$ and $\beta_n = 2n - 2\mu + 1$ and

(3.7)

$$\widehat{w}_n = w(z; \widehat{\alpha}_n, \widehat{\beta}_n, 1, -1) = 1 + \frac{d}{dz} \left\{ \ln \left[\frac{S_{n-1}(z; \mu)}{S_n(z; \mu - 1)} \right] \right\} \equiv \frac{S_n(z; \mu) S_{n-1}(z; \mu - 1)}{S_n(z; \mu - 1) S_{n-1}(z; \mu)},$$

satisfies P_{III} with $\widehat{\alpha}_n = -2n + 2\mu - 1$ and $\widehat{\beta}_n = -2n - 2\mu + 1$.

The rational solutions of P_{III} defined by (3.6) and (3.7) can be generalized using the Bäcklund transformation (3.2e) to include all those described in Theorem 3.1 satisfying the condition $\alpha + \beta = 4n$. Rational solutions of P_{III} satisfying the condition $\alpha - \beta = 4n$ are obtained by letting $w \rightarrow iw$ and $z \rightarrow iz$ in (3.6) and (3.7), and then using the Bäcklund transformation (3.2e).

We remark that the polynomials $S_n(z; \mu)$ and $T_n(z; \mu)$, defined by (3.5) and (3.3), respectively, are related through

$$(3.8) \quad S_n(z; \mu) = z^{n(n+1)/2} T_n(z; \mu).$$

Also the polynomials $S_n(z; \mu)$ have the symmetry property

$$(3.9) \quad S_n(z; \mu) = S_n(-z; -\mu).$$

Plots of the roots of the polynomials $S_n(z; \mu)$ for various μ are given in [17]. Initially for μ sufficiently large and negative, the $\frac{1}{2}n(n+1)$ roots form an approximate triangle with n roots on each side. Then as μ increases, the roots in turn coalesce and eventually for μ sufficiently large and positive they form another approximate triangle, similar to the original triangle, though with its orientation reversed. It is straightforward to determine when the roots of $S_n(z; \mu)$ coalesce using discriminants of polynomials. Suppose that $f(z) = z^m + a_{m-1}z^{m-1} + \dots + a_1z + a_0$ is a monic polynomial of degree m with roots $\alpha_1, \alpha_2, \dots, \alpha_m$, so $f(z) = \prod_{j=1}^m (z - \alpha_j)$. Then the discriminant of $f(z)$ is

$$(3.10) \quad \text{Dis}(f) = \prod_{1 \leq j < k \leq m} (\alpha_j - \alpha_k)^2.$$

Hence the polynomial f has a multiple root when $\text{Dis}(f) = 0$. It is straightforward to show that

$$\text{Dis}(S_3(z; \mu)) = 3^{12} 5^5 \mu^6 (\mu^2 - 1)^2,$$

$$\text{Dis}(S_4(z; \mu)) = 3^{27} 5^{20} 7^7 \mu^{14} (\mu^2 - 1)^6 (\mu^2 - 4)^2,$$

$$\text{Dis}(S_5(z; \mu)) = 3^{66} 5^{45} 7^{28} \mu^{26} (\mu^2 - 1)^{14} (\mu^2 - 4)^6 (\mu^2 - 9)^2,$$

$$\text{Dis}(S_6(z; \mu)) = -3^{147} 5^{80} 7^{63} 11^{11} \mu^{44} (\mu^2 - 1)^{26} (\mu^2 - 4)^{14} (\mu^2 - 9)^6 (\mu^2 - 16)^2.$$

Thus $S_3(z; \mu)$ has multiple roots when $\mu = 0, \pm 1$, $S_4(z; \mu)$ when $\mu = 0, \pm 1, \pm 2$, $S_5(z; \mu)$ when $\mu = 0, \pm 1, \pm 2, \pm 3$, and $S_6(z; \mu)$ when $\mu = 0, \pm 1, \pm 2, \pm 3, \pm 4$. In all cases the multiple roots occur at $z = 0$. This naturally leads to the following conjecture.

Conjecture 3.5 ([6]). — *The polynomial $S_n(z; \mu)$ has multiple roots at $z = 0$ when $\mu = 0, \pm 1, \pm 2, \dots, \pm(n - 2)$.*

3.3. Hamiltonian theory for P_{III} . — The Hamiltonian associated with P_{III} is [36, 60] (see also [25])

$$(3.11) \quad H_{\text{III}} = p^2 q^2 - z p q^2 - (\beta - 1) p q + z p + \frac{1}{2} (\beta - 2 - \alpha) z q,$$

and so from Hamilton's equations we have

$$(3.12) \quad z q' = 2 p q^2 - z q^2 - (\beta - 1) q + z, \quad z p' = -2 p^2 q + 2 z p q + (\beta - 1) p - \frac{1}{2} (\beta - 2 - \alpha) z.$$

Setting $q = w$ and eliminating p in this system yields P_{III} (3.1). Next, define the auxiliary Hamiltonian function σ by

$$(3.13) \quad \sigma = \frac{1}{2} H_{\text{III}} + \frac{1}{2} p q + \frac{1}{8} (\beta - 2)^2 - \frac{1}{4} z^2,$$

where p and q satisfy the Hamiltonian system (3.12). Then σ satisfies the second order, second degree equation given by

$$(3.14) \quad (z \sigma'' - \sigma')^2 + 4 (\sigma')^2 (z \sigma' - 2 \sigma) + 4 z \lambda_1 \sigma' - z^2 (z \sigma' - 2 \sigma + 2 \lambda_0) = 0,$$

with $\lambda_1 = -\frac{1}{4} \alpha (\beta - 2)$ and $\lambda_0 = \frac{1}{8} \alpha^2 + \frac{1}{8} (\beta - 2)^2$ [36, 60]. Conversely if σ is a solution of (3.14) then

$$(3.15) \quad q = \frac{2 z \sigma'' + 2 (1 - \beta) \sigma' - \alpha z}{z^2 - 4 (\sigma')^2}, \quad p = \sigma' + \frac{1}{2} z,$$

are solution of (3.12). Due to the relationship between the Hamiltonian and the τ -function (see [60]), it can be shown that solutions of (3.14) have the form

$$\sigma(z) = z \frac{d}{dz} \ln \left\{ z^{1/8} \exp\left(\frac{1}{8} z^2\right) \tau_n(z) \right\} = \frac{1}{4} z^2 + \frac{1}{8} + z \frac{d}{dz} \ln \tau_n(z)$$

where τ_n satisfies the Toda equation (2.8), with $' \equiv z \frac{d}{dz}$. Hence, since $\tau_n(z) = \exp\left(-\frac{1}{4} z^2 - \mu z\right) S_n(z; \mu)$, then rational solutions of (3.14) have the form

$$(3.16) \quad \sigma_n(z; \mu) = -\frac{1}{4} z^2 - \mu z + \frac{1}{8} + z \frac{d}{dz} \ln S_n(z; \mu),$$

with $\lambda_1 = \mu^2 - (n + \frac{1}{2})^2$ and $\lambda_0 = \mu^2 + (n + \frac{1}{2})^2$.

Using this Hamiltonian formalism for P_{III} , it can be shown that the polynomials $S_n(z; \mu)$ satisfy an fourth order bilinear ordinary differential equation and a sixth order, hexa-linear difference equation [17]. Multiplying (3.14) by $1/z^2$ and the differentiating with respect to z yields

$$(3.17) \quad z^2 \sigma''' - z \sigma'' + 6 z (\sigma')^2 - 8 \sigma \sigma' + \sigma' - \frac{1}{2} z^3 + 2 z \lambda_1 = 0.$$

Then substituting (3.16) and $\lambda_1 = \mu^2 - (n + \frac{1}{2})^2$ into this yields the fourth order, bilinear equation

$$(3.18) \quad \begin{aligned} & z^2 \left[S_n S_n'''' - 4S_n' S_n''' + 3(S_n'')^2 \right] + 2z (S_n S_n''' - S_n' S_n'') \\ & - 4z(z + \mu) \left[S_n S_n'' - (S_n')^2 \right] - 2S_n S_n'' + 4\mu S_n S_n' = 2n(n + 1)S_n^2. \end{aligned}$$

As for the case for the ordinary differential equations satisfied by the Yablonskii–Vorob'ev polynomials, i.e. equations (2.11) and (2.12), it seems reasonable to expect that the ordinary differential equation (3.18) will be useful for the derivation of properties of the polynomials $S_n(z; \mu)$. For example, using (3.18) it is straightforward to show that the polynomials $S_n(z; \mu)$ has degree $\frac{1}{2}n(n + 1)$.

4. Special Polynomials Associated with Rational Solutions of P_{IV}

4.1. Rational solutions and Bäcklund transformations for P_{IV} . — Rational solutions of P_{IV} (1.3) are classified in the following theorem due to Lukashevich [47], Gromak [31] and Murata [53] (see also [8, 32, 78]).

Theorem 4.1. — P_{IV} has rational solutions if and only if either

$$(4.1) \quad \alpha = m, \quad \beta = -2(2n - m + 1)^2,$$

or

$$(4.2) \quad \alpha = m, \quad \beta = -2(2n - m + \frac{1}{3})^2,$$

with $m, n \in \mathbb{Z}$. Further the rational solutions for these parameter values are unique.

Some simple rational solutions of P_{IV} are

$$(4.3) \quad w_1(z; \pm 2, -2) = \pm 1/z, \quad w_2(z; 0, -2) = -2z, \quad w_3(z; 0, -\frac{2}{9}) = -\frac{2}{3}z.$$

It is known that there are three families of unique rational solutions of P_{IV} , which have the solutions (4.3) as the simplest members. These are summarized in the following theorem due to Bassom, Clarkson and Hicks [8] (see also Murata [53] and Umemura and Watanabe [78]).

Theorem 4.2. — There are three families of rational solutions of P_{IV} , which have the forms

$$(4.4a) \quad w_1(z; \alpha_1, \beta_1) = p_{1,n-1}(z)/q_{1,n}(z),$$

$$(4.4b) \quad w_2(z; \alpha_2, \beta_2) = -2z + p_{2,n-1}(z)/q_{2,n}(z),$$

$$(4.4c) \quad w_3(z; \alpha_3, \beta_3) = -\frac{2}{3}z + p_{3,n-1}(z)/q_{3,n}(z),$$

where $p_{j,n}(z)$ and $q_{j,n}(z)$, $j = 1, 2, 3$, are polynomials of degree n , and

$$(4.5a) \quad (\alpha_1, \beta_1) = (\pm m, -2(1 + 2n + m)^2), \quad n \leq -1, \quad m \geq -2n,$$

$$(4.5b) \quad (\alpha_2, \beta_2) = (m, -2(1 + 2n + m)^2), \quad n \geq 0, \quad m \geq -n,$$

$$(4.5c) \quad (\alpha_3, \beta_3) = (m, -\frac{2}{9}(1 + 6n - 3m)^2),$$

with $m, n \in \mathbb{Z}$.

The three families given in this theorem are known as the “ $-1/z$ hierarchy”, the “ $-2z$ hierarchy” and the “ $-\frac{2}{3}z$ hierarchy”, respectively (see [8] where the terminology was introduced). The “ $-1/z$ hierarchy” and the “ $-2z$ hierarchy” form the set of rational solutions of P_{IV} with parameter values given by (4.1) and the “ $-\frac{2}{3}z$ hierarchy” forms the set with parameter values given by (4.2). The rational solutions of P_{IV} with parameter values given by (4.1) lie at the vertexes of the “Weyl chambers” and those with parameter values given by (4.2) lie at the centres of the “Weyl chamber” [78].

The Bäcklund transformations of P_{IV} are described in the following theorem due to Lukashovich [47], Gromak [30, 31] (see also [8, 32]).

Theorem 4.3. — Let $w_0 = w(z; \alpha_0, \beta_0)$ and $w_j^\pm = w(z; \alpha_j^\pm, \beta_j^\pm)$, $j = 1, 2, 3, 4$, be solutions of P_{IV} with

$$(4.6a) \quad \alpha_1^\pm = \frac{1}{4} \left(2 - 2\alpha_0 \pm 3\sqrt{-2\beta_0} \right), \quad \beta_1^\pm = -\frac{1}{2} \left(1 + \alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0} \right)^2,$$

$$(4.6b) \quad \alpha_2^\pm = -\frac{1}{4} \left(2 + 2\alpha_0 \pm 3\sqrt{-2\beta_0} \right), \quad \beta_2^\pm = -\frac{1}{2} \left(1 - \alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0} \right)^2,$$

$$(4.6c) \quad \alpha_3^\pm = \frac{3}{2} - \frac{1}{2}\alpha_0 \mp \frac{3}{4}\sqrt{-2\beta_0}, \quad \beta_3^\pm = -\frac{1}{2} \left(1 - \alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0} \right)^2,$$

$$(4.6d) \quad \alpha_4^\pm = -\frac{3}{2} - \frac{1}{2}\alpha_0 \mp \frac{3}{4}\sqrt{-2\beta_0}, \quad \beta_4^\pm = -\frac{1}{2} \left(-1 - \alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0} \right)^2.$$

Then

$$(4.7a) \quad \mathcal{T}_1^\pm : \quad w_1^\pm = \frac{w_0' - w_0^2 - 2zw_0 \mp \sqrt{-2\beta_0}}{2w_0},$$

$$(4.7b) \quad \mathcal{T}_2^\pm : \quad w_2^\pm = -\frac{w_0' + w_0^2 + 2zw_0 \mp \sqrt{-2\beta_0}}{2w_0},$$

$$(4.7c) \quad \mathcal{T}_3^\pm : \quad w_3^\pm = w_0 + \frac{2(1 - \alpha_0 \mp \frac{1}{2}\sqrt{-2\beta_0})w_0}{w_0' \pm \sqrt{-2\beta_0} + 2zw_0 + w_0^2},$$

$$(4.7d) \quad \mathcal{T}_4^\pm : \quad w_4^\pm = w_0 + \frac{2(1 + \alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0})w_0}{w_0' \mp \sqrt{-2\beta_0} - 2zw_0 - w_0^2},$$

valid when the denominators are non-zero, and where the upper signs or the lower signs are taken throughout each transformation.

4.2. Okamoto polynomials. — In a comprehensive study of the fourth Painlevé equation P_{IV} , Okamoto [60] (see also [26, 41, 58]) defined two sets of polynomials analogous to the Yablonskii–Vorob’ev polynomials, which are defined in Theorems 4.4 and 4.5 below. These have been scaled compared to Okamoto’s original definition, where the polynomials are monic, so that they are for the standard P_{IV} .

Theorem 4.4. — Suppose $Q_n(z)$ satisfies the recursion relation

$$(4.8) \quad Q_{n+1}Q_{n-1} = \frac{9}{2} [Q_n Q_n'' - (Q_n')^2] + [2z^2 + 3(2n-1)] Q_n^2,$$

with $Q_0(z) = Q_1(z) = 1$. Then

$$(4.9) \quad w_n = w(z; \alpha_n, \beta_n) = -\frac{2}{3}z + \frac{d}{dz} \left\{ \ln \left[\frac{Q_{n+1}(z)}{Q_n(z)} \right] \right\},$$

for $n \geq 0$, satisfies P_{IV} with $(\alpha_n, \beta_n) = (2n, -\frac{2}{9})$.

Theorem 4.5. — Suppose $R_n(z)$ satisfies the recursion relation

$$(4.10) \quad R_{n+1}R_{n-1} = \frac{9}{2} [R_n R_n'' - (R_n')^2] + 2(z^2 + 3n)R_n^2,$$

with $R_0(z) = 1$ and $R_1(z) = \sqrt{2}z$. Then

$$(4.11) \quad \hat{w}_n = w(z; \hat{\alpha}_n, \hat{\beta}_n) = -\frac{2}{3}z + \frac{d}{dz} \left\{ \ln \left[\frac{R_{n+1}(z)}{R_n(z)} \right] \right\},$$

for $n \geq 0$, satisfies P_{IV} with $(\hat{\alpha}_n, \hat{\beta}_n) = (2n+1, -\frac{8}{9})$.

The polynomials $Q_n(z)$ are polynomials of degree $n(n-1)$, in fact they are monic polynomials in $\zeta = \sqrt{2}z$ with integer coefficients, which is the form in which Okamoto [60] originally defined these polynomials. Further the polynomials $Q_n(z)$ are even polynomials, i.e. monic polynomials in $\zeta^2 = 2z^2$ of degree $\frac{1}{2}n(n-1)$. The polynomials $R_n(z)$ are polynomials of degree n^2 , in fact they are monic polynomials in $\zeta = 2z$ with integer coefficients, which is the form in which Okamoto [60] originally defined these polynomials. In [16] plots of the locations of the zeros, in the complex plane, for the Okamoto polynomials $Q_n(z) = 0$, defined by (4.8), and $R_n(z) = 0$, defined by (4.10), are given. These both take the form of two “triangles” with the polynomials $R_n(z)$ having an additional row of zeros on a straight line, the real axis, between the two “triangles”. The term “triangles” is used since the zeros lie on arcs, rather than straight lines and so are only approximately triangular.

4.3. Generalized Hermite polynomials and generalized Okamoto polynomials. — Noumi and Yamada [58] generalized the results of Okamoto [60] described above and introduced the *generalized Hermite polynomials* $H_{m,n}(z)$, which are defined in Theorem 4.6, and the *generalized Okamoto polynomials* $Q_{m,n}(z)$, which are defined in Theorem 4.7. Noumi and Yamada [58] expressed both the generalized Hermite polynomials and the generalized Okamoto polynomials in terms of Schur functions related to the so-called modified Kadomtsev–Petviashvili (mKP) hierarchy. Kajiwara

and Ohta [41] also expressed rational solutions of P_{IV} in terms of Schur functions by expressing the solutions in the form of determinants.

Theorem 4.6. — Suppose $H_{m,n}(z)$ satisfies the recurrence relations

$$(4.12a) \quad 2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - (H'_{m,n})^2 + 2mH_{m,n}^2,$$

$$(4.12b) \quad 2nH_{m,n+1}H_{m,n-1} = -H_{m,n}H''_{m,n} + (H'_{m,n})^2 + 2nH_{m,n}^2,$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1$ and $H_{1,1} = 2z$, then

$$(4.13a) \quad w_{m,n}^{(I)} = \frac{d}{dz} \left\{ \ln \left(\frac{H_{m+1,n}}{H_{m,n}} \right) \right\},$$

$$(4.13b) \quad w_{m,n}^{(II)} = -\frac{d}{dz} \left\{ \ln \left(\frac{H_{m,n+1}}{H_{m,n}} \right) \right\},$$

$$(4.13c) \quad w_{m,n}^{(III)} = -2z + \frac{d}{dz} \left\{ \ln \left(\frac{H_{m,n+1}}{H_{m+1,n}} \right) \right\},$$

where $w_{m,n}^{(J)} = w(z; \alpha_{m,n}^{(J)}, \beta_{m,n}^{(J)})$ for $J=I,II,III$, is a solution of P_{IV} , respectively for

$$(4.14a) \quad \alpha_{m,n}^{(I)} = 2m + n + 1, \quad \beta_{m,n}^{(III)} = -2n^2,$$

$$(4.14b) \quad \alpha_{m,n}^{(II)} = -(m + 2n + 1), \quad \beta_{m,n}^{(I)} = -2m^2,$$

$$(4.14c) \quad \alpha_{m,n}^{(III)} = n - m, \quad \beta_{m,n}^{(II)} = -2(m + n + 1)^2$$

The rational solutions of P_{IV} defined by (4.13) include all the solutions in the “ $-1/z$ ” and “ $-2z$ ” hierarchies, as is easily verified by comparing the parameters in (4.14) with those in (4.5a) and (4.5b). Further they are the set of rational solutions of P_{IV} with parameter values given by (4.1). The rational solutions of P_{IV} generated by the generalized Hermite polynomials $H_{m,n}(z)$ are special cases of the special function solutions, often called *one-parameter families of solutions*, which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$, or a special case of the Whittaker functions $M_{\kappa,\mu}(\zeta)$ and $W_{\kappa,\mu}(\zeta)$ (cf. [16]; see, for example, [3, §19.12] for the relationship between parabolic cylinder functions and Whittaker functions).

Plots of the locations of the zeros of the polynomials $H_{m,n}(z)$ for various choices of m and n , are given in [16]. These plots, which are invariant under reflections in the real and imaginary z -axes, take the form of $m \times n$ “rectangles”, though these are only approximate rectangles as can be seen by looking at the actual values of the zeros. A plot of the complex roots of the generalized Hermite polynomial $H_{20,20}(z)$ is given in Figure 4.3.

Theorem 4.7. — Suppose $Q_{m,n}(z)$ satisfies the recurrence relations

$$(4.15a) \quad Q_{m+1,n}Q_{m-1,n} = \frac{9}{2} \left[Q_{m,n}Q''_{m,n} - (Q'_{m,n})^2 \right] + [2z^2 + 3(2m + n - 1)] Q_{m,n}^2,$$

$$(4.15b) \quad Q_{m,n+1}Q_{m,n-1} = \frac{9}{2} \left[Q_{m,n}Q''_{m,n} - (Q'_{m,n})^2 \right] + [2z^2 + 3(1 - m - 2n)] Q_{m,n}^2,$$

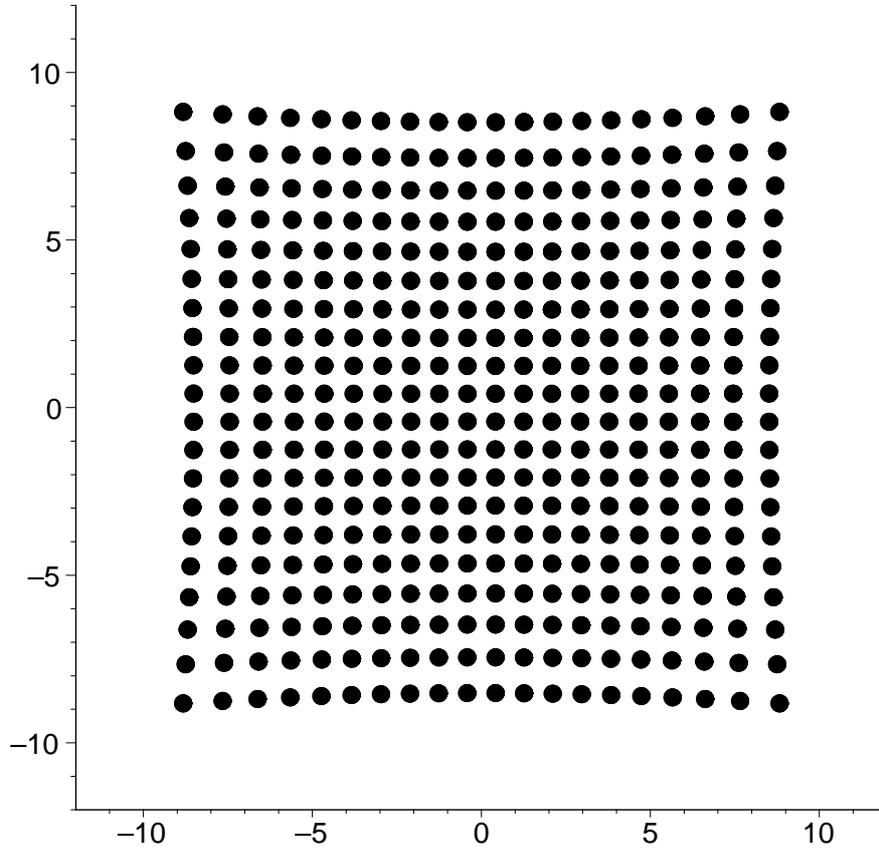


FIGURE 4.1. Roots of the generalized Hermite polynomial $H_{20,20}(z)$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}z$, then

$$(4.16a) \quad \tilde{w}_{m,n}^{(I)} = -\frac{2}{3}z + \frac{d}{dz} \left\{ \ln \left(\frac{Q_{m+1,n}}{Q_{m,n}} \right) \right\},$$

$$(4.16b) \quad \tilde{w}_{m,n}^{(II)} = -\frac{2}{3}z - \frac{d}{dz} \left\{ \ln \left(\frac{Q_{m,n+1}}{Q_{m,n}} \right) \right\},$$

$$(4.16c) \quad \tilde{w}_{m,n}^{(III)} = -\frac{2}{3}z + \frac{d}{dz} \left\{ \ln \left(\frac{Q_{m,n+1}}{Q_{m+1,n}} \right) \right\},$$

where $\tilde{w}_{m,n}^{(J)} = w(z; \tilde{\alpha}_{m,n}^{(J)}, \tilde{\beta}_{m,n}^{(J)})$ for $J=I,II,III$, are solutions of P_{IV} , respectively for

$$(4.17a) \quad \tilde{\alpha}_{m,n}^{(I)} = 2m + n, \quad \tilde{\beta}_{m,n}^{(I)} = -2\left(n - \frac{1}{3}\right)^2,$$

$$(4.17b) \quad \tilde{\alpha}_{m,n}^{(II)} = -(m + 2n), \quad \tilde{\beta}_{m,n}^{(II)} = -2\left(m - \frac{1}{3}\right)^2,$$

$$(4.17c) \quad \tilde{\alpha}_{m,n}^{(III)} = n - m, \quad \tilde{\beta}_{m,n}^{(III)} = -2\left(m + n + \frac{1}{3}\right)^2.$$

The rational solutions of P_{IV} defined by (4.16) include all the solutions in the “ $-\frac{2}{3}z$ ” hierarchy, as is easily verified by comparing the parameters in (4.17) with those in (4.5c). Further they are the set of rational solutions of P_{IV} with parameter values given by (4.2).

Examples of generalized Okamoto polynomials and plots of the locations of their complex roots are given in [16]. Plots of the complex roots of the generalized Okamoto polynomials $Q_{10,10}(z)$ and $Q_{-8,-8}(z)$ are given in Figures 4.3 and 4.3, respectively. The roots of the polynomial $Q_{m,n}(z)$, with $m, n \geq 1$, take the form of $m \times n$ “rectangle” with an “equilateral triangle”, which have either $m - 1$ or $n - 1$ roots, on each of its sides. The roots of the polynomial $Q_{-m,-n}(z)$, with $m, n \geq 1$, take the form of $m \times n$ “rectangle” with an “equilateral triangle”, which now have either m or n roots, on each of its sides. These are only approximate rectangles and equilateral triangles as can be seen by looking at the actual values of the roots. We remark that as for the generalized Hermite polynomials above, the plots are invariant under reflections in the real and imaginary z -axes.

Due to the symmetries

$$(4.18a) \quad Q_{n,m}(z) = \exp(-\frac{1}{2}\pi i d_{m,n}) Q_{m,n}(iz),$$

$$(4.18b) \quad Q_{1-m-n,n}(z) = \exp(-\frac{1}{2}\pi i d_{m,n}) Q_{m,n}(iz),$$

where $d_{m,n} = m^2 + n^2 + mn - m - n$ is the degree of $Q_{m,n}(z)$, the roots of the polynomials $Q_{-m,n}(z)$ and $Q_{m,-n}(z)$, with $m, n \geq 1$ take similar forms as these polynomials they can be expressed in terms of $Q_{M,N}(z)$ and $Q_{-M,-N}(z)$ for suitable $M, N \geq 1$. Specifically, the roots of the polynomial $Q_{-m,n}(z)$, with $m \geq n \geq 1$, has the form of a $n \times (m - n + 1)$ “rectangle” with an “equilateral triangle”, which have either $n - 1$ or $n - m - 1$ roots, on each of its sides. Also the roots of the polynomial $Q_{-m,n}(z)$ with $n > m \geq 1$, has the form of a $m \times (n - m - 1)$ “rectangle” with an “equilateral triangle”, which have either m or $n - m - 1$ roots, on each of its sides. Further, we note that $Q_{-m,m}(z) = Q_{m,1}(z)$ and $Q_{1-m,m}(z) = Q_{m,0}(z)$, for all $m \in \mathbb{Z}$, where $Q_{m,0}(z)$ and $Q_{m,0}(z)$ are the original polynomials introduced by Okamoto [60]. Analogous results hold for $Q_{m,-n}(z)$, with $m, n \geq 1$.

4.4. Hamiltonian Theory P_{IV} . — The Hamiltonian for P_{IV} is [60]

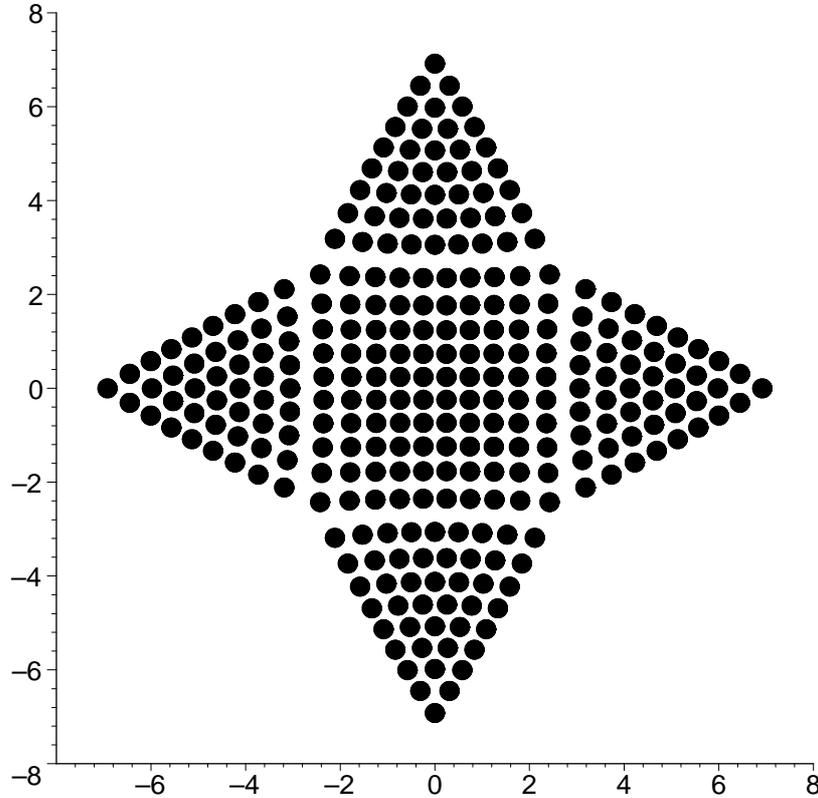
$$(4.19) \quad H_{\text{IV}}(q, p, z; \theta_0, \theta_\infty) = 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q,$$

then from Hamilton’s equation we have

$$(4.20) \quad q' = \frac{\partial H_{\text{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\theta_0, \quad p' = -\frac{\partial H_{\text{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \theta_\infty.$$

Eliminating p in (4.20), then $q = w$ satisfies P_{IV} with $(\alpha, \beta) = (1 - \theta_0 + 2\theta_\infty, -2\theta_0^2)$, and eliminating q in (4.20), then $w = -2p$ satisfies P_{IV} with $(\alpha, \beta) = (-1 + 2\theta_0 - \theta_\infty, -2\theta_\infty^2)$. The Hamiltonian function $\sigma(z; \theta_0, \theta_\infty) = H_{\text{IV}}(q, p, z; \theta_0, \theta_\infty)$ satisfies

$$(4.21) \quad (\sigma'')^2 - 4(z\sigma' - \sigma)^2 + 4\sigma'(\sigma' + 2\theta_0)(\sigma' + 2\theta_\infty) = 0.$$

FIGURE 4.2. Roots of the generalized Okamoto polynomial $Q_{10,10}(z)$

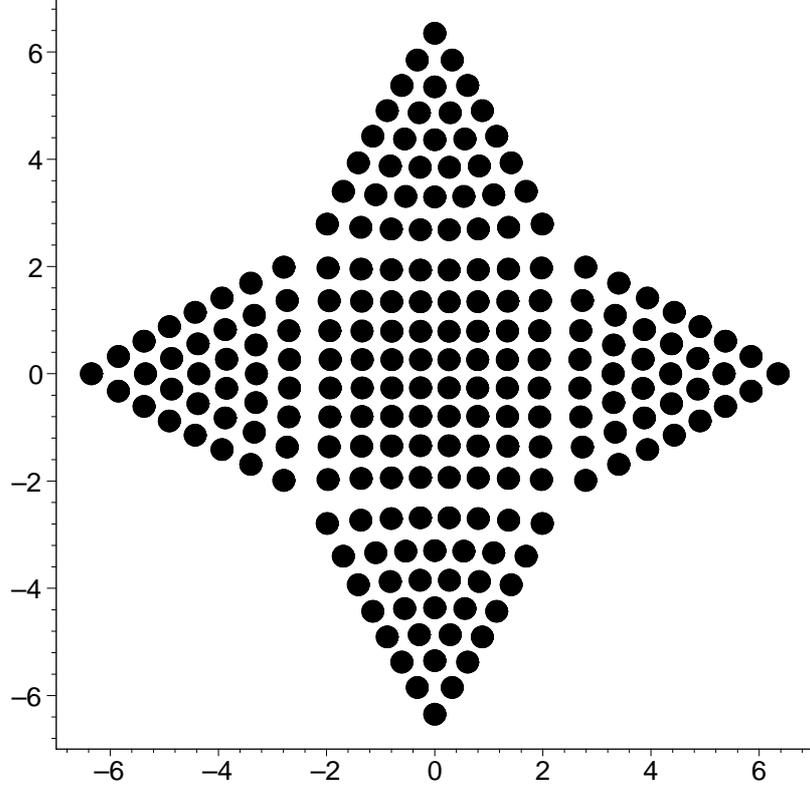
This equation is equivalent to equation SD-I.c in the classification of second order, second degree ordinary differential equations with the Painlevé property due to Cosgrove and Scoufis [23], an equation first derived and solved by Chazy [12] and rederived by Bureau [10, 9, 11]. It was also derived by Jimbo and Miwa [36] and Okamoto [60] in a Hamiltonian description of P_{IV} . Further equation (4.21) arises in various applications including random matrix theory (cf. [24, 72]). Conversely, if σ is a solution of (4.21), then

$$(4.22) \quad q = -\frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\theta_\infty)}, \quad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{2(\sigma' + 2\theta_0)},$$

are solutions of (4.20).

Due to the relationship between the Hamiltonian function σ and the associated τ -functions given by [60]

$$(4.23) \quad \frac{d}{dz} \ln \tau(z; \theta_0, \theta_\infty) = \sigma(z; \theta_0, \theta_\infty),$$

FIGURE 4.3. Roots of the generalized Okamoto polynomial $Q_{-8,-8}(z)$

then it can be shown that rational solutions of (4.21) have the form

$$(4.24a) \quad h_{m,n} = \frac{d}{dz} \ln H_{m,n}, \quad \theta_0 = -n, \quad \theta_\infty = m,$$

$$(4.24b) \quad \sigma_{m,n} = \frac{4}{27}z^3 - \frac{2}{3}(m-n)z + \frac{d}{dz} \ln Q_{m,n}, \quad \theta_0 = -n + \frac{1}{3}, \quad \theta_\infty = m - \frac{1}{3},$$

where $H_{m,n}(z)$ are the generalized Hermite polynomials and $Q_{m,n}(z)$ the generalized Okamoto polynomials.

Using this Hamiltonian formalism for P_{IV} , it can be shown that the generalized Hermite polynomials $H_{m,n}(z)$ and generalized Okamoto polynomials $Q_{m,n}(z)$, which are defined by differential-difference equations, also satisfy fourth order bilinear ordinary differential equations and homogeneous difference equations [18]. Differentiating (4.22) with respect to z yields

$$(4.25) \quad \sigma''' + 6(\sigma')^2 - 4(z^2 + 2\theta_0 + 2\theta_\infty)\sigma' + 4z\sigma + 8\theta_0\theta_\infty = 0.$$

Then substituting (4.24) into (4.25) yields the fourth order, bilinear equations

$$\begin{aligned}
 (4.26) \quad & H_{m,n}H_{m,n}'''' - 4H'_{m,n}H_{m,n}''' + 3(H''_{m,n})^2 \\
 & - 4(z^2 + 2n - 2m) \left[H_{m,n}H_{m,n}'' - (H'_{m,n})^2 \right] \\
 & + 4zH_{m,n}H'_{m,n} - 8mnH_{m,n}^2 = 0, \\
 (4.27) \quad & Q_{m,n}Q_{m,n}'''' - 4Q'_{m,n}Q_{m,n}''' + 3(Q''_{m,n})^2 + \frac{4}{3}z^2 \left[Q_{m,n}Q_{m,n}'' - (Q'_{m,n})^2 \right] \\
 & + 4zQ_{m,n}Q'_{m,n} - \frac{8}{3}(m^2 + n^2 + mn - m - n)Q_{m,n}^2 = 0.
 \end{aligned}$$

As for the case for the ordinary differential equations satisfied by the Yablonskii–Vorob'ev polynomials, i.e. equations (2.11) and (2.12), it seems reasonable to expect that the ordinary differential equations (4.26) and (4.27) will be useful for the derivation of properties of the generalized Hermite and generalized Okamoto polynomials. For example, using (4.26) and (4.27) it is straightforward to show that the polynomials $H_{m,n}(z)$ and $Q_{m,n}(z)$ have degree mn and $m^2 + n^2 + mn - m - n$, respectively.

5. Special Polynomials Associated with Algebraic Solutions of P_{III}

In this section we consider the special case of P_{III} when either (i), $\gamma = 0$ and $\alpha\delta \neq 0$, or (ii), $\delta = 0$ and $\beta\gamma \neq 0$. In case (i), we make the transformation

$$(5.1) \quad w(z) = \left(\frac{2}{3}\right)^{1/2}u(\zeta), \quad z = \left(\frac{2}{3}\right)^{3/2}\zeta^3,$$

and set $\alpha = 1$, $\beta = 2\mu$ and $\delta = -1$, with μ an arbitrary constant, without loss of generality, which yields

$$(5.2) \quad \frac{d^2u}{d\zeta^2} = \frac{1}{u} \left(\frac{du}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + 4\zeta u^2 + 12\mu\zeta - \frac{4\zeta^4}{u}.$$

In case (ii), we make the transformation

$$(5.3) \quad w(z) = \left(\frac{3}{2}\right)^{1/2}/u(\zeta), \quad z = \left(\frac{2}{3}\right)^{3/2}\zeta^3,$$

and set $\alpha = 2\mu$, $\beta = -1$ and $\gamma = 1$, with μ an arbitrary constant, without loss of generality, which again yields (5.2). The scalings in (5.1) and (5.3) have been chosen so that the associated special polynomials are monic polynomials. We remark that equation (5.2) is of type D_7 in the terminology of Sakai [67], and we shall refer to it as $P_{\text{III}}^{(7)}$. Further, Ramani *et al.* [64] argue that $P_{\text{III}}^{(7)}$ (5.2) should be considered as a different canonical form from P_{III} with $\gamma\delta \neq 0$, which is of type D_6 in Sakai's classification since (i), the structure of the Bäcklund transformation is quite different with a different associated Weyl group as shown below, (ii), there are no solutions expressible in terms of classical special functions, and (iii), the coalescence limit of $P_{\text{III}}^{(7)}$ yields P_{I} , whereas the coalescence limit of P_{III} with $\gamma\delta \neq 0$ yields P_{II} . Tsuda, Okamoto and Sakai [74] state that “from the viewpoint of algebraic geometry

and of Hamiltonian structure, it is necessary and quite natural to study these cases separately”.

Rational solutions of (5.2) correspond to algebraic solutions of P_{III} with $\gamma = 0$ and $\alpha\delta \neq 0$, or $\delta = 0$ and $\beta\gamma \neq 0$. Lukashovich [46, 48] obtained algebraic solutions of P_{III} , which are classified in the following theorem.

Theorem 5.1. — *Equation (5.2) has rational solutions if and only if $\mu = n$, with $n \in \mathbb{Z}$. These rational solutions have the form $u(\zeta) = P_{n^2+1}(\zeta)/Q_{n^2}(\zeta)$, where $P_{n^2+1}(\zeta)$ and $Q_{n^2}(\zeta)$ are monic polynomials of degree $n^2 + 1$ and n^2 , respectively.*

Proof. See Gromak, Laine and Shimomura [32, p. 164] (see also [28, 52, 54]).

A straightforward method for generating rational solutions of (5.2) is through the Bäcklund transformation

$$(5.4) \quad u_{\mu\pm 1} = \frac{\zeta^3}{u_\mu^2} \pm \frac{\zeta}{2u_\mu^2} \frac{du_\mu}{d\zeta} - \frac{3(2\mu \pm 1)}{2u_\mu},$$

where u_μ is the solution of (5.2) for parameter μ , using the “seed solution” $u_0(\zeta) = \zeta$ for $\mu = 0$ (see Gromak, Laine and Shimomura [32, p. 164] — see also [28, 52, 54]). Therefore the transformation group for (5.2) is isomorphic to the affine Weyl group $A_1^{(1)}$, which also is the transformation group for P_{II} [60, 76, 78]; the transformation group for P_{III} with $\gamma\delta \neq 0$ is isomorphic to the affine Weyl group $B_2^{(1)}$.

5.1. Associated special polynomials. — Ohyaama [59] derived special polynomials associated with the rational solutions of (5.2). These are essentially described in Theorem 5.2 below, though here the variables have been scaled and the expression of the rational solutions of (5.2) in terms of these special polynomials is explicitly given.

Theorem 5.2. — *Suppose that $R_n(\zeta)$ satisfies the recursion relation*

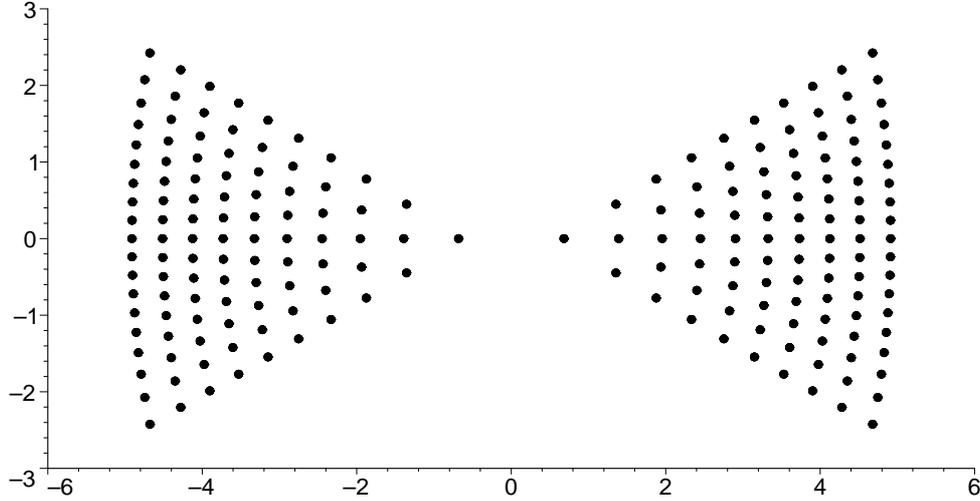
$$(5.5) \quad 2\zeta R_{n+1}R_{n-1} = -R_n \frac{d^2 R_n}{d\zeta^2} + \left(\frac{dR_n}{d\zeta} \right)^2 - \frac{R_n}{\zeta} \frac{dR_n}{d\zeta} + 2(\zeta^2 - n)R_n^2,$$

with $R_0(\zeta) = 1$ and $R_1(\zeta) = \zeta^2$. Then

$$(5.6) \quad u_n(\zeta) = \frac{R_{n+1}(\zeta)R_{n-1}(\zeta)}{R_n^2(\zeta)} \equiv \frac{\zeta^2 - n}{\zeta} - \frac{1}{2\zeta^2} \frac{d}{d\zeta} \left\{ \zeta \frac{d}{d\zeta} \ln R_n(\zeta) \right\},$$

satisfies (5.2) with $\mu = n$. Additionally $u_{-n}(\zeta) = -iu_n(i\zeta)$.

Plots of the locations of the roots of the polynomials $R_n(\zeta)$ are given in [17]. These plots show that the locations of the poles also have a very symmetric, regular structure and take the form of two “triangles” in a “bow-tie” shape. A plot of the complex roots of $R_{20}(\zeta)$ is given in Figure 5.1.

FIGURE 5.1. Roots of the polynomial $R_{20}(\zeta)$

5.2. Hamiltonian theory for $P_{\text{III}}^{(7)}$. — A Hamiltonian associated with $P_{\text{III}}^{(7)}$ (5.2) is [59, 67]

$$(5.7) \quad H_{\text{III}}^{(7)}(p, q; \kappa) = p^2 q^2 + 6(\kappa - \frac{1}{2})pq - 2\zeta^3(p + q),$$

and so from Hamilton's equations we have

$$(5.8) \quad \zeta \frac{dq}{d\zeta} = 2pq^2 + 6(\kappa - \frac{1}{2})q - 2\zeta^3, \quad \zeta \frac{dp}{d\zeta} = -2p^2q - 6(\kappa - \frac{1}{2})p + 2\zeta^3.$$

Setting $p = u$ and eliminating q in this system yields $P_{\text{III}}^{(7)}$ (5.2) with $\mu = \kappa$, whilst setting $q = u$ and eliminating p yields (5.2) with $\mu = \kappa - 1$, and so $p = u_\mu$ and $q = u_{\mu-1}$. Now define the auxiliary Hamiltonian function

$$(5.9) \quad \sigma = \frac{1}{6}H_{\text{III}}^{(7)}(p, q; \mu) + \frac{1}{2}pq + \frac{3}{2}\mu^2 = \frac{1}{6}p^2q^2 - \frac{1}{3}(p + q)\zeta^3 + \mu pq + \frac{3}{2}\mu^2,$$

where p and q satisfy (5.8). Then σ satisfies the second order, second degree equation

$$(5.10) \quad \left(\zeta \frac{d^2\sigma}{d\zeta^2} - 5 \frac{d\sigma}{d\zeta} \right)^2 + 4 \left(\frac{d\sigma}{d\zeta} \right)^2 \left(\zeta \frac{d\sigma}{d\zeta} - 6\sigma \right) - 48\mu\zeta^5 \frac{d\sigma}{d\zeta} = 16\zeta^{10}.$$

Conversely, if σ is a solution of (5.10), then

$$p = -\frac{1}{2\zeta^2} \frac{d\sigma}{d\zeta}, \quad q = \zeta^2 \left[\zeta \frac{d^2\sigma}{d\zeta^2} + (6\mu - 5) \frac{d\sigma}{d\zeta} + 4\zeta^5 \right] / \left(\frac{d\sigma}{d\zeta} \right)^2,$$

are solutions of (5.8). Since $p = u_\mu$ and $q = u_{\mu-1}$, where u_μ satisfies (5.2), then rational solutions of the Hamiltonian system (5.8) with $\kappa = n$ have the form

$$(5.11) \quad p_n(\zeta) = \frac{R_{n+1}(\zeta) R_{n-1}(\zeta)}{R_n^2(\zeta)}, \quad q_n(\zeta) = p_{n-1}(\zeta) = \frac{R_n(\zeta) R_{n-2}(\zeta)}{R_{n-1}^2(\zeta)}.$$

It is straightforward to show, using the relationship between solutions of (5.8) and (5.10) together with (5.5), that rational solutions of (5.10) with $\mu = n$ have the form (5.12)

$$\sigma_n = \frac{1}{6}p_n^2q_n^2 - \frac{1}{3}(p_n + q_n)\zeta^3 + np_nq_n + \frac{3}{2}n^2 = -\frac{1}{2}\zeta^4 + n\zeta^2 - \frac{3}{2}n + \frac{1}{6} + \zeta \frac{d}{d\zeta} \ln R_n.$$

Using this Hamiltonian formalism for $P_{\text{III}}^{(7)}$, it can be shown that the polynomials $R_n(\zeta)$ satisfy an fourth order bilinear ordinary differential equation and a fifth order, tri-linear difference equation [17]. Dividing (5.10) by ζ^{10} , setting $\mu = n$ and then differentiating with respect to ζ yields the third order equation

$$(5.13) \quad \zeta^2 \frac{d^3\sigma}{d\zeta^3} - 9\zeta \frac{d^2\sigma}{d\zeta^2} + 6\zeta \left(\frac{d\sigma}{d\zeta} \right)^2 + (25 - 24\sigma) \frac{d\sigma}{d\zeta} = 24n\zeta^5.$$

Substituting (5.12) into this equation yields the fourth order, bilinear equation

$$(5.14) \quad \zeta^3 \left[R_n \frac{d^4 R_n}{d\zeta^4} - 4 \frac{dR_n}{d\zeta} \frac{d^3 R_n}{d\zeta^3} + 3 \left(\frac{d^2 R_n}{d\zeta^2} \right)^2 \right] - 6\zeta^2 \left(R_n \frac{d^3 R_n}{d\zeta^3} - \frac{dR_n}{d\zeta} \frac{d^2 R_n}{d\zeta^2} \right) \\ - 12\zeta(\zeta^4 - 3n - 1) \left[R_n \frac{d^2 R_n}{d\zeta^2} - \left(\frac{dR_n}{d\zeta} \right)^2 \right] - 9\zeta \left[R_n \frac{d^2 R_n}{d\zeta^2} + \left(\frac{dR_n}{d\zeta} \right)^2 \right] \\ + 3(12\zeta^4 - 16n\zeta^2 + 12n + 7)R_n \frac{dR_n}{d\zeta} - 24n\zeta[(n+3)\zeta^2 - 3n - 1]R_n^2 = 0.$$

Additionally $R_n(\zeta)$ satisfies the fifth order, tri-linear difference equation

$$(5.15) \quad R_{n+2}R_{n-1}^2 + R_{n-2}R_{n+1}^2 = 2\zeta^3 R_n^3 - 6nR_{n+1}R_nR_{n-1}$$

(see [17] for details).

As for the ordinary differential equations satisfied by the special polynomials associated with rational solutions of $P_{\text{II}}-P_{\text{IV}}$, it seems reasonable to expect that the ordinary differential equation (5.14) will be useful for the derivation of properties of the polynomials $R_n(\zeta)$.

6. Special Polynomials Associated with Algebraic solutions of P_{V}

It is well-known that there is a relationship between solutions of P_{III}

$$(6.1) \quad \frac{d^2v}{d\zeta^2} = \frac{1}{v} \left(\frac{dv}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{dv}{d\zeta} + \frac{av^2 + b}{\zeta} + cv^3 + \frac{d}{v},$$

where a, b, c and d are arbitrary constants, in the generic case when $cd \neq 0$ (then we set $c = 1$ and $d = -1$, without loss of generality), and solutions of the special case of P_{V} (1.4) with $\delta = 0$ and $\gamma \neq 0$ [29] (see also [32]). This is given in the following theorem.

Theorem 6.1. — Suppose that $v = v(\zeta; a, b, 1, -1)$ is a solution of P_{III} and

$$(6.2) \quad \eta(\zeta) = \frac{dv}{d\zeta} - \varepsilon v^2 + \frac{(1 - \varepsilon a)v}{\zeta},$$

with $\varepsilon^2 = 1$. Then

$$(6.3) \quad w(z; \alpha, \beta, \gamma, \delta) = \frac{\eta(\zeta) - 1}{\eta(\zeta) + 1}, \quad z = \frac{1}{2}\zeta^2,$$

satisfies P_{V} with

$$(6.4) \quad (\alpha, \beta, \gamma, \delta) = ((b - \varepsilon a + 2)^2/32, -(b + \varepsilon a - 2)^2/32, -\varepsilon, 0).$$

Making the change of variables $w(z) = u(\zeta)$, with $z = \frac{1}{2}\zeta^2$, in P_{V} with $\delta = 0$ yields

$$(6.5) \quad \frac{d^2u}{d\zeta^2} = \left(\frac{1}{2u} + \frac{1}{u-1} \right) \left(\frac{du}{d\zeta} \right)^2 - \frac{1}{\zeta} \frac{du}{d\zeta} + \frac{4(u-1)^2}{\zeta^2} \left(\alpha u + \frac{\beta}{u} \right) + 2\gamma u.$$

Algebraic solutions of P_{V} with $\delta = 0$ and $\gamma \neq 0$ are equivalent to rational solutions of (6.5) and so henceforth we shall only discuss rational solutions of (6.5). These are obtained by substituting the rational solutions of P_{III} , which are classified in Theorem 3.1, into equations (6.2) and (6.3). Consequently we have the following classification of rational solutions for equation (6.5); for details see [32, §38], also [52, 54].

Theorem 6.2. — Necessary and sufficient conditions for the existence of rational solutions of (6.5) are either

$$(6.6) \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}\mu^2, -\frac{1}{8}(2n-1)^2, -1 \right),$$

or

$$(6.7) \quad (\alpha, \beta, \gamma) = \left(\frac{1}{8}(2n-1)^2, -\frac{1}{2}\mu^2, 1 \right),$$

where $n \in \mathbb{Z}$ and μ is arbitrary.

We remark that the solutions of (6.5) satisfying (6.6) are related to those satisfying (6.7) by the Bäcklund transformation for P_{V} given by the transformation

$$(6.8) \quad \tilde{\mathcal{S}}: \quad \tilde{w}(\tilde{z}) = 1/w(z), \quad \tilde{z} = z, \quad (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (-\beta, -\alpha, -\gamma, \delta).$$

Thus we shall be concerned only with rational solutions of (6.5) satisfying (6.6).

As shown above, there are special polynomials associated with the rational solutions of P_{III} given in Theorem 3.1. Finally rational solutions of (6.5) are obtained by substituting the rational solutions of P_{III} given by (3.6) into (6.2) and (6.3). Hence, in the case when $\varepsilon = 1$, rational solutions of (6.5) have the form

$$u_n(\zeta; \mu) = \frac{\zeta v'_n(\zeta; \mu) - \zeta v_n^2(\zeta; \mu) - 2(n + \mu)v_n(\zeta; \mu) - \zeta}{\zeta v'_n(\zeta; \mu) - \zeta v_n^2(\zeta; \mu) - 2(n + \mu)v_n(\zeta; \mu) + \zeta},$$

with $v_n(\zeta; \mu)$ given by (3.6). Consequently we obtain the following result.

Theorem 6.3 ([21]). — Suppose that $S_n(\zeta; \mu)$ satisfies the recursion relation (3.5) with $S_{-1}(\zeta; \mu) = S_0(\zeta; \mu) = 1$. Then, for $n \geq 1$, the rational solution

$$(6.9) \quad u_n(\zeta; \mu) = \frac{S_n(\zeta; \mu)S_{n-2}(\zeta; \mu)}{\mu S_{n-1}(\zeta; \mu + 1)S_{n-1}(\zeta; \mu - 1)},$$

satisfies (6.5) with parameters given by (6.6).

It is straightforward to any specific value of n that (6.9) satisfies (6.5) with parameters given by (6.6). However, at present, Theorem 6.3 should be regarded as a conjecture rather than a theorem since we do not yet have a proof.

7. Interlacing of roots?

An important, well-known property of classical orthogonal polynomials, such as the Hermite, Laguerre or Legendre polynomials whose roots all lie on the real line, is that the roots of successive polynomials interlace (cf. [3, 7, 71]). Thus for a set of orthogonal polynomials $\varphi_n(z)$, for $n = 0, 1, 2, \dots$, if $z_{n,m}$ and $z_{n,m+1}$ are two successive roots of $\varphi_n(z)$, i.e. $\varphi_n(z_{n,m}) = 0$ and $\varphi_n(z_{n,m+1}) = 0$, then $\varphi_{n-1}(\zeta_{n-1}) = 0$ and $\varphi_{n+1}(\zeta_{n+1}) = 0$ for some ζ_{n-1} and ζ_{n+1} such that $z_{n,m} < \zeta_{n-1}, \zeta_{n+1} < z_{n,m+1}$. An interesting question is whether there are analogous results for the special polynomials $P_n(z)$ associated with rational solutions of the Painlevé equations. Clearly there are notable differences since the special polynomials $P_n(z)$ are polynomials with complex roots whereas classical orthogonal polynomials $\varphi_n(z)$ have real roots. The pattern of the roots of the special polynomials are highly symmetric and structured, suggesting that they have interesting properties. An particularly intriguing question is whether there is any “interlacing of roots” (in the complex plane), analogous to that for classical polynomials (on the real line); though we do not expect any specific relationship between the roots of the special polynomials with roots of any classical polynomial. Further it is necessary to define what is meant by “interlacing of roots in the complex plane”. There have been some preliminary numerical investigations using MAPLE of the “interlacing of roots” of the special polynomials associated with rational solutions of P_{II} [22], algebraic solutions of P_{III} [17] and rational solutions of P_{IV} [18]. These studies give experimental evidence which suggests that there is structure to the relative positions of the roots. A plot of the roots of the Yablonskii–Vorob’ev polynomials $Q_{25}(z)$, denoted by \bullet , and $Q_{26}(z)$, denoted by \circ , are given in Figure 7.

Some properties of the roots of the Yablonskii–Vorob’ev polynomials $Q_n(z)$ are given in the following theorems.

Theorem 7.1. — For every positive integer n , the polynomial $Q_n(z)$ has simple roots. Further the polynomials $Q_n(z)$ and $Q_{n+1}(z)$ do not have a common root.

Proof. — See Fukutani, Okamoto and Umemura [26]. □

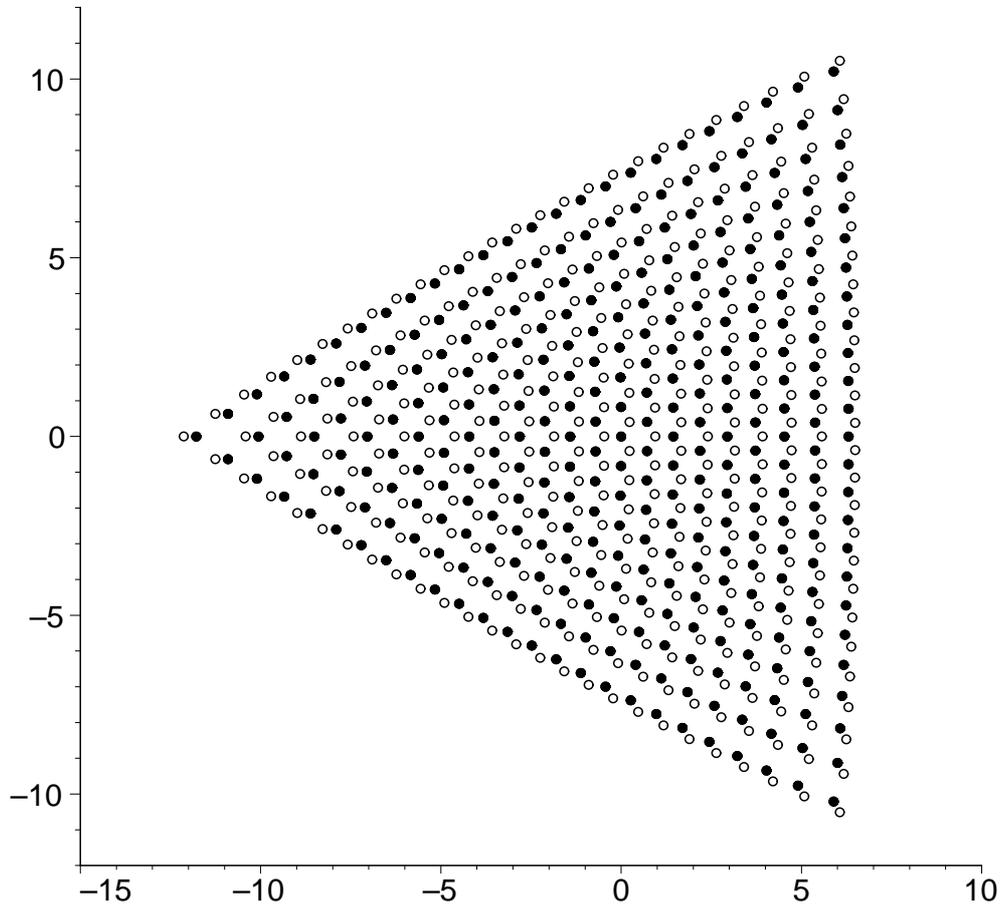


FIGURE 7.1. Roots of the Yablonskii-Vorob'ev polynomials $Q_{25}(z)$ (\bullet) and $Q_{26}(z)$ (\circ)

Theorem 7.2. — *The polynomial $Q_n(z)$ is divisible by z if and only if $n \equiv 1 \pmod{3}$. Further $Q_n(z)$ is a polynomial in z^3 if $n \not\equiv 1 \pmod{3}$ and $Q_n(z)/z$ is a polynomial in z^3 if $n \equiv 1 \pmod{3}$.*

Proof. — See Taneda [68]. □

Theorem 7.3. — *The real roots of the Yablonskii-Vorob'ev polynomials $Q_{n-1}(z)$ and $Q_{n+1}(z)$ interlace.*

Proof. — Suppose that a and b are successive real roots of $Q_{n-1}(z)$, i.e. $Q_{n-1}(a) = Q_{n-1}(b) = 0$, with $Q_{n-1}(z) > 0$ for $a < z < b$, so that $Q'_{n-1}(a) > 0$ and $Q'_{n-1}(b) < 0$; the case when $Q_{n-1}(z) < 0$ for $a < z < b$, so that $Q'_{n-1}(a) < 0$ and $Q'_{n-1}(b) > 0$, is

treated analogously. It is known that $Q_n(z)$ satisfies

$$(7.1) \quad Q'_{n+1}Q_{n-1} - Q_{n+1}Q'_{n-1} = (2n+1)Q_n^2,$$

(cf. [26, 43, 68]). Evaluating this at $z = a$ yields

$$(7.2) \quad Q_{n+1}(a)Q'_{n-1}(a) = (2n+1)Q_n^2(a).$$

We know from Theorem 7.1 that $Q_n(z)$ and $Q_{n-1}(z)$ have no common roots and the roots of $Q_{n-1}(z)$ are simple. Hence if $Q_{n-1}(a) = 0$ then $Q_n(a) \neq 0$ and $Q'_{n-1}(a) \neq 0$ and so from (7.2) we have

$$(7.3) \quad Q_{n+1}(a) = (2n+1)Q_n^2(a)/Q'_{n-1}(a) > 0.$$

Similarly by setting $z = b$ in (7.1) gives

$$(7.4) \quad Q_{n+1}(b) = (2n+1)Q_n^2(b)/Q'_{n-1}(b) < 0.$$

Therefore $Q_{n+1}(\xi) = 0$ for some $\xi \in (a, b)$ and hence between any two real roots of $Q_{n-1}(z)$ there is a real root of $Q_{n+1}(z)$. Similarly it can be shown that between any two real roots of $Q_{n+1}(z)$ there is a real root of $Q_{n-1}(z)$. \square

The plots of the roots of the Yablonskii–Vorob’ev polynomials $Q_n(z)$ suggest the following conjecture.

Conjecture 7.4. — *The Yablonskii–Vorob’ev polynomials $Q_{2n-1}(z)$ and $Q_{2n}(z)$ have n real roots.*

We feel that this “interlacing of roots” for the special polynomials warrants further analytical and numerical studies, though we shall not pursue these questions any further here.

Another indication that the Yablonskii–Vorob’ev polynomials are special is given by studying their discriminants, which are defined by (3.10).

Theorem 7.5. — *The discriminant of the Yablonski–Vorob’ev polynomial $Q_n(z)$ is given by*

$$|\text{Dis}(Q_n)| = 2^{m(m^2-1)(m+2)/6} \prod_{j=1}^m (2j+1)^{(2j+1)(m-j)^2},$$

where $\text{Dis}(Q_n) < 0$ if and only $n = 2 \pmod{4}$.

Proof. — See Roberts [65], whose results have to be scaled. \square

Roberts [65] also derives expressions for the discriminants for the generalized Hermite polynomials $H_{m,n}(z)$ and the generalized Okamoto polynomials $Q_{m,n}(z)$. These results show that the discriminants are expressed as products of small integers to large powers.

8. Discussion

In this paper we have studied properties of special polynomials associated with rational solutions of P_{II} , P_{III} and P_{IV} and algebraic solutions of P_{III} and P_V , which are related to rational solutions of P_{III} . In particular the zeroes of these polynomials have a very symmetric, regular structure. Further using the Hamiltonian formalism for P_{II} – P_{IV} , it is shown that these special polynomials, which are defined by second order bilinear differential-difference equations, which are equivalent to the Toda equation, also satisfy fourth order bilinear ordinary differential equations and homogeneous difference equations. It seems reasonable to expect that these ordinary differential equations will be useful in proving properties of the associated polynomials since there are more techniques for studying solutions of ordinary differential equations rather than differential-difference equations. Regular, symmetric structures also arise for the roots of special polynomials associated with rational solutions of the equations in the P_{II} hierarchy [22]. This seems to be yet another remarkable property of the Painlevé equations.

Open questions related to special polynomials associated with solutions of the Painlevé equations discussed in this paper include the following.

1. What is the structure of the roots of the special polynomials associated with rational and algebraic solutions of P_{VI} and rational solutions of the discrete Painlevé equations? It should be noted that most of these special polynomials have yet to be derived.
2. What is the structure of the roots of special polynomials associated with rational solutions of soliton equations? Airault, McKean and Moser [5] studied the motion of the poles of rational solutions of the Korteweg-de Vries (KdV) equation and a related many-body problem; see also [2, 4, 13]. Subsequently there has been studies of other soliton equations, including the Boussinesq equation [27], the classical Boussinesq system [66], the Kadomtsev-Petviashvili equation [62, 63] and the nonlinear Schrödinger (NLS) equation

$$(8.1) \quad iu_t = u_{xx} - 2|u|^2u,$$

[33, 55]. A recent study of the roots of special polynomials associated with rational and rational-oscillatory solutions of the NLS equation (8.1) is given in [20], which includes some new rational-oscillatory solutions that are expressed in terms of the generalized Okamoto polynomials.

3. Do these special polynomials have applications, for example in numerical analysis? The classical orthogonal polynomials, such as Hermite, Laguerre, Legendre and Tchebychev polynomials which are associated with rational solutions classical special functions, play an important role in a variety of applications (cf. [7, 71]). Hence it seems probable that the polynomials discussed here which are

associated with rational solutions of nonlinear special functions, i.e. the Painlevé equations, will also arise in variety of applications.

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