

POINT CONFIGURATIONS, CREMONA
TRANSFORMATIONS AND THE ELLIPTIC DIFFERENCE
PAINLEVÉ EQUATION

by

Kenji Kajiwara, Tetsu Masuda, Masatoshi Noumi, Yasuhiro Ohta
& Yasuhiko Yamada

Abstract. — A theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions is provided in the framework of birational Weyl group action on the space of point configurations in general position in a projective space. By introducing an elliptic parametrization of point configurations, a realization of the Weyl group is proposed as a group of Cremona transformations containing elliptic functions in the coefficients. For this elliptic Cremona system, a theory of τ -functions is developed to translate it into a system of bilinear equations of Hirota-Miwa type for the τ -functions on the lattice. Application of this approach is also discussed to the elliptic difference Painlevé equation.

Résumé (Configurations de points, transformations de Cremona et équation de Painlevé aux différences elliptique)

Dans le cadre de l'action birationnelle du groupe de Weyl sur l'espace des configurations de points en position générale dans un espace projectif on établit des fondements théoriques en vue d'une généralisation aux dimensions supérieures de l'équation de Painlevé aux différences elliptique. On réalise le groupe de Weyl comme un groupe de transformations de Cremona à coefficients fonctions elliptiques grâce à une paramétrisation elliptique des configurations de points. Une théorie des fonctions τ permet de traduire ce système de Cremona en un système d'équations bilinéaires de type Hirota-Miwa pour les fonctions τ sur le réseau. On en donne une application à l'équation de Painlevé aux différences elliptique.

1. Introduction

The main purpose of this paper is to provide a theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions in the framework of birational Weyl group actions on the spaces of point configurations in general position in projective spaces.

2000 Mathematics Subject Classification. — Primary: 39A20 Secondary: 14E07, 14N20, 14H52, 33E17.
Key words and phrases. — discrete Painlevé equation, Cremona transformation, configuration space, elliptic function.

Since the pioneering work of Grammaticos, Ramani, Papageorgiou and Hietarinta [5, 19], discrete Painlevé equations have been studied from various viewpoints. A large class of second order discrete Painlevé equations, as well as their generalizations, has been discovered through the studies of singularity confinement property, bilinear equations, affine Weyl group symmetries and spaces of initial conditions (see [20, 21, 15, 22]...). For historical aspects of discrete Painlevé equations, we refer the reader to the review of Grammaticos-Ramani [4].

Among many others, we mention here the geometric approach proposed by Sakai [22] for a class of discrete Painlevé equations arising from rational surfaces. Each equation in this class is defined by the group of Cremona transformations on a certain family of surfaces obtained from the projective plane $\mathbb{P}^2(\mathbb{C})$ by blowing-up. According to the types of rational surfaces, those discrete Painlevé equations are classified in terms of affine root systems. Also, their symmetries are described by means of affine Weyl groups. The *elliptic difference Painlevé equation*, which is regarded as the master equation for all discrete Painlevé equations of this class, is a discrete dynamical system defined on a family of surfaces parametrized by the 9-point configurations in general position in $\mathbb{P}^2(\mathbb{C})$; the corresponding group of Cremona transformations is the affine Weyl group of type $E_8^{(1)}$. As we have shown in [8], this system of difference equations can be transformed into the eight-parameter discrete Painlevé equation of Ohta-Ramani-Grammaticos [16], constructed from a completely different viewpoint of bilinear equations for the τ -functions on the E_8 lattice. It is also known by [8] that the elliptic difference Painlevé equation has special Riccati type solutions obtained by linearization to the *elliptic difference hypergeometric equation*. This gives a new perspective of nonlinear special functions to the elliptic hypergeometric functions which have been studied for instance by Frenkel-Turaev [3] in the context of elliptic 6- j symbols and by Spiridonov-Zhedanov [23] in the theory of biorthogonal rational functions on elliptic grids.

Generalizing the geometric approach to the elliptic difference Painlevé equations, in this paper we investigate the configuration space $\mathbb{X}_{m,n}$ of n points p_1, \dots, p_n in general position in the projective space $\mathbb{P}^{m-1}(\mathbb{C})$. It is well-known [2] that the Weyl group $W_{m,n}$ associated with the tree $T_{2,m,n-m}$ can be realized as a group of birational transformations on the configuration space $\mathbb{X}_{m,n}$. Through the $W_{m,n}$ -equivariant projection $\mathbb{X}_{m,n+1} \rightarrow \mathbb{X}_{m,n}$ that maps $[p_1, \dots, p_n, q]$ to $[p_1, \dots, p_n]$, from the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of the Weyl group $W_{m,n}$ as a group of Cremona transformations on $q \in \mathbb{P}^{m-1}(\mathbb{C})$ parameterized by the configuration space $\mathbb{X}_{m,n}$. Note that in the case when $(m, n) = (3, 9), (4, 8)$ or $(6, 9)$, the Weyl group $W_{m,n}$ is the affine Weyl group of type $E_8^{(1)}, E_7^{(1)}$ or $E_8^{(1)}$, respectively; this group $W_{m,n} = W(E_l^{(1)})$ decomposes into the semidirect product of the root lattice $Q(E_l)$ and the finite Weyl group $W(E_l)$. In each of the three cases, through the birational

action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$, the lattice part of the affine Weyl group provides a *discrete Painlevé system* on $\mathbb{P}^{m-1}(\mathbb{C})$ with parameter space $\mathbb{X}_{m,n}$. The discrete Painlevé system of type (3,9) thus obtained contains the three discrete Painlevé equations, elliptic, trigonometric and rational, with $W(E_8^{(1)})$ symmetry in Sakai's table.

In this framework of configuration spaces, in Section 4 we construct a $W_{m,n}$ -equivariant meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ by means of elliptic functions, where $\mathfrak{h}_{m,n}$ denotes the Cartan subalgebra of the Kac-Moody Lie algebra associated with the tree $T_{2,m,n-m}$. If we regard the birational $W_{m,n}$ -action on $\mathbb{X}_{m,n}$ as a system of functional equations for the coordinate functions, a 'canonical' elliptic solution is provided by the meromorphic mapping $\varphi_{m,n}$. Its image also specifies a $W_{m,n}$ -stable class of n -point configurations in $\mathbb{P}^{m-1}(\mathbb{C})$ in which the n points are on an elliptic curve. By restricting the point configurations to this class, from the birational Weyl group action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of $W_{m,n}$ as a group of Cremona transformations on $\mathbb{P}^{m-1}(\mathbb{C})$ parametrized by elliptic functions, which we call the *elliptic Cremona system* of type (m,n) . In Section 5 we develop a theory of τ -functions for this elliptic Cremona system of type (m,n) , and show that it is translated into a system of bilinear equations of Hirota-Miwa type for the τ -functions on the lattice. After that we reconsider the case of the elliptic difference Painlevé system of type (3,9) in the scope of the general setting of this paper. There we give explicit description for some of the discrete time evolutions, in terms of homogeneous coordinates in Section 6, and in the language of geometry of plane curves in Section 7.

The τ -function approach developed in this paper can be applied effectively to the study of special hypergeometric solutions of the elliptic Painlevé equation and its degenerations. Also, it is an important problem to *complete* the framework of $\mathbb{X}_{m,n}$ of point configurations in general position, so that it should contain all reasonable degenerate configurations as in Sakai's table. These subjects will be investigated in our subsequent papers.

2. Point configurations and Cremona transformations

Let $\mathbb{X}_{m,n}$ be the configuration space of n points in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ ($n > m > 1$). We say that an n -tuple of points (p_1, \dots, p_n) in $\mathbb{P}^{m-1}(\mathbb{C})$ is *in general position* if p_1, \dots, p_n are mutually distinct, and $\#(H \cap \{p_1, \dots, p_n\}) < m$ for any hyperplane H in $\mathbb{P}^{m-1}(\mathbb{C})$. We denote by $[p_1, \dots, p_n]$ the corresponding *configuration*, namely, the equivalence class of (p_1, \dots, p_n) under the diagonal $PGL_m(\mathbb{C})$ -action. By fixing a system of homogeneous coordinates for $\mathbb{P}^{m-1}(\mathbb{C})$, the configuration space $\mathbb{X}_{m,n}$ may be identified with the double coset space

$$(1) \quad \mathbb{X}_{m,n} = GL_m(\mathbb{C}) \backslash \text{Mat}_{m,n}^*(\mathbb{C}) / T_n,$$

where $\text{Mat}_{m,n}^*(\mathbb{C})$ stands for the space of all $m \times n$ complex matrices whose $m \times m$ minor determinants are all nonzero, and $T_n = (\mathbb{C}^*)^n$ for the diagonal subgroup of $GL_n(\mathbb{C})$. The configuration space $\mathbb{X}_{m,n}$ has the structure of an affine algebraic variety, isomorphic to a Zariski open subset of $\mathbb{C}^{(m-1)(n-m-1)}$ (see [25], for instance). Also, it is known [1], [2] that the Weyl group associated with the tree

$$(2) \quad T_{2,m,n-m} : \begin{array}{c} \circ \alpha_0 \\ | \\ \circ \alpha_1 \text{---} \circ \alpha_2 \text{---} \dots \text{---} \circ \alpha_m \text{---} \circ \alpha_{m+1} \text{---} \dots \text{---} \circ \alpha_{n-1} \end{array}$$

acts birationally on $\mathbb{X}_{m,n}$. This Weyl group $W_{m,n} = W(T_{2,m,n-m})$ is generated by the simple reflections s_0, s_1, \dots, s_{n-1} with the following fundamental relations.

$$(3) \quad W_{m,n} = \langle s_0, s_1, \dots, s_{n-1} \rangle : \begin{array}{l} s_i^2 = 1 \\ s_i s_j = s_j s_i \\ s_i s_j s_i = s_j s_i s_j \end{array} \quad \begin{array}{cc} \alpha_i & \alpha_j \\ \circ & \circ \\ \text{---} & \text{---} \end{array}$$

As we will recall below, $W_{m,n}$ is realized as a group of birational transformations of $\mathbb{X}_{m,n}$ by the standard Cremona transformations with respect to m points among p_1, \dots, p_n .

Given a set of m points p_1, \dots, p_m in general position, choose a system of homogeneous coordinates $x = (x_1, \dots, x_m)$ such that

$$(4) \quad p_1 = (1 : 0 : \dots : 0), \quad p_2 = (0 : 1 : \dots : 0), \quad \dots, \quad p_m = (0 : \dots : 0 : 1).$$

Then the standard Cremona transformation with respect to (p_1, \dots, p_m) is the birational transformation $p \rightarrow \tilde{p}$ of $\mathbb{P}^{m-1}(\mathbb{C})$ defined by $\tilde{p} = (x_1^{-1} : \dots : x_m^{-1})$ for any $p = (x_1 : \dots : x_m)$ with $x_i \neq 0$ ($i = 1, \dots, m$). Note that this transformation depends on the choice of homogeneous coordinates, and is determined only up to the action of $(\mathbb{C}^*)^m$. The birational (right) action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ is then defined as follows. Firstly, the symmetric group \mathfrak{S}_n acts on $\mathbb{X}_{m,n}$ by the permutation of n points:

$$(5) \quad [p_1, \dots, p_n] \cdot \sigma = [p_{\sigma(1)}, \dots, p_{\sigma(n)}] \quad (\sigma \in \mathfrak{S}_n).$$

The adjacent transpositions $s_j = (j, j + 1)$ ($j = 1, \dots, n - 1$) provide the simple reflections attached to the subdiagram of type A_{n-1} in $T_{2,m,n-m}$. The remaining simple reflection s_0 is given by the (well-defined) birational transformation

$$(6) \quad [p_1, \dots, p_n] \cdot s_0 = [p_1, \dots, p_m, \tilde{p}_{m+1}, \dots, \tilde{p}_n],$$

in terms of the standard Cremona transformation $p \rightarrow \tilde{p}$ with respect to the first m points (p_1, \dots, p_m) . These birational transformations s_0, s_1, \dots, s_{n-1} in fact satisfy the fundamental relations for the simple reflections of $W_{m,n}$. We also remark that, for each subset $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$ of mutually distinct m indices, the standard Cremona transformation with respect to $(p_{j_1}, \dots, p_{j_m})$ is determined as $\text{cr}_{j_1, \dots, j_m} = \sigma s_0 \sigma^{-1} \in W_{m,n}$ by a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma(a) = j_a$ for $a = 1, \dots, m$.

The right birational action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ induces a left action of $W_{m,n}$ on the field $\mathcal{K}(\mathbb{X}_{m,n})$ of rational functions on $\mathbb{X}_{m,n}$ as a group of automorphisms: For each $\varphi \in \mathcal{K}(\mathbb{X}_{m,n})$ and $w \in W_{m,n}$, we define $w(\varphi) \in \mathcal{K}(\mathbb{X}_{m,n})$ by

$$(7) \quad w(\varphi)([p_1, \dots, p_n]) = \varphi([p_1, \dots, p_n].w)$$

for any generic $[p_1, \dots, p_n] \in \mathbb{X}_{m,n}$. Let us consider the set $\mathcal{U}_{m,n}$ of all matrices $U \in \text{Mat}_{m,n}(\mathbb{C})^*$ of the form

$$(8) \quad U = \begin{pmatrix} 1 & \dots & 0 & 0 & 1 & u_{1,m+2} & \dots & u_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & 1 & u_{m-1,m+2} & \dots & u_{m-1,n} \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

It is easily shown that each $(GL_m(\mathbb{C}), T_n)$ -orbit in $\text{Mat}_{m,n}^*(\mathbb{C})$ intersects with $\mathcal{U}_{m,n}$ at one point. By using this transversal $\mathcal{U}_{m,n} \xrightarrow{\sim} GL_m(\mathbb{C}) \backslash \text{Mat}_{m,n}^*(\mathbb{C}) / T_n$, we identify $\mathbb{X}_{m,n}$ with a Zariski open subset of the affine space $\mathbb{C}^{(m-1)(n-m-1)}$ with canonical coordinates $u = (u_{i,j})_{1 \leq i \leq m-1; m+2 \leq j \leq n}$. Through the isomorphism $\mathcal{K}(\mathbb{X}_{m,n}) \xrightarrow{\sim} \mathbb{C}(u)$, the action of $W_{m,n}$ on $\mathcal{K}(\mathbb{X}_{m,n})$ can be described explicitly in terms of the u variables. The following table shows how the simple reflections s_k ($k = 0, 1, \dots, n-1$) transform the coordinates $u_{i,j}$ ($i = 1, \dots, m-1; j = m+2, \dots, n$):

$$(9) \quad \begin{aligned} k = 0 : & \quad s_0(u_{ij}) = \frac{1}{u_{ij}}, \\ k = 1, \dots, m-2 : & \quad s_k(u_{ij}) = u_{s_k(i),j}, \\ k = m-1 : & \quad s_{m-1}(u_{ij}) = \begin{cases} \frac{u_{ij}}{u_{m-1,j}} & (i = 1, \dots, m-2), \\ \frac{1}{u_{m-1,j}} & (i = m-1), \end{cases} \\ k = m : & \quad s_m(u_{ij}) = 1 - u_{ij}, \\ k = m+1 : & \quad s_{m+1}(u_{ij}) = \begin{cases} \frac{1}{u_{i,m+2}} & (j = m+2), \\ \frac{u_{ij}}{u_{i,m+2}} & (j = m+3, \dots, n), \end{cases} \\ k = m+2, \dots, n-1 : & \quad s_k(u_{ij}) = u_{i,s_k(j)}, \end{aligned}$$

where $s_k(i)$ stands for the action of the adjacent transposition $(k, k+1)$ on the index $i \in \{1, \dots, n\}$. From this representation, for each $w \in W_{m,n}$ we obtain a family of rational functions

$$(10) \quad w(u_{i,j}) = S_{i,j}^w(u) \quad (i = 1, \dots, m-1; j = m+2, \dots, n)$$

in the u variables; these functions satisfy the consistency relations

$$(11) \quad S_{i,j}^1(u) = u_{ij}, \quad S_{i,j}^{ww'}(u) = S_{i,j}^{w'}(S^w(u))$$

for any i, j and $w, w' \in W_{m,n}$.

If we regard the u variables as dependent variables (unknown functions), (10) or equivalently (9) can be regarded as a system of functional equations for u_{ij} . Typically, we take a vector space V with canonical coordinates $t = (t_1, \dots, t_N)$, assuming that $W_{m,n}$ acts linearly on V (from the right). If we regard the coordinate functions t_j of V as the independent variables, then a solution of the system (10) is nothing but a $W_{m,n}$ -equivariant mapping $\varphi : V \rightarrow \mathbb{X}_{m,n}$. In Section 4, we construct an elliptic solution of the system (10) in this sense, with V being the Cartan subalgebra $\mathfrak{h}_{m,n}$ of the Kac-Moody Lie algebra associated with $T_{2,m,n-m}$.

3. Tracing the Cremona transformations

Given a generic configuration $[p_1, \dots, p_n]$ of n points, let us ask how a general point of $\mathbb{P}^{m-1}(\mathbb{C})$, as well as the configuration itself, is transformed by a successive application of standard Cremona transformations. In what follows, by a *Cremona transformation* we mean a birational transformation of $\mathbb{P}^{m-1}(\mathbb{C})$ obtained by a successive application of standard Cremona transformations.

We now consider the relative situation with respect to the projection $\pi : \mathbb{X}_{m,n+1} \rightarrow \mathbb{X}_{m,n}$ that maps $[p_1, \dots, p_n, p_{n+1}]$ to $[p_1, \dots, p_n]$. This projection is $W_{m,n}$ -equivariant relative to the inclusion $W_{m,n} \subset W_{m,n+1}$ of Weyl groups. We regard $\mathbb{X}_{m,n}$ as the parameter space for Cremona transformations belonging to $W_{m,n}$, and the last point $q = p_{n+1}$ as the general point in $\mathbb{P}^{m-1}(\mathbb{C})$ that should be transformed by such Cremona transformations. (This formulation has been used by [10] in the case $(m, n) = (3, 9)$.) Then, our question is how to describe $[p_1, \dots, p_n, q].w$ for each $[p_1, \dots, p_n, q] \in \mathbb{X}_{m,n+1}$ and $w \in W_{m,n}$. By using the coordinates

$$(12) \quad \tilde{U} = \begin{pmatrix} 1 & \dots & 0 & 0 & 1 & u_{1,m+2} & \dots & u_{1,n} & z_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 1 & u_{m-1,m+2} & \dots & u_{m-1,n} & z_{m-1} \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

for $\mathcal{U}_{m,n+1}$, we parametrize the configurations $[p_1, \dots, p_n] \in \mathbb{X}_{m,n}$ and the general points $q \in \mathbb{P}^{m-1}(\mathbb{C})$ as

$$(13) \quad \begin{aligned} p_1 &= (1 : 0 : \dots : 0), \dots, p_m = (0 : \dots : 0 : 1), p_{m+1} = (1 : \dots : 1 : 1), \\ p_j &= (u_{1,j} : \dots : u_{m-1,j} : 1) \quad (j = m + 2, \dots, n), \quad q = (z_1 : \dots : z_{m-1} : 1). \end{aligned}$$

Then for any $w \in W_{m,n}$ the configuration $[p_1, \dots, p_n, q].w = [\tilde{p}_1, \dots, \tilde{p}_n, \tilde{q}]$ is given by

$$(14) \quad \begin{aligned} \tilde{p}_j &= p_j && (j = 1, \dots, m + 1), \\ \tilde{p}_j &= (w(u_{1,j}) : \dots : w(u_{m-1,j}) : 1) && (j = m + 2, \dots, n), \\ \tilde{q} &= (w(z_1) : \dots : w(z_{m-1}) : 1). \end{aligned}$$

In this sense, for each $w \in W_{m,n}$ the corresponding Cremona transformation of $q = p_{n+1}$ is determined as

$$(15) \quad w(z_i) = R_i^w(u; z) \quad (i = 1, \dots, m - 1),$$

in terms of rational functions $R_i^w(u; z)$ in the variables $u = (u_{ij})_{1 \leq i \leq m-1; m+2 \leq j \leq n}$ and $z = (z_1, \dots, z_{m-1})$. Note also that $R_i^w(u; z)$ satisfy

$$(16) \quad R_i^1(u; z) = z_i, \quad R_i^{ww'}(u; z) = R_i^{w'}(S^w(u); R^w(u; z))$$

for any i and $w, w' \in W_{m,n}$. As we will see below, these $R_i^w(u; z)$, regarded as rational functions in the variable $z = (z_1, \dots, z_{m-1})$, have a characteristic property concerning their multiplicities of zero at p_1, \dots, p_n .

Consider a free \mathbb{Z} -module

$$(17) \quad L_{m,n} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$$

of rank $n + 1$ with basis $\{e_0, e_1, \dots, e_n\}$, and define a symmetric bilinear form $(|) : L_{m,n} \times L_{m,n} \rightarrow \mathbb{Z}$ by

$$(18) \quad \begin{aligned} (e_0 | e_0) &= -(m - 2), & (e_j | e_j) &= 1 \quad (j = 1, \dots, n), \\ (e_i | e_j) &= 0 & (i, j &= 0, 1, \dots, n; i \neq j). \end{aligned}$$

This lattice $L_{m,n}$ admits a natural linear action of the Weyl group $W_{m,n}$ defined by

$$(19) \quad s_k \cdot \Lambda = \Lambda - (h_k | \Lambda) h_k \quad (\Lambda \in L_{m,n})$$

for each $k = 0, 1, \dots, n - 1$, where

$$(20) \quad h_0 = e_0 - e_1 - \dots - e_m, \quad h_k = e_k - e_{k+1} \quad (k = 1, \dots, n).$$

Note that $(h_j | h_j) = 2$ ($j = 0, 1, \dots, n - 1$) and that $((h_i | h_j))_{i,j=0}^{n-1}$ is the (generalized) Cartan matrix associated with the tree $T_{2,m,n-m}$. For each

$$(21) \quad \Lambda = de_0 - \nu_1 e_1 - \dots - \nu_n e_n \in L_{m,n} \quad (d, \nu_1, \dots, \nu_n \in \mathbb{Z})$$

we denote by $L(\Lambda)$ the vector space over $\mathcal{K}(\mathbb{X}_{m,n})$ consisting of all homogeneous polynomials $f(x) \in \mathcal{K}(\mathbb{X}_{m,n})[x]$ of degree d in m variables $x = (x_1, \dots, x_m)$ that have a zero of multiplicity $\geq \nu_j$ at each p_j ($j = 1, \dots, n$):

$$(22) \quad \deg f(x) = d, \quad \text{ord}_{p_j} f(x) \geq \nu_j \quad (j = 1, \dots, n).$$

Here we regard $x = (x_1, \dots, x_m)$ as the homogeneous coordinate system for $\mathbb{P}^{m-1}(\mathcal{K}(\mathbb{X}_{m,n}))$ such that $(z_1 : \dots : z_{m-1} : 1) = (a_1 x_1 : \dots : a_m x_m)$ for some nonzero constants $a_i \in \mathcal{K}(\mathbb{X}_{m,n})$ ($i = 1, \dots, m - 1$). Then we have

Theorem 3.1. — (1) Let $M_{m,n} = W_{m,n}\{e_1, \dots, e_n\}$ be the $W_{m,n}$ -orbit of $\{e_1, \dots, e_n\}$ in $L_{m,n}$. Then for any $\Lambda \in M_{m,n}$, one has $\dim_{\mathcal{K}(\mathbb{X}_{m,n})} L(\Lambda) = 1$.

(2) Given any element $w \in W_{m,n}$, take nonzero polynomials

$$(23) \quad F_i(x) \in L(w \cdot e_i), \quad G_i(x) \in L(ws_0 \cdot e_i) \quad (i = 1, \dots, m).$$

Then one has

$$(24) \quad (w(z_1) : \dots : w(z_{m-1}) : 1) = \left(c_1 \frac{G_1(x)}{F_1(x)} : \dots : c_m \frac{G_m(x)}{F_m(x)} \right)$$

for some nonzero constants $c_i \in \mathcal{K}(\mathbb{X}_{m,n})$ ($i = 1, \dots, m$).

This theorem can be proved by decomposing each $w \in W_{m,n}$ into a product of simple reflections $w = s_{j_1} \cdots s_{j_p}$, and then by lifting each s_{j_k} to the level of homogeneous coordinates. We remark that there is no canonical way *a priori* to define the action of s_j on homogeneous polynomials. This is the reason why we cannot specify the choice of F_i and G_i . We will return to this point later in Section 5 in the context of τ -functions for the elliptic Cremona system. In the case $(m, n) = (3, 9)$, for any $\Lambda \in M_{m,n}$ a nontrivial element in $L(\Lambda)$ can be constructed as a certain interpolation determinant (see Section 6).

As we have seen above, the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ can be expressed in the form

$$(25) \quad w(u_{i,j}) = S_{i,j}^w(u), \quad w(z_i) = R_i^w(u; z).$$

Formula (25) can be thought of as a system of functional equations for the dependent variables $z = (z_1, \dots, z_{m-1})$ including the u variables as parameters. Theorem 3.1 then implies that such a system of equations can be expressed as

$$(26) \quad w(z_i) = R_i^w(z) = \frac{c_i}{c_m} \frac{G_i^w(x) F_m^w(x)}{F_i^w(x) G_m^w(x)} \quad (i = 1, \dots, m-1)$$

for each $w \in W_{m,n}$, by means of homogeneous polynomials $F_i^w \in L(w.e_i)$ and $G_i^w \in L(ws_0.e_i)$ that are characterized by the degree and the multiplicities of zero at p_1, \dots, p_n . (The dependence on the u variables is suppressed in this formula.) Note that any abelian subgroup of the Weyl group $W_{m,n}$ gives rise to a commuting family of birational transformations on $\mathbb{P}^{m-1}(\mathbb{C})$ parameterized by configurations of n points. Such a birational dynamical system (for the z -variables) could be called a *discrete Painlevé system* associated with point configurations in $\mathbb{P}^{m-1}(\mathbb{C})$.

When the number $4 - (m-2)(n-m-2)$ has the sign $+$, 0 , or $-$, the root system associated with the tree $T_{2,m,n-m}$ is of finite type, of affine type, or of indefinite type, respectively, in the sense of [6]. In particular there are three cases $(m, n) = (3, 9), (4, 8)$ and $(6, 9)$ of affine type; the corresponding root systems are of type $E_8^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$, respectively. In these three cases, the configuration of n points can be parametrized generically by means of elliptic functions. Note also that the Weyl group $W_{m,n} = W(E_l^{(1)})$ is then expressed as the semidirect product of the root lattice $Q(E_l)$ of rank l and the finite Weyl group $W(E_l)$ acting on it. We call the discrete dynamical system arising from the lattice part of $W_{m,n}$ the *elliptic difference Painlevé equation* of type (m, n) .

4. Linearization of the $W_{m,n}$ -action in terms of elliptic functions

Let $\mathfrak{h}_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification of the lattice $L_{m,n}$. In this section we construct a $W_{m,n}$ -equivariant meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ by means of elliptic functions. This mapping specifies a class of configurations of n points on an elliptic curve in $\mathbb{P}^{m-1}(\mathbb{C})$, which is preserved by the action of $W_{m,n}$. In order to simplify the presentation, we assume $m \geq 3$. In this case the symmetric bilinear form $(\mid) : \mathfrak{h}_{m,n} \times \mathfrak{h}_{m,n} \rightarrow \mathbb{C}$ is nondegenerate, and $\mathfrak{h}_{m,n}$ is identified with the Cartan subalgebra of the Kac-Moody Lie algebra associated with the tree $T_{2,m,n-m}$.

We define the linear functions ε_j ($j = 0, 1, \dots, n$) and α_j ($j = 0, 1, \dots, n - 1$) on $\mathfrak{h}_{m,n}$ by

$$(27) \quad \varepsilon_j(h) = (e_j \mid h), \quad \alpha_j(h) = (h_j \mid h) \quad (h \in \mathfrak{h}_{m,n}).$$

We regard $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) : \mathfrak{h}_{m,n} \xrightarrow{\sim} \mathbb{C}^{n+1}$ as the canonical coordinates for $\mathfrak{h}_{m,n}$. The linear functions $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are the *simple roots* of the root system associated with $T_{2,m,n-m}$. Note also that the dual space $\mathfrak{h}_{m,n}^* = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \dots \oplus \mathbb{C}\varepsilon_n$ has the induced symmetric bilinear form: $(\varepsilon_i \mid \varepsilon_j) = (e_i \mid e_j)$ for any $i, j \in \{0, 1, \dots, n\}$. The Weyl group $W_{m,n}$ acts on $\mathfrak{h}_{m,n}$ and $\mathfrak{h}_{m,n}^*$ in a standard way: For each $k = 0, 1, \dots, n - 1$,

$$(28) \quad s_k \cdot h = h - \langle h, \alpha_k \rangle \alpha_k \quad (h \in \mathfrak{h}_{m,n}), \quad s_k \cdot \lambda = \lambda - \langle h_k, \lambda \rangle \alpha_k \quad (\lambda \in \mathfrak{h}_{m,n}^*),$$

where $\langle \cdot, \cdot \rangle : \mathfrak{h}_{m,n} \times \mathfrak{h}_{m,n}^* \rightarrow \mathbb{C}$ is the canonical pairing. When we consider the right action of $W_{m,n}$ on $\mathfrak{h}_{m,n}$, we use the convention $h \cdot w = w^{-1} \cdot h$ for $h \in \mathfrak{h}_{m,n}$ and $w \in W_{m,n}$.

We now fix a nonzero holomorphic function on \mathbb{C} , denoted by $[x]$, assuming that $[x]$ is odd ($[-x] = -[x]$ for any $x \in \mathbb{C}$), and satisfies the *Riemann relation*:

$$(29) \quad [x+y][x-y][u+v][u-v] = [x+u][x-u][y+v][y-v] - [x+v][x-v][y+u][y-u]$$

for any $x, y, u, v \in \mathbb{C}$. If this condition is satisfied, the set $\Omega = \{a \in \mathbb{C} \mid [a] = 0\}$ of zeros of $[x]$ forms a \mathbb{Z} -submodule of \mathbb{C} , and the function $[x]$ is quasi-periodic with respect to Ω . There are three classes of such functions; elliptic, trigonometric and rational, according to the rank of Ω :

Elliptic case:	$[x] = c e^{ax^2} \sigma(x; \Omega)$	$(\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2),$
Trigonometric case:	$[x] = c e^{ax^2} \sin(\pi x / \omega_0)$	$(\Omega = \mathbb{Z}\omega_0),$
Rational case:	$[x] = c e^{ax^2} x$	$(\Omega = \{0\}).$

Here $\sigma(x; \Omega)$ denotes the Weierstrass sigma function associated with the period lattice Ω . In the context of discrete Painlevé equations, these three cases correspond to the three types of difference equations (elliptic, trigonometric and rational). We will use this symbol $[x]$ whenever the three cases can be treated simultaneously.

Taking constants $\lambda \in \mathbb{C}$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{C}^m$ such that $[\lambda] \neq 0$ and $[\mu_i - \mu_j] \neq 0$ ($1 \leq i < j \leq m$), we define a holomorphic mapping $p_{\lambda, \mu} : \mathbb{C} \rightarrow \mathbb{P}^{m-1}(\mathbb{C})$ by

$$(30) \quad \begin{aligned} p_{\lambda, \mu}(t) &= \left(\frac{[\lambda + \mu_1 - t]}{[\lambda][\mu_1 - t]} : \dots : \frac{[\lambda + \mu_m - t]}{[\lambda][\mu_m - t]} \right) \\ &= \left([\lambda + \mu_1 - t] \prod_{k=2}^m [\mu_k - t] : \dots : [\lambda + \mu_m - t] \prod_{k=1}^{m-1} [\mu_k - t] \right) \end{aligned}$$

for any $t \in \mathbb{C}$. Thanks to the quasi-periodicity of $[x]$, this mapping induces a holomorphic mapping $\bar{p}_{\lambda, \mu} : E_\Omega = \mathbb{C}/\Omega \rightarrow \mathbb{P}^{m-1}(\mathbb{C})$. We denote by $C_{\lambda, \mu} = \overline{p_{\lambda, \mu}(\mathbb{C})} \subset \mathbb{P}^{m-1}(\mathbb{C})$ the curve obtained as the closure of the image of $p_{\lambda, \mu}$. Note that this curve passes the m coordinate origins in $\mathbb{P}^{m-1}(\mathbb{C})$; in fact we have

$$(31) \quad p_{\lambda, \mu}(\mu_1) = (1 : 0 : \dots : 0), \quad \dots, \quad p_{\lambda, \mu}(\mu_m) = (0 : \dots : 0 : 1).$$

We also remark that $C_{\lambda, \mu}$ is a smooth elliptic curve when $\text{rank } \Omega = 2$, and a singular elliptic curve with a node (resp. a cusp) when $\text{rank } \Omega = 1$ (resp. $\text{rank } \Omega = 0$) at $(1 : \dots : 1)$.

We now consider a configuration of n points on $C_{\lambda, \mu} \subset \mathbb{P}^{m-1}(\mathbb{C})$ defined as

$$(32) \quad [p_1, \dots, p_n] = [p_{\lambda, \mu}(\varepsilon_1), \dots, p_{\lambda, \mu}(\varepsilon_n)]$$

by the coordinates $\varepsilon_j \in \mathbb{C}$ ($j = 1, \dots, n$). Setting $\varepsilon_0 = \lambda + \mu_1 + \dots + \mu_m$, we assume that the parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ are generic in the sense

$$(33) \quad \begin{aligned} [\varepsilon_i - \varepsilon_j] &\neq 0, & (1 \leq i < j \leq n), \\ [\varepsilon_0 - \varepsilon_{j_1} - \dots - \varepsilon_{j_m}] &\neq 0 & (1 \leq j_1 < \dots < j_m \leq n). \end{aligned}$$

Then by the Frobenius formula

$$(34) \quad \det \left(\frac{[\lambda + x_i - y_j]}{[\lambda][x_i - y_j]} \right)_{i, j=1}^m = \frac{[\lambda + \sum_{k=1}^m (x_k - y_k)] \prod_{1 \leq i < j \leq m} [x_j - x_i][y_i - y_j]}{[\lambda] \prod_{1 \leq i, j \leq m} [x_i - y_j]}$$

for the function $[x]$, we see that the configuration $[p_1, \dots, p_n]$ defined as above is in general position, and that its u coordinates are given explicitly by

$$(35) \quad \begin{aligned} u_{i, j} = u_{i, j}(\varepsilon) &= \frac{[\alpha_0 + \varepsilon_{m, m+1}][\varepsilon_{i, m+1}]}{[\varepsilon_{m, m+1}][\alpha_0 + \varepsilon_{i, m+1}]} \frac{[\alpha_0 + \varepsilon_{i, j}][\varepsilon_{m, j}]}{[\varepsilon_{i, j}][\alpha_0 + \varepsilon_{m, j}]} \\ &(i = 1, \dots, m - 1; j = m + 2, \dots, n) \end{aligned}$$

where $\varepsilon_{i, j} = \varepsilon_i - \varepsilon_j$ for $i, j \in \{1, \dots, n\}$, and $\alpha_0 = \varepsilon_0 - \varepsilon_1 - \dots - \varepsilon_m$. Note that, by the passage to the double coset space $\mathbb{X}_{m, n}$, the dependence of the configuration on the parameters λ and $\mu = (\mu_1, \dots, \mu_m)$ has been confined in the parameter $\varepsilon_0 = \lambda + \mu_1 + \dots + \mu_m$. Observe also that these functions $u_{i, j}(\varepsilon)$ are Ω -periodic in all the variables ε_j ($j = 0, 1, \dots, n$).

Under the identification of the parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ with the canonical coordinates for $\mathfrak{h}_{m,n}$, the construction described above implies two meromorphic mappings

$$(36) \quad \begin{aligned} \varphi_{m,n} : \mathfrak{h}_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} \mathbb{C} &\dashrightarrow \mathbb{X}_{m,n}, \quad \text{and} \\ \bar{\varphi}_{m,n} : E_{m,n} = L_{m,n} \otimes_{\mathbb{Z}} (\mathbb{C}/\Omega) &\dashrightarrow \mathbb{X}_{m,n}. \end{aligned}$$

Note that, when $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathfrak{h}_{m,n}$ is generic, the corresponding configuration $\varphi_{m,n}(\varepsilon) = [p_1, \dots, p_n] \in \mathbb{X}_{m,n}$ is realized by an n -tuple of points on the elliptic curve $C_{\lambda,\mu} \subset \mathbb{P}^{m-1}(\mathbb{C})$ for any choice of λ and $\mu = (\mu_1, \dots, \mu_m)$ such that $\varepsilon_0 = \lambda + \mu_1 + \dots + \mu_m$. The meromorphic mapping $\varphi_{m,n}$ is equivariant by construction under the action of the symmetric group $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$. Also, the equivariance with respect to s_0 is clearly seen by the explicit formula (35) for the u coordinates. Hence we obtain

Theorem 4.1. — *The meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ (resp. $\bar{\varphi}_{m,n} : E_{m,n} \dashrightarrow \mathbb{X}_{m,n}$) defined as above is $W_{m,n}$ -equivariant with respect to the canonical linear action of the Weyl group $W_{m,n}$ on $\mathfrak{h}_{m,n}$ and its nonlinear birational action on $\mathbb{X}_{m,n}$ by Cremona transformations.*

This theorem means that the Ω -periodic functions (35) satisfy the equations

$$(37) \quad u_{i,j}(w(\varepsilon)) = S_{i,j}^w(u(\varepsilon)) \quad (i = 1, \dots, m-1; j = m+2, \dots, n)$$

for any $w \in W_{m,n}$, where $w(\varepsilon) = (w(\varepsilon_0), w(\varepsilon_1), \dots, w(\varepsilon_n))$. Namely, (35) give a solution of the system of functional equations (10) for the u variables.

As we discussed in the previous section, in the relative situation $\mathbb{X}_{m,n+1} \rightarrow \mathbb{X}_{m,n}$, the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ is expressed as

$$(38) \quad w(u_{i,j}) = S_{i,j}^w(u), \quad w(z_i) = R_i^w(u; z).$$

By substituting $u_{ij} = u_{ij}(\varepsilon)$ as in (35), we obtain a realization of $W_{m,n}$ as a group of automorphisms of the field $\mathcal{M}(E_{m,n})(z)$ of rational functions in z with coefficients in the field of meromorphic functions on $E_{m,n}$:

$$(39) \quad w(z_i) = R_i^w(\varepsilon; z) \quad (i = 1, \dots, m-1),$$

where we have used the notation $R_i^w(\varepsilon; z)$ again instead of $R_i^w(u(\varepsilon); z)$. We will refer to this system (39) of functional equations for the z variables as the *elliptic Cremona*

system of type (m, n) . The action of the simple reflection s_k ($k = 0, 1, \dots, n-1$) on the z variables is now given as follows:

$$(40) \quad \begin{aligned} k = 0 : & & s_0(z_i) &= \frac{1}{z_i}, \\ k = 1, \dots, m-2 : & & s_k(z_i) &= z_{s_k(i)}, \\ k = m-1 : & & s_{m-1}(z_i) &= \begin{cases} \frac{z_i}{z_{m-1}} & (i = 1, \dots, m-2), \\ \frac{1}{z_{m-1}} & (i = m), \end{cases} \\ k = m : & & s_m(z_i) &= 1 - z_i, \\ k = m+1 : & & s_{m+1}(z_i) &= \frac{z_i}{u_{i,m+2}(\varepsilon)} \\ k = m+2, \dots, n-1 : & & s_k(z_i) &= z_i. \end{aligned}$$

Note that the dependence on the ε variables enters in the action of s_{m+1} . Also, from Theorem 4.1 for the case of $\mathbb{X}_{m,n+1}$ with $\varepsilon_{n+1} = t$, we see that the functions

$$(41) \quad z_i(\varepsilon; t) = \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]}{[\varepsilon_{m,m+1}][\alpha_0 + \varepsilon_{i,m+1}]} \frac{[\alpha_0 + \varepsilon_i - t][\varepsilon_m - t]}{[\varepsilon_i - t][\alpha_0 + \varepsilon_m - t]} \quad (i = 1, \dots, m-1)$$

satisfy the functional equations

$$(42) \quad z_i(w(\varepsilon); t) = R_i^w(\varepsilon; z(\varepsilon; t)) \quad (i = 1, \dots, m-1),$$

for any $w \in W_{m,n}$. Namely, (41) provides a one-parameter family of solutions to the elliptic Cremona system (39) of type (m, n) . This solution will be called the *canonical solution* of the elliptic Cremona system, which corresponds to the *vertical solutions* in the context of differential Painlevé equations.

5. Tau functions for the elliptic Cremona system

In the case of the elliptic Cremona systems as we introduced above, the homogeneous polynomials $F_i^w(x)$ and $G_i^w(x)$ in (26) can be determined without ambiguity by means of the action of the Weyl group on the τ -functions. In this section, we introduce a framework of τ -functions for the elliptic Cremona systems, and show that the Weyl group action is translated into bilinear equations of Hirota-Miwa type for τ -functions on the lattice.

By using the natural linear action of $W_{m,n}$ on $\mathfrak{h}_{m,n}^*$, we define the set of *real roots* by $\Delta_{m,n}^{\text{Re}} = W_{m,n}\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$, and denote by $\mathbb{K} = \mathbb{C}([\alpha]; \alpha \in \Delta_{m,n}^{\text{Re}})$ the field of meromorphic functions on $\mathfrak{h}_{m,n}$ generated by $[\alpha]$ for all real roots $\alpha \in \Delta_{m,n}^{\text{Re}}$.

In order to formulate τ -functions for the elliptic Cremona system of type (m, n) , we use two kinds of variables (indeterminates) f_1, \dots, f_m , which will be identified with a

system of homogeneous coordinates for $\mathbb{P}^{m-1}(\mathbb{C})$, and τ_1, \dots, τ_n corresponding to the n points p_1, \dots, p_n in the configuration $[p_1, \dots, p_n]$. We denote by

$$(43) \quad \mathcal{L} = \mathbb{K}(f_1, \dots, f_m; \tau_1, \dots, \tau_n)$$

the field of rational functions in the f variables and the τ variables with coefficients in \mathbb{K} . On this field \mathcal{L} , we define the automorphisms s_0, s_1, \dots, s_{n-1} as follows. These elements act on the coefficient field \mathbb{K} through the canonical $W_{m,n}$ -action on the real roots: For each $k = 0, 1, \dots, n - 1$,

$$(44) \quad s_k([\alpha]) = [s_k(\alpha)] = [\alpha - \langle h_k, \alpha \rangle \alpha_k] \quad (\alpha \in \Delta_{m,n}^{\text{Re}}).$$

The action of s_k on τ_j ($j = 1, \dots, n$) is defined by

$$(45) \quad \begin{aligned} s_0(\tau_j) &= \begin{cases} \tau_j f_j & (j = 1, \dots, m), \\ \tau_j & (j = m + 1, \dots, n), \end{cases} \\ s_k(\tau_j) &= \tau_{s_k(j)} \quad (k = 1, \dots, n - 1; j = 1, \dots, n). \end{aligned}$$

The action of s_k on f_i ($i = 1, \dots, m$) is defined by

$$(46) \quad \begin{aligned} s_0(f_i) &= \frac{1}{f_i} \\ s_k(f_i) &= f_{s_k(i)} \quad (k = 1, \dots, m - 1), \\ s_k(f_i) &= f_i \quad (k = m + 1, \dots, n), \end{aligned}$$

and

$$(47) \quad \begin{aligned} s_m(f_i) &= \frac{\tau_m}{\tau_{m+1}} \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]f_i - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}]f_m}{[\alpha_0][\varepsilon_{i,m}]} \\ & \quad (i = 1, \dots, m - 1), \\ s_m(f_m) &= \frac{\tau_m}{\tau_{m+1}} f_m. \end{aligned}$$

Theorem 5.1. — *The automorphisms s_0, s_1, \dots, s_{n-1} of \mathcal{L} defined as above satisfy the fundamental relations for the simple reflections of the Weyl group $W_{m,n}$.*

This theorem can be proved directly by using the Riemann relation for $[x]$.

We remark that the action of the Weyl group $W_{m,n}$ preserves the subalgebra

$$(48) \quad \mathcal{R} = S[\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}] \subset \mathcal{L},$$

where

$$(49) \quad S = \bigoplus_{d \in \mathbb{Z}} S_d, \quad S_d = f_m^d \mathbb{K}(f_1/f_m, \dots, f_{m-1}/f_m).$$

Consider the following elements in S_0 :

$$(50) \quad z_i = \frac{a_i}{a_m} \frac{f_i}{f_m}, \quad a_i = \frac{[\varepsilon_{i,m+1}]}{[\alpha_0 + \varepsilon_{i,m+1}]} \quad (i = 1, \dots, m),$$

so that $S_0 = \mathbb{K}(z_1, \dots, z_{m-1})$ and that $(z_1 : \dots : z_{m-1} : 1) = (a_1 f_1 : \dots : a_m f_m)$. Then the action of s_k ($k = 0, 1, \dots, n - 1$) on these z variables coincides with the

one given earlier in (40). In this sense the f variables are thought of as *normalized* homogeneous coordinates. We also remark that the f variables and the τ variables for the canonical solution are given by

$$(51) \quad f_i^C = \frac{[\alpha_0 + \varepsilon_i - t]}{[\varepsilon_i - t]} \quad (i = 1, \dots, m), \quad \tau_j^C = [\varepsilon_j - t] \quad (j = 1, \dots, n).$$

(We have used the superscript C to indicate that they are “canonical”, and also associated with the elliptic curve $C_{\lambda, \mu}$.)

It is also convenient to introduce another τ variable τ_0 and define x_i ($i = 1, \dots, m$) by the formula

$$(52) \quad f_i = \frac{\tau_0 x_i}{\tau_1 \cdots \tau_m} \quad (i = 1, \dots, m).$$

To be more precise, consider the field of rational functions

$$(53) \quad \tilde{\mathcal{L}} = \mathbb{K}(x_1, \dots, x_m; \tau_0, \tau_1, \dots, \tau_m)$$

and identify \mathcal{L} with its subfield

$$(54) \quad \mathcal{L} = \mathbb{K}(\tau_0 x_1, \dots, \tau_0 x_m; \tau_1, \dots, \tau_m)$$

by the formula (52). The action of the symmetric group $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$ can be extended to $\tilde{\mathcal{L}}$ by setting

$$(55) \quad s_k(x_i) = x_{s_k(i)} \quad (k = 1, \dots, m-1), \quad s_k(x_i) = x_i \quad (k = m+1, \dots, n-1)$$

for $i = 1, \dots, m$, and

$$(56) \quad s_m(x_i) = \frac{[\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]x_i - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}]x_m}{[\alpha_0][\varepsilon_{i,m}]}$$

for $i = 1, \dots, m-1$ and $s_m(x_m) = x_m$. The action of \mathfrak{S}_n on the τ variables are defined as

$$(57) \quad s_k(\tau_0) = \tau_0, \quad s_k(\tau_j) = \tau_{s_k(j)} \quad (j = 1, \dots, n)$$

for any $k = 1, \dots, n-1$. Note that the action of $s_0 \in W_{m,n}$ is defined only on the subfield $\mathcal{L} \subset \tilde{\mathcal{L}}$. The products $\tau_0 x_i \in \mathcal{L}$ are transformed by s_0 as follows:

$$(58) \quad s_0(\tau_0 x_i) = \frac{\tau_0^{m-1} x_1 \cdots \hat{x}_i \cdots x_m}{(\tau_1 \cdots \tau_m)^{m-2}} \quad (i = 1, \dots, m).$$

We regard $x = (x_1, \dots, x_m)$ as a normalized homogeneous coordinate system such that $(z_1 : \dots : z_{m-1} : 1) = (a_1 x_1 : \dots : a_m x_m)$. Note that the canonical solution is given by

$$(59) \quad \tau_0 x_i = x_i^C(t) = [\alpha_0 + \varepsilon_i - t] \prod_{1 \leq k \leq m; k \neq i} [\varepsilon_k - t] \quad (i = 1, \dots, m).$$

Accordingly, we define the x coordinates of the reference points p_1, \dots, p_n by

$$(60) \quad x(p_j) = (x_1^C(\varepsilon_j) : \dots : x_m^C(\varepsilon_j)), \quad x_i^C(\varepsilon_j) = [\alpha_0 + \varepsilon_{i,j}] \prod_{1 \leq k \leq m; k \neq i} [\varepsilon_{k,j}],$$

for $j = 1, \dots, n$. For each element

$$(61) \quad \Lambda = de_0 - \nu_1 e_1 - \dots - \nu_n e_n \in L_{m,n} \quad (d, \nu_1, \dots, \nu_n \in \mathbb{Z}),$$

we define the *formal exponential* τ^Λ by

$$(62) \quad \tau^\Lambda = \tau_0^d \tau_1^{-\nu_1} \dots \tau_n^{-\nu_n}.$$

Since $f_i = \tau^{h_0} x_i$ ($i = 1, \dots, m$), the algebra \mathcal{R} can be expressed alternatively as

$$(63) \quad \mathcal{R} = \bigoplus_{\Lambda \in L_{m,n}} \mathbb{K}(x)_{\deg(\Lambda)} \tau^\Lambda, \quad \mathbb{K}(x)_d = x_m^d \mathbb{K}(x_1/x_m, \dots, x_{m-1}/x_m),$$

where $\deg(\Lambda)$ stands for the coefficient d of e_0 in (61). In the x and τ variables, the action of the Weyl group on \mathcal{R} is described as follows:

$$(64) \quad s_0(\tau^\Lambda \varphi(x)) = \tau^{s_0 \cdot \Lambda} x_1^{d-\nu_1} \dots x_m^{d-\nu_m} s_0 \varphi(x^{-1})$$

and

$$(65) \quad s_k(\tau^\Lambda \varphi(x)) = \tau^{s_k \cdot \Lambda} s_k \varphi(s_k(x)) \quad (k = 1, \dots, n-1)$$

for any $\varphi(x) = \varphi(x_1, \dots, x_m) \in \mathbb{K}(x)_d$, where ${}^w\varphi(x)$ stands for the polynomial obtained from $\varphi(x)$ by applying w to its coefficients. The expression $s_k(x) = (s_k(x_1), \dots, s_k(x_m))$ ($k = 1, \dots, n-1$) should be understood in the sense of the action of \mathfrak{S}_n on $\tilde{\mathcal{L}}$ described in (55) and (56).

We are now ready to introduce the lattice τ -functions for the elliptic Cremona system of type (m, n) . We consider the $W_{m,n}$ -orbit of the lattice point e_n in $L_{m,n}$:

$$(66) \quad M_{m,n} = W_{m,n} e_n = W_{m,n} \{e_1, \dots, e_n\} \subset L_{m,n}.$$

Notice that the stabilizer of e_n is $W_{m,n-1}$, and that τ_n is $W_{m,n-1}$ -invariant. Hence, for each $\Lambda \in M_{m,n}$ we can define an element $\tau(\Lambda) = w(\tau_n) \in \mathcal{R}$ by taking any element $w \in W_{m,n}$ such that $w.e_n = \Lambda$. This family of $(\tau(\Lambda))_{\Lambda \in M_{m,n}}$ of elements of \mathcal{R} , indexed by the lattice points $M_{m,n} \subset L_{m,n}$ will be called the *lattice τ -functions* for the elliptic Cremona system of type (m, n) . These τ -functions are determined by the condition $\tau(e_j) = \tau_j$ ($j = 1, \dots, n$) together with the compatibility condition

$$(67) \quad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_{m,n}; w \in W_{m,n})$$

with respect to the action of $W_{m,n}$ on $M_{m,n}$. By using

$$(68) \quad f_i = \frac{s_0(\tau_i)}{\tau_i} = \frac{\tau(e_i + h_0)}{\tau(e_i)} \quad (i = 1, \dots, m),$$

from the action of s_m (47) on the f variables we obtain the following bilinear relations for the lattice τ -functions:

$$(69) \quad \begin{aligned} & [\alpha_0][\varepsilon_{i,m}]\tau(h_0 + e_i + e_m - e_{m+1})\tau(e_{m+1}) \\ &= [\alpha_0 + \varepsilon_{m,m+1}][\varepsilon_{i,m+1}]\tau(h_0 + e_i)\tau(e_m) \\ &\quad - [\alpha_0 + \varepsilon_{i,m+1}][\varepsilon_{m,m+1}]\tau(h_0 + e_m)\tau(e_i). \end{aligned}$$

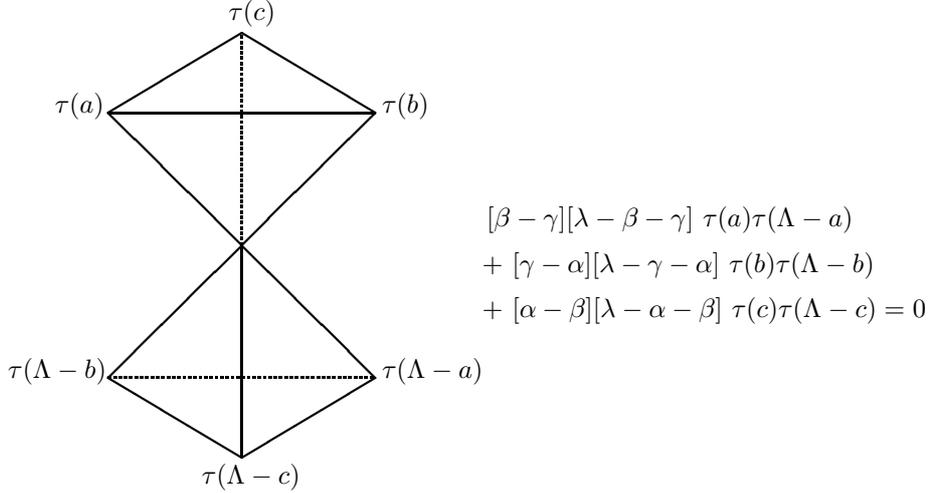


FIGURE 1. Bilinear equations for the lattice τ -functions

From this we obtain

Theorem 5.2. — *The lattice τ -functions $(\tau(\Lambda))_{\Lambda \in M_{m,n}}$ defined as above satisfy the following bilinear equations of Hirota-Miwa type: For any choice of mutually distinct indices $i, j, k, l_1, \dots, l_{m-2} \in \{1, \dots, n\}$,*

$$(70) \quad \begin{aligned} & [\varepsilon_{j,k}][\lambda - \varepsilon_j - \varepsilon_k]\tau(e_i)\tau(\Lambda - e_i) + [\varepsilon_{k,i}][\lambda - \varepsilon_k - \varepsilon_i]\tau(e_j)\tau(\Lambda - e_j) \\ & + [\varepsilon_{i,j}][\lambda - \varepsilon_i - \varepsilon_j]\tau(e_k)\tau(\Lambda - e_k) = 0, \end{aligned}$$

where $\Lambda = e_0 - e_{l_1} - \dots - e_{l_{m-2}}$ and $\lambda = (\Lambda | \cdot) = \varepsilon_0 - \varepsilon_{l_1} - \dots - \varepsilon_{l_{m-2}}$.

The lattice τ -functions for the canonical solution are given simply by

$$(71) \quad \tau^C(\Lambda) = [\lambda - t] \quad \text{with} \quad \lambda = (\Lambda | \cdot) \in \mathfrak{h}_{m,n}^*$$

for any $\Lambda \in M_{m,n}$. In this case the bilinear equations in Theorem 5.2 recover the Riemann relation for the function $[x]$.

We also remark that these bilinear equations of Hirota-Miwa type characterize the lattice τ -functions for the elliptic Cremona system. To be more precise, suppose that the natural action of $W_{m,n}$ on \mathbb{K} is extended to a field \mathcal{K} containing \mathbb{K} (as that of a group of automorphisms). If a family of nonzero elements $(\tau(\Lambda))_{\Lambda \in M_{m,n}}$ of \mathcal{K} , indexed by $M_{m,n}$, satisfies the two conditions (67) and (70), then the elements

$$(72) \quad f_i = \frac{\tau(e_i + h_0)}{\tau(e_i)} \quad (i = 1, \dots, m), \quad \tau_j = \tau(e_j) \quad (j = 1, \dots, n)$$

of \mathcal{K} recover the relations (45)–(47) for the action of $W_{m,n}$ on \mathcal{L} .

Recall that the algebra \mathcal{R} is expressed as

$$(73) \quad \mathcal{R} = \bigoplus_{\Lambda \in L_{m,n}} \mathbb{K}(x)_{\deg(\Lambda)} \tau^\Lambda.$$

As in Section 3, for each $\Lambda \in L_{m,n}$ of the form (61) we define the \mathbb{K} -vector subspace $L(\Lambda) \subset \mathbb{K}[x]_d$ by

$$(74) \quad L(\Lambda) = \{f(x) \in \mathbb{K}[x]_d \mid \text{ord}_{p_j} f(x) \geq \nu_j \quad (j = 1, \dots, n)\},$$

with the reference points p_j specified by the x coordinates as in (60). Then one can show that the subalgebra

$$(75) \quad \mathcal{S} = \bigoplus_{\Lambda \in L_{m,n}} L(\Lambda) \tau^\Lambda \subset \mathcal{R}$$

is preserved under the action of $W_{m,n}$. In fact each $w \in W_{m,n}$ induces a \mathbb{C} -isomorphism

$$(76) \quad w : L(\Lambda) \tau^\Lambda \xrightarrow{\sim} L(w.\Lambda) \tau^{w.\Lambda}$$

for any $\Lambda \in L_{m,n}$. (This fact can be established by examining the cases of simple reflections s_0, s_1, \dots, s_{n-1} .) Since $\tau_j \in L(e_j) \tau^{e_j}$ ($j = 1, \dots, n$), the lattice τ -function $\tau(\Lambda)$ for $\Lambda \in M_{m,n}$ can be expressed in the form

$$(77) \quad \tau(\Lambda) = \tau^\Lambda \phi(\Lambda; x), \quad \phi(\Lambda; x) \in L(\Lambda)$$

in terms of the original τ variables $\tau_0, \tau_1, \dots, \tau_n$ and the homogeneous coordinates x_1, \dots, x_m . We remark that this family of homogeneous polynomials $\phi(\Lambda; x)$ ($\Lambda \in M_{m,n}$) is determined uniquely by the following properties:

$$(78) \quad \begin{aligned} \phi(e_j; x) &= 1 & (j = 1, \dots, n), \\ \phi(s_0.\Lambda; x) &= x_1^{d-\nu_1} \dots x_m^{d-\nu_m} s_0 \phi(\Lambda; x^{-1}), \\ \phi(s_k.\Lambda; x) &= s_k \phi(\Lambda; s_k(x)) & (k = 1, \dots, n-1), \end{aligned}$$

where $\Lambda = de_0 - \nu_1 e_1 - \dots - \nu_n e_n$. We sometimes refer to this family of polynomials $\phi(\Lambda; x)$ as the τ -cocycle. They correspond to what are called ϕ -factors in [14].

Since $f_i = \tau^{h_0} x_i$ ($i = 1, \dots, m$), the formula (77) is rewritten also into

$$(79) \quad \tau(\Lambda) = \tau^{\Lambda - \deg(\Lambda)e_0} \phi(\Lambda; \tau_0 x) = \tau^{\Lambda - \deg(\Lambda)h_0} \phi(\Lambda; f)$$

in terms of the variables $\tau_0 x_i$ or f_i . For Λ as in (61), we have

$$(80) \quad \tau(\Lambda) = \frac{\phi(\Lambda; \tau_0 x)}{\tau_1^{\nu_1} \dots \tau_n^{\nu_n}} = \frac{\tau_1^{d-\nu_1} \dots \tau_m^{d-\nu_m}}{\tau_{m+1}^{\nu_{m+1}} \dots \tau_n^{\nu_n}} \phi(\Lambda; f).$$

Note here that $\phi(\Lambda; x)$ is a homogeneous polynomial of degree d having a zero of multiplicity $\geq \nu_j$ at each p_j ($j = 1, \dots, n$). In the case of the canonical solution, the above formula gives rise to

$$(81) \quad \phi(\Lambda; x^C(t)) = [\lambda - t][\varepsilon_1 - t]^{\nu_1} \dots [\varepsilon_n - t]^{\nu_n}, \quad \lambda = (\Lambda \mid \cdot).$$

In this form, it is clearly seen that $\phi(\Lambda; x^C(t))$ has zeros at p_1, \dots, p_n with expected multiplicities.

Note that

$$(82) \quad f_i = \frac{\tau(e_i + h_0)}{\tau(e_i)} = \frac{\tau(s_0 \cdot e_i)}{\tau(e_i)} \quad (i = 1, \dots, m)$$

implies

$$(83) \quad w(f_i) = \frac{\tau(ws_0 \cdot e_i)}{\tau(w \cdot e_i)} = \frac{\tau^{ws_0 \cdot e_i} \phi(ws_0 \cdot e_i; x)}{\tau^{w \cdot e_i} \phi(w \cdot e_i; x)} = \tau^{w \cdot h_0} \frac{\phi(ws_0 \cdot e_i; x)}{\phi(w \cdot e_i; x)}$$

for any $w \in W_{m,n}$. Hence, the action of $w \in W_{m,n}$ on the variables f_1, \dots, f_m and τ_1, \dots, τ_n can be written in a closed form as follows:

$$(84) \quad \begin{aligned} w(f_i) &= \tau^{w \cdot h_0 - \deg(w \cdot h_0)h_0} \frac{\phi(ws_0 \cdot e_i; f)}{\phi(w \cdot e_i; f)} & (i = 1, \dots, m), \\ w(\tau_j) &= \tau^{w \cdot e_j - \deg(w \cdot e_j)h_0} \phi(w \cdot e_j; f) & (j = 1, \dots, n). \end{aligned}$$

Also, from

$$(85) \quad (z_1 : \dots : z_{m-1} : 1) = (a_1 f_1 : \dots : a_m f_m), \quad a_i = \frac{[\varepsilon_{i,m+1}]}{[\alpha_0 + \varepsilon_{i,m+1}]},$$

we obtain

$$(86) \quad (w(z_1) : \dots : w(z_{m-1}) : 1) = \left(w(a_1) \frac{\phi(ws_0 \cdot e_1; x)}{\phi(w \cdot e_1; x)} : \dots : w(a_m) \frac{\phi(ws_0 \cdot e_m; x)}{\phi(w \cdot e_m; x)} \right)$$

for any $w \in W_{m,n}$. This formula provides a refinement of Theorem 3.1 for the elliptic Cremona system of type (m, n) .

6. Elliptic difference Painlevé equation

In the rest of this article, we confine ourselves to the elliptic Cremona system of type (3, 9) which has the affine Weyl group symmetry of type $E_8^{(1)}$. The discrete dynamical system arising from the lattice part of $W(E_8^{(1)})$ is the elliptic difference Painlevé equation.

The parameter space for the elliptic Cremona system of type (3, 9) is the 10-dimensional vector space

$$(87) \quad \mathfrak{h}_{3,9} = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_9$$

with the nondegenerate symmetric bilinear form $(\cdot | \cdot) : \mathfrak{h}_{3,9} \times \mathfrak{h}_{3,9} \rightarrow \mathbb{C}$ such that

$$(88) \quad \begin{aligned} (e_0 | e_0) &= -1, & (e_j | e_j) &= 1 \quad (j = 1, \dots, 9), \\ (e_i | e_j) &= 0 & (i, j \in \{1, \dots, 9\}; i \neq j), \end{aligned}$$

which we regard as the complexification of the lattice $L_{3,9} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_9$. We take the linear functions $\varepsilon_j = (e_j | \cdot)$ ($j = 0, 1, \dots, 9$) as the canonical coordinates

For any real root $\alpha \in \Delta_{3,9}^{\text{Re}}$, the translation T_α can be expressed as the composition of two reflections $T_\alpha = s_{\delta-\alpha}s_\alpha$. From this fact it follows that $T_\alpha \in W_{3,9}$ for any $\alpha \in Q_{3,9}$. Furthermore, it is known that the affine Weyl group $W_{3,9}$ is decomposed into the semidirect product of the root lattice

$$(98) \quad Q_{3,8} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_7 \subset Q_{3,9}, \quad Q_{3,8} \xrightarrow{\sim} Q(E_8),$$

and the finite Weyl group $W_{3,8} = \langle s_0, s_1, \dots, s_7 \rangle$ of type E_8 acting on $Q_{3,8}$. In fact the correspondence $(\alpha, w) \mapsto T_\alpha w$ induces the isomorphism $Q_{3,8} \rtimes W_{3,8} \xrightarrow{\sim} W_{3,9}$ (see [6]). We remark that the action of T_α on $\mathfrak{h}_{3,9}$ is expressed in the form

$$(99) \quad T_\alpha(\Lambda) = \Lambda + (c|\Lambda)h - \left(\frac{1}{2}(h|h)(c|\Lambda) + (h|\Lambda)\right)c \quad (\Lambda \in \mathfrak{h}_{3,9})$$

by using the element $h \in \mathfrak{h}_{3,9}$ such that $\alpha = (h|\cdot)$. When $\alpha = \varepsilon_{ij}$ or ε_{ijk} , we write the translation T_α simply as T_{ij} or T_{ijk} , respectively. In this case of $(m, n) = (3, 9)$, the $W_{3,9}$ -orbit $M_{3,9} = W_{3,9}\{e_1, \dots, e_9\} \subset L_{3,9}$ can be characterized as follows:

$$(100) \quad M_{3,9} = \{\Lambda \in L_{3,9} \mid (c|\Lambda) = -1, \quad (\Lambda|\Lambda) = 1\}.$$

Also, the correspondence $\alpha \mapsto T_\alpha.e_9$ induces a bijection $Q_{3,8} \xrightarrow{\sim} M_{3,9}$. Any element $\Lambda \in M_{3,9}$ can be expressed in the form

$$(101) \quad \Lambda = de_0 - \nu_1e_1 - \cdots - \nu_9e_9, \quad d \geq 0, \quad \nu_j \geq -1 \quad (j = 1, \dots, 9).$$

In fact, except for the cases $\Lambda = e_k$ ($k = 1, \dots, 9$), the coefficients ν_j are all nonnegative.

As in the previous section, we consider the \mathbb{K} -algebra

$$(102) \quad \mathcal{R} = S[\tau_1^{\pm 1}, \dots, \tau_9^{\pm 1}], \quad S = \bigoplus_{d \in \mathbb{Z}} f_3^d \mathbb{K}(f_1/f_3, f_2/f_3)$$

of f variables and τ variables. The standard Cremona transformation s_0 with respect to (p_1, p_2, p_3) acts on the f variables and τ variables as

$$(103) \quad s_0(f_i) = \frac{1}{f_i} \quad (i = 1, 2, 3), \quad s_0(\tau_j) = \begin{cases} \tau_j f_j & (j = 1, 2, 3), \\ \tau_j & (j = 4, 5, 6, 7, 8, 9). \end{cases}$$

Among the 8 adjacent transpositions s_k ($k = 1, \dots, 8$), s_3 acts nontrivially on the f variables:

$$(104) \quad \begin{aligned} s_3(f_1) &= \frac{\tau_3 [\varepsilon_{124}][\varepsilon_{14}]f_1 - [\varepsilon_{234}][\varepsilon_{34}]f_3}{\tau_4 [\varepsilon_{123}][\varepsilon_{13}]}, \\ s_3(f_2) &= \frac{\tau_3 [\varepsilon_{124}][\varepsilon_{24}]f_1 - [\varepsilon_{134}][\varepsilon_{34}]f_3}{\tau_4 [\varepsilon_{123}][\varepsilon_{23}]}, \\ s_3(f_3) &= \frac{\tau_3}{\tau_4} f_3. \end{aligned}$$

(Note that $\alpha_0 = \varepsilon_{123}$.) Recall that the lattice τ -functions $(\tau(\Lambda))_{\Lambda \in M_{3,9}}$ are defined as a family of dependent variables indexed by the $W_{3,9}$ -orbit

$$(105) \quad M_{3,9} = W_{3,9}\{e_1, \dots, e_9\} = \{\Lambda \in L_{3,9} \mid (c|\Lambda) = -1, \quad (\Lambda|\Lambda) = 1\}.$$

These τ -functions $\tau(\Lambda)$ are characterized by the consistency condition

$$(106) \quad \tau(e_j) = \tau_j \quad (j = 1, \dots, 9), \quad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_{3,9}; w \in W_{3,9})$$

with respect to the action of $W_{3,9}$ on $M_{3,9}$, and the bilinear equations

$$(107) \quad \begin{aligned} & [\varepsilon_{jk}][\varepsilon_{jkl}]\tau(e_i)\tau(e_0 - e_i - e_l) + [\varepsilon_{ki}][\varepsilon_{kil}]\tau(e_j)\tau(e_0 - e_j - e_l) \\ & + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau(e_k)\tau(e_0 - e_k - e_l) = 0 \end{aligned}$$

for any mutually distinct $i, j, k, l \in \{1, \dots, 9\}$. These bilinear equations guarantee the equivalence of our formulation of the elliptic difference Painlevé equation of type $E_8^{(1)}$ to that of Ohta-Ramani-Grammaticos [16] on the E_8 lattice.

We already know that the lattice τ -functions can be expressed in the form

$$(108) \quad \tau(\Lambda) = \tau^\Lambda \phi(\Lambda; x) = \tau^{\Lambda - \deg(\Lambda)h_0} \phi(\Lambda; f), \quad \phi(\Lambda; x) \in L(\Lambda),$$

where the x coordinates are defined by $f_i = \tau^{h_0} x_i$ ($i = 1, 2, 3$). When $\Lambda = de_0 - \nu_1 e_1 - \dots - \nu_9 e_9$, $\phi(\Lambda; x)$ is a homogeneous polynomial of degree $d = \deg(\Lambda)$, and for each $j = 1, \dots, 9$ it has a zero of multiplicity $\geq \nu_j$ at p_j ; the x coordinates of p_j are now given by

$$(109) \quad x(p_j) = ([\varepsilon_{23j}][\varepsilon_{2j}][\varepsilon_{3j}] : [\varepsilon_{13j}][\varepsilon_{1j}][\varepsilon_{3j}] : [\varepsilon_{12j}][\varepsilon_{1j}][\varepsilon_{2j}]) \quad (j = 1, \dots, 9).$$

By the homogeneous polynomials $\phi(\Lambda; x)$, the action of $W_{3,9}$ on the algebra \mathcal{R} is described as

$$(110) \quad \begin{aligned} w(f_i) &= \tau^{w.h_0 - \deg(w.h_0)h_0} \frac{\phi(ws_0.e_i; f)}{\phi(w.e_i; f)} & (i = 1, 2, 3), \\ w(\tau_j) &= \tau^{w.e_j - \deg(w.e_j)h_0} \phi(w.e_j; f) & (j = 1, \dots, 9). \end{aligned}$$

Recall that the z variables $z = (z_1, z_2)$ are recovered from the f variables by the formula

$$(111) \quad (z_1 : z_2 : 1) = \left(\frac{[\varepsilon_{14}]}{[\varepsilon_{234}]} f_1 : \frac{[\varepsilon_{24}]}{[\varepsilon_{134}]} f_2 : \frac{[\varepsilon_{34}]}{[\varepsilon_{124}]} f_3 \right),$$

and hence

$$(112) \quad (w(z_1) : w(z_2) : 1) = \left(\frac{[w(\varepsilon_{14})]}{[w(\varepsilon_{234})]} w(f_1) : \frac{[w(\varepsilon_{24})]}{[w(\varepsilon_{134})]} w(f_2) : \frac{[w(\varepsilon_{34})]}{[w(\varepsilon_{124})]} w(f_3) \right)$$

for any $w \in W_{3,9}$. This implies

$$(113) \quad \begin{aligned} w(z_1) &= \frac{[w(\varepsilon_{14})][w(\varepsilon_{124})]}{[w(\varepsilon_{234})][w(\varepsilon_{34})]} \frac{G_1^w(z)F_3^w(z)}{F_1^w(z)G_3^w(z)}, \\ w(z_2) &= \frac{[w(\varepsilon_{24})][w(\varepsilon_{124})]}{[w(\varepsilon_{134})][w(\varepsilon_{34})]} \frac{G_2^w(z)F_3^w(z)}{F_2^w(z)G_3^w(z)}, \end{aligned}$$

where $F_i^w(z)$ and $G_i^w(z)$ are the inhomogenizations of $\phi(w.e_i; f)$ and $\phi(ws_0.e_i; f)$, respectively, by (111). The formulas (110) (resp. (113)) for the translations $w = T_\alpha$

($\alpha \in Q_{3,9}$) give the elliptic difference Painlevé equation of type (3, 9) in the homogeneous form of f and τ variables (resp. in the inhomogeneous form in z variables). In the following, we will mainly work with the homogeneous form (110).

Note that, for each $\Lambda \in M_{3,9}$, the homogeneous polynomial $\phi(\Lambda; x)$ is characterized by its degree and the multiplicity of zeros at p_1, \dots, p_9 . Thanks to this fact, we are able to express $\phi(\Lambda; x)$ by means of an interpolation determinant. For each $d = 0, 1, 2, \dots$, we denote by

$$(114) \quad \mathbf{m}_d(x) = (x^\mu)_{|\mu|=d} = (x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3})_{\mu_1+\mu_2+\mu_3=d}$$

the column vector of monomials of degree d in $x = (x_1, x_2, x_3)$. Note that the number of such monomials is given by

$$(115) \quad \dim_{\mathbb{K}} \mathbb{K}[x]_d = \binom{d+2}{2} = \frac{1}{2}(d+1)(d+2).$$

For each $k = 0, 1, \dots, d$ we denote by

$$(116) \quad \mathbf{m}_d^{(k)}(x) = (\partial_x^\kappa(x^\mu))_{|\mu|=d, |\kappa|=k} = (\partial_{x_1}^{\kappa_1} \partial_{x_2}^{\kappa_2} \partial_{x_3}^{\kappa_3} (x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3}))_{|\mu|=d, |\kappa|=k}$$

the $\binom{d+2}{2} \times \binom{k+2}{2}$ matrix defined by the partial derivatives $\mathbf{m}_d(x)$ of order k . (For $k < 0$, we consider $\mathbf{m}_d^{(k)}(x)$ as an empty matrix.) Given an element

$$(117) \quad \Lambda = de_0 - \nu_1 e_1 - \dots - \nu_9 e_9 \in M_{3,9},$$

we consider the homogeneous polynomial

$$(118) \quad F(\Lambda; x) = \det \left(\mathbf{m}_d^{(\nu_1-1)}(p_1), \dots, \mathbf{m}_d^{(\nu_9-1)}(p_9), \mathbf{m}_d(x) \right)$$

of degree d in $x = (x_1, x_2, x_3)$, where $\mathbf{m}_d^{(k)}(p_j)$ stands for the matrix obtained from $\mathbf{m}_d^{(k)}(x)$ by the substitution $x = x(p_j)$ as in (109). Note here that

$$(119) \quad \binom{d+2}{2} - \sum_{j=1}^9 \binom{\nu_j+1}{2} - 1 = -\frac{1}{2}((c|\Lambda) + (\Lambda|\Lambda)) = 0$$

for any $\Lambda \in M_{3,9}$; this means that the number of column vectors in (118) is equal to $\binom{d+2}{2}$. Also, from $\dim_{\mathbb{K}} L(\Lambda) = 1$ it follows that $F(\Lambda; x)$ is a nonzero polynomial. Combining this fact with the normalization condition (71), we obtain the following theorem.

Theorem 6.1. — *For each $\Lambda \in M_{3,9}$, define the homogeneous polynomial $F(\Lambda; x)$ in $x = (x_1, x_2, x_3)$ by the determinant (118). Then, the specialization of $F(\Lambda; x)$ to the canonical solution $x^C(t)$ is expressed in the form*

$$(120) \quad F(\Lambda; x^C(t)) = C_\Lambda [\lambda - t] \prod_{j=1}^9 [\varepsilon_j - t]^{\nu_j}, \quad \lambda = (\Lambda | \cdot) \in \mathfrak{h}_{3,9}^*,$$

with a nonzero constant $C_\Lambda \in \mathbb{K}$. With this normalization constant C_Λ , the homogeneous polynomial $\phi(\Lambda; x) \in L(\Lambda)$ is expressed as the determinant

$$(121) \quad \phi(\Lambda; x) = C_\Lambda^{-1} F(\Lambda; x).$$

Let us consider the action of translations $w = T_\alpha$ ($\alpha \in \Delta_{3,9}^{\text{Re}}$) on \mathcal{R} :

$$(122) \quad \begin{aligned} T_\alpha(f_i) &= \tau^{T_\alpha \cdot h_0 - \deg(T_\alpha \cdot h_0)h_0} \frac{\phi(T_\alpha \cdot (e_i + h_0); f)}{\phi(T_\alpha \cdot e_i; f)} & (i = 1, 2, 3), \\ T_\alpha(\tau_j) &= \tau^{T_\alpha \cdot e_j - \deg(T_\alpha \cdot e_j)h_0} \phi(T_\alpha \cdot e_j; f) & (j = 1, \dots, 9). \end{aligned}$$

These formulas are the elliptic difference Painlevé equations of type (3, 9) for the f and τ variables. The polynomials $\phi(\Lambda; x)$ can be determined either recursively by (78), or by using the determinants of Theorem 6.1. In this sense, these polynomials are computable in principle. We will show later some explicit formulas for small $\phi(\Lambda; x)$.

For $\alpha \in \Delta_{3,9}^{\text{Re}}$, the translation T_α is defined by

$$(123) \quad T_\alpha(\Lambda) = \Lambda - h + (1 - (h | \Lambda))c$$

for any $\Lambda \in M_{3,9}$, where $h \in L_{3,9}$, $\alpha = (h | \cdot)$. (Note that $(\alpha | \alpha) = 2$.) As an example, we consider the case $T_{78} = T_{\alpha_7}$. (The argument below applies to any $T_{ij} = T_{\varepsilon_{ij}}$ for mutually distinct $i, j \in \{4, 5, 6, 7, 8, 9\}$). In this case, the discrete time evolution of the f variables by T_{78} can be written in the form

$$(124) \quad T_{78}(f_i) = \frac{G_i(f_1, f_2, f_3)}{F_i(f_1, f_2, f_3)} \quad (i = 1, 2, 3)$$

where $F_i(x) = \phi(T_{78}(e_i); x)$ and $G_i(x) = \phi(T_{78}(e_i + h_0); x)$. For $i = 1$ we have

$$(125) \quad T_{78}(e_1) = 3e_0 - e_2 - e_3 - e_4 - e_5 - e_6 - 2e_7 - e_9.$$

This means that $F_1(x) = \phi(T_{78}(e_1); x)$ is a homogeneous polynomial of degree 3 with multiplicities of zeros (0, 1, 1) at (p_1, p_2, p_3) , and (1, 1, 1, 2, 0, 1) at $(p_4, p_5, p_6, p_7, p_8, p_9)$. In particular $F_1(x)$ can be written in the form

$$(126) \quad F_1(x) = a_0x_1^3 + a_1x_1^2x_2 + a_2x_1^2x_3 + a_3x_1x_2^2 + a_4x_1x_2x_3 + a_5x_1x_3^2 + a_6x_2^2x_3 + a_7x_2x_3^2.$$

Observe that the monomials x_2^3 and x_3^3 are missing in this formula; this is because $F_1(x)$ should have zeros at $p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$. (The coefficients a_k could be determined in principle from the pattern of multiplicities of zeros at (p_4, \dots, p_9) .) Similarly from

$$(127) \quad T_{78}(e_1 + h_0) = 4e_0 - e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6 - 2e_7 - e_9,$$

we see that $G_1(x)$ is of degree 4. Since $s_0T_{78}(e_1) = T_{78}(e_1 + h_0)$, we know that $G_1(x)$ is in fact determined from $F_1(x)$ as $G_1(x) = x_1^3x_2^2x_3^2 s_0F(x^{-1})$. These polynomials $F_i(x)$ and $G_i(x)$ are in fact too big to write down explicitly. A way to see this time evolution of the f variables is to decompose T_{78} into two steps by using the expression

$$(128) \quad T_{78} = w^2, \quad w = s_{11}'7s_{22}'7s_{33}'7s_{89}s_{78},$$

where $\{1', 2', 3'\} = \{4, 5, 6\}$. If we set $g_i = w(f_i)$, we obtain

$$(129) \quad g_i = \frac{Q_i(f_1, f_2, f_3)}{P_i(f_1, f_2, f_3)}, \quad T_{78}(f_i) = \frac{S_i(g_1, g_2, g_3)}{R_i(g_1, g_2, g_3)} \quad (i = 1, 2, 3),$$

where $P_i(x) = \phi(w(e_i); x)$ and $Q_i(x) = \phi(w(e_i + h_0); x)$; $R_i(x)$ and $S_i(x)$ are determined as $R_i(x) = {}^w P_i(x)$ and $S_i(x) = {}^w Q_i(x)$, by applying w to the coefficients. Since

$$(130) \quad w(e_i) = e_0 - e_{j'} - e_{k'}, \quad w(e_i + h_0) = 2e_0 - e_1 - e_2 - e_3 - e_{j'} - e_{k'}$$

for $\{i, j, k\} = \{1, 2, 3\}$, we see that the corresponding $\phi(\Lambda; x)$ have degree 1 and 2, respectively. In fact we have

$$(131) \quad \phi(e_0 - e_a - e_b; x) = \frac{[\varepsilon_{1a}][\varepsilon_{1b}][\varepsilon_{1ab}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]} x_1 - \frac{[\varepsilon_{2a}][\varepsilon_{2b}][\varepsilon_{2ab}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]} x_2 + \frac{[\varepsilon_{3a}][\varepsilon_{3b}][\varepsilon_{3ab}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]} x_3$$

for $1 \leq a < b \leq 9$, and

$$(132) \quad \begin{aligned} & \phi(2e_0 - e_1 - e_2 - e_3 - e_a - e_b; x) \\ &= -\frac{[\varepsilon_{23a}][\varepsilon_{23b}][\varepsilon_{1ab}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]} x_2 x_3 + \frac{[\varepsilon_{13a}][\varepsilon_{13b}][\varepsilon_{2ab}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]} x_1 x_3 - \frac{[\varepsilon_{12a}][\varepsilon_{12b}][\varepsilon_{3ab}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]} x_1 x_2 \end{aligned}$$

for $4 \leq a < b \leq 9$. Hence we obtain the explicit formulas for P_i, Q_i, R_i, S_i :

$$(133) \quad \begin{aligned} P_i(x) &= \frac{[\varepsilon_{1i'}][\varepsilon_{17}][\varepsilon_{1i'7}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]} x_1 - \frac{[\varepsilon_{2i'}][\varepsilon_{27}][\varepsilon_{2i'7}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]} x_2 + \frac{[\varepsilon_{3i'}][\varepsilon_{37}][\varepsilon_{3i'7}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]} x_3, \\ Q_i(x) &= -\frac{[\varepsilon_{23i'}][\varepsilon_{237}][\varepsilon_{1i'7}]}{[\varepsilon_{12}][\varepsilon_{13}][\varepsilon_{123}]} x_2 x_3 + \frac{[\varepsilon_{13i'}][\varepsilon_{137}][\varepsilon_{2i'7}]}{[\varepsilon_{12}][\varepsilon_{23}][\varepsilon_{123}]} x_1 x_3 - \frac{[\varepsilon_{12i'}][\varepsilon_{127}][\varepsilon_{3i'7}]}{[\varepsilon_{13}][\varepsilon_{23}][\varepsilon_{123}]} x_1 x_2, \\ R_i(x) &= -\frac{[\varepsilon_{i1'}][\varepsilon_{1'79}][\varepsilon_{i1'8}]}{[\varepsilon_{1'2'}][\varepsilon_{1'3'}][\varepsilon_{123}]} x_1 + \frac{[\varepsilon_{i2'}][\varepsilon_{2'79}][\varepsilon_{i2'8}]}{[\varepsilon_{1'2'}][\varepsilon_{2'3'}][\varepsilon_{123}]} x_2 - \frac{[\varepsilon_{i3'}][\varepsilon_{3'79}][\varepsilon_{i3'8}]}{[\varepsilon_{1'3'}][\varepsilon_{2'3'}][\varepsilon_{123}]} x_3, \\ S_i(x) &= -\frac{[\varepsilon_{jk1'}][\varepsilon_{2'3'8}][\varepsilon_{i1'8}]}{[\varepsilon_{1'2'}][\varepsilon_{1'3'}][\varepsilon_{123}]} x_2 x_3 + \frac{[\varepsilon_{jk2'}][\varepsilon_{1'3'8}][\varepsilon_{i2'8}]}{[\varepsilon_{1'2'}][\varepsilon_{2'3'}][\varepsilon_{123}]} x_1 x_3 - \frac{[\varepsilon_{jk3'}][\varepsilon_{1'2'8}][\varepsilon_{i3'8}]}{[\varepsilon_{1'3'}][\varepsilon_{2'3'}][\varepsilon_{123}]} x_1 x_2 \end{aligned}$$

for $\{i, j, k\} = \{1, 2, 3\}$, where $\varepsilon_{ij8}^- = \varepsilon_{ij8} - \delta$.

We remark that the translation T_α for any $\alpha \in \Delta_{3,9}^{\text{Re}}$ can be expressed as $T_\alpha = v T_{78} v^{-1}$ for some $v \in W_{3,9}$ such that $v(\alpha_7) = \alpha$. If we introduce the dependent variables $\varphi_i = v(f_i)$, $\psi_i = v(g_i)$ ($i = 1, 2, 3$), the discrete time evolution of these variables by T_α can be expressed in the same form as in the case of T_{78} .

Explicit description for the time evolutions of the elliptic difference Painlevé equation is discussed also in [17] and [11] from different viewpoints.

7. In terms of geometry of plane curves

The discrete time evolution of type T_{ij} ($i, j \in \{1, \dots, 9\}; i \neq j$) for the elliptic difference Painlevé equation can be described by means of geometry of plane cubic

curves. In this final section we give an explanation of this fact in the scope of this paper.

Take three constants $c_1, c_2, c_3 \in \mathbb{C}$ such that

$$(134) \quad [c_1 + c_2 + c_3] \neq 0, \quad [c_i - c_j] \neq 0 \quad (1 \leq i < j \leq 3),$$

and set $c_0 = -c_1 - c_2 - c_3$. With these constants fixed, let us consider the holomorphic mapping $p : \mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C})$ defined by

$$(135) \quad \begin{aligned} p(u) &= (x_1(u) : x_2(u) : x_3(u)) \quad (u \in \mathbb{C}), \\ x_i(u) &= [c_0 + c_i - u][c_j - u][c_k - u] \quad (\{i, j, k\} = \{1, 2, 3\}). \end{aligned}$$

(By the quasi-periodicity of the function $[u]$, this mapping induces a holomorphic mapping $\bar{p} : E = \mathbb{C}/\Omega \rightarrow \mathbb{P}^2(\mathbb{C})$ as well.) We denote by $C_0 = \bar{p}(\mathbb{C})$ the plane curve obtained by the parametrization (135). The defining equation for this curve C_0 is given explicitly by

$$(136) \quad \begin{aligned} & -\frac{[c_0 + c_3 - c_1]}{[c_3 - c_1]}x_1^2x_2 + \frac{[c_0 + c_2 - c_1]}{[c_2 - c_1]}x_1^2x_3 + \frac{[c_0 + c_3 - c_2]}{[c_3 - c_2]}x_1x_2^2 \\ & + 2\frac{[c_0]}{[0]'} \left(\frac{[c_0 + c_2 - c_3]'}{[c_2 - c_3]} + \frac{[c_0 + c_3 - c_1]'}{[c_3 - c_1]} + \frac{[c_0 + c_1 - c_2]'}{[c_1 - c_2]} \right) x_1x_2x_3 \\ & - \frac{[c_0 + c_2 - c_3]}{[c_2 - c_3]}x_1x_3^2 - \frac{[c_0 + c_1 - c_2]}{[c_1 - c_2]}x_2^2x_3 + \frac{[c_0 + c_1 - c_3]}{[c_1 - c_3]}x_2x_3^2 = 0, \end{aligned}$$

where $[u]'$ stands for the derivative of $[u]$. (The coefficient of $x_1x_2x_3$ can be written in various ways.) We remark that this definition of p and C_0 is related to that of $p_{\lambda,\mu}$ and $C_{\lambda,\mu}$ for the case $m = 3$ in Section 4 by the change of variables

$$(137) \quad \lambda = c_0, \quad \mu_i = c_i + \frac{\varepsilon_0}{3} \quad (i = 1, 2, 3), \quad t = u + \frac{\varepsilon_0}{3}.$$

In particular, the $W_{3,n}$ -equivariant meromorphic mapping $\varphi_{3,n} : \mathfrak{h}_{3,n} \dashrightarrow \mathbb{X}_{3,n}$ can be realized by means of point configurations on one single curve $C_0 \subset \mathbb{P}^2(\mathbb{C})$:

$$(138) \quad \varphi_{3,n}(\varepsilon) = [p(u_1), \dots, p(u_n)], \quad u_j = \varepsilon_j - \frac{\varepsilon_0}{3} \quad (j = 1, \dots, n)$$

for each generic $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathfrak{h}_{3,n}$. Thanks to the condition $c_0 + c_1 + c_2 + c_3 = 0$, we see that a set of $3d$ points $p(a_1), \dots, p(a_{3d})$ on C_0 is realized as the intersection of C_0 and a curve of degree d if and only if $[a_1 + \dots + a_{3d}] = 0$. In particular, three points $p(a_1), p(a_2), p(a_3)$ on C_0 are colinear if and only if $[a_1 + a_2 + a_3] = 0$. The affine Weyl group $W_{3,n}$ acts on these variables u_1, \dots, u_n as follows:

$$(139) \quad \begin{aligned} s_0(u_j) &= \begin{cases} u_j - \frac{2}{3}(u_1 + u_2 + u_3) & (j = 1, 2, 3), \\ u_j + \frac{1}{3}(u_1 + u_2 + u_3) & (j = 4, \dots, n). \end{cases} \\ s_k(u_j) &= u_{s_k(j)} \quad (k = 1, \dots, n - 1; j = 1, \dots, n). \end{aligned}$$

Before going further, we remark that any irreducible cubic curve in $\mathbb{P}^2(\mathbb{C})$ can be obtained by a projective linear transformation from a curve of the form C_0 with $[u]$

and c_1, c_2, c_3 appropriately chosen. In fact the curve C_0 is related with the Weierstrass canonical form of a cubic curve in the following way. Consider the case when $\text{rank } \Omega = 2$, and for $[u]$ take the Weierstrass sigma function $\sigma(u) = \sigma(u; \Omega)$ associated with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. If we set

$$(140) \quad c_0 = -\frac{\omega_1 + \omega_2}{2}, \quad c_1 = \frac{\omega_1}{2}, \quad c_2 = \frac{\omega_2}{2}, \quad c_3 = 0,$$

then the parametrization of C_0 is given by

$$(141) \quad \begin{aligned} x_1 &= \sigma(u)\sigma(c_2 + u)\sigma(c_2 - u) = \sigma(c_2)^2\sigma(u)^3(\wp(u) - \wp(c_2)), \\ x_2 &= \sigma(u)\sigma(c_1 + u)\sigma(c_1 - u) = \sigma(c_1)^2\sigma(u)^3(\wp(u) - \wp(c_1)), \\ x_3 &= \sigma(c_0 - u)\sigma(c_1 - u)\sigma(c_2 - u) = -\frac{1}{2}\sigma(c_0)\sigma(c_1)\sigma(c_2)\sigma(u)^3\wp'(u), \end{aligned}$$

where $\wp(u) = \wp(u; \Omega)$ is the Weierstrass \wp function associated with Ω . Hence the curve C_0 is transformed into the canonical form

$$(142) \quad y_1 y_3^2 = 4(y_2 - \wp(c_0)y_1)(y_2 - \wp(c_1)y_1)(y_2 - \wp(c_2)y_1)$$

by the projective linear transformation

$$(143) \quad \begin{aligned} x_1 &= \sigma(c_2)^2(y_2 - \wp(c_2)y_1), & x_2 &= \sigma(c_1)^2(y_2 - \wp(c_1)y_1), \\ x_3 &= -\frac{1}{2}\sigma(c_0)\sigma(c_1)\sigma(c_2)y_3. \end{aligned}$$

This implies that any smooth cubic curve can be expressed in the form of C_0 up to a projective linear transformation. Note that through this transformation to the Weierstrass canonical form, formula (139) recovers the same Weyl group action in the parametrization by the \wp function as in [22].

In what follows, we consider the translation $T_{89} \in W_{3,9}$ as an example, and describe the corresponding Cremona transformation in the language of geometry of plane curves. Namely, given an generic configuration $[p_1, \dots, p_9, q] \in \mathbb{X}_{3,10}$, we explain in geometric terms how to specify $\bar{p}_1, \dots, \bar{p}_9$ and \bar{q} in $\mathbb{P}^2(\mathbb{C})$ such that

$$(144) \quad [p_1, \dots, p_9, q].T_{89} = [\bar{p}_1, \dots, \bar{p}_9, \bar{q}].$$

We first consider the case where all the 10 points $p_1, \dots, p_9, q = p_{10}$ are on a smooth cubic curve C . Given four points $p, q, p', q' \in C$, we say that $p + q = p' + q'$ under the addition of C if the third intersection point of the line $L_{p,q}$, passing through p, q , with C coincides with that of the line $L_{p',q'}$.

Lemma 7.1. — *Let $[p_1, \dots, p_9, p_{10}] \in \mathbb{X}_{3,10}$ be generic and assume that the 10 points p_1, \dots, p_9, p_{10} are on a smooth cubic curve C . Then the action of T_{89} on $[p_1, \dots, p_9, p_{10}]$ is expressed in the form*

$$(145) \quad [p_1, \dots, p_7, p_8, p_9, p_{10}].T_{89} = [p_1, \dots, p_7, \bar{p}_8, \bar{p}_9, \bar{p}_{10}]$$

by using the three points $\bar{p}_8, \bar{p}_9, \bar{p}_{10} \in C$ that are determined by the following three conditions.

- (1) The 9 points $p_1, \dots, p_8, \bar{p}_9$ form the base points of a pencil of cubic curves.
- (2) $\bar{p}_8 + \bar{p}_9 = p_8 + p_9$ under the addition of C .
- (3) $\bar{p}_9 + \bar{p}_{10} = p_8 + p_{10}$ under the addition of C .

In order to prove Lemma 7.1, by a projective linear transformation, we may assume that this curve C is of the form C_0 , and that the 10 points are parametrized as

$$(146) \quad [p_1, \dots, p_9, p_{10}] = [p(u_1), \dots, p(u_9), p(u_{10})].$$

As we already know, such a configuration is transformed by T_{89} into

$$(147) \quad [p_1, \dots, p_9, p_{10}].T_{89} = [\bar{p}_1, \dots, \bar{p}_9, \bar{p}_{10}],$$

$$\bar{p}_j = p(\bar{u}_j), \quad \bar{u}_j = T_{89}(u_j) \quad (j = 1, \dots, 10).$$

Since

$$(148) \quad \begin{aligned} T_{89}(u_j) &= u_j & (j = 1, \dots, 7), \\ T_{89}(u_8) &= u_8 - \delta, & T_{89}(u_9) &= u_9 + \delta, \\ T_{89}(u_{10}) &= u_{10} + u_8 - u_9 - \delta, \\ u_1 + \dots + u_9 &= -\delta, \end{aligned}$$

the new coordinates \bar{u}_j ($j = 1, \dots, 10$) are determined by the conditions

$$(149) \quad \begin{aligned} (0) \quad &\bar{u}_j = u_j & (j = 1, \dots, 7), \\ (1) \quad &u_1 + \dots + u_8 + \bar{u}_9 = 0, \\ (2) \quad &\bar{u}_8 + \bar{u}_9 = u_8 + u_9, \\ (3) \quad &\bar{u}_9 + \bar{u}_{10} = u_8 + u_{10}. \end{aligned}$$

Lemma 7.1 is a paraphrase of this characterization of \bar{u}_j ($j = 1, \dots, 10$) in geometric terms. We remark that the point \bar{p}_9 is determined only from p_1, \dots, p_8 , and does not depend on the position of p_9 , while \bar{p}_8 depends essentially on p_9 .

Lemma 7.1 can be extended to the general case as follows.

Theorem 7.2. — *Let $[p_1, \dots, p_9, q]$ be a configuration of 10 points in $\mathbb{P}^2(\mathbb{C})$ in general position. Suppose that this configuration is generic, and take two smooth cubic curves C_0 and C such that*

$$(150) \quad p_1, \dots, p_8, p_9 \in C_0, \quad \text{and} \quad p_1, \dots, p_8, q \in C,$$

respectively. Then the action of the translation T_{89} on the configuration $[p_1, \dots, p_9, q] \in \mathbb{X}_{3,10}$ is expressed as

$$(151) \quad [p_1, \dots, p_9, q].T_{89} = [p_1, \dots, p_7, \bar{p}_8, \bar{p}_9, \bar{q}],$$

in terms of the points \bar{p}_8, \bar{p}_9 on C_0 and $\bar{q} \in C$ that are determined by the following conditions:

- (1) The 9 points $p_1, \dots, p_8, \bar{p}_9 \in C_0$ form the base points of a pencil of cubic curves.
- (2) Under the addition of C_0 , $\bar{p}_8 + \bar{p}_9 = p_8 + p_9$.
- (3) Under the addition of C , $\bar{p}_9 + \bar{q} = p_8 + q$.

In particular \bar{p}_9 is determined as the ninth point in the intersection of C_0 and C .

From Lemma 7.1 applied to C_0 , we have

$$(152) \quad [p_1, \dots, p_7, p_8, p_9].T_{89} = [p_1, \dots, p_7, \bar{p}_8, \bar{p}_9]$$

with $\bar{p}_8, \bar{p}_9 \in C_0$ determined by the conditions (1), (2) of Theorem 7.2. (This part does not depend on the tenth point.) Hence, the action of T_{89} on $[p_1, \dots, p_8, p_9, q]$ can be written as

$$(153) \quad [p_1, \dots, p_7, p_8, p_9, q].T_{89} = [p_1, \dots, p_7, \bar{p}_8, \bar{p}_9, \bar{q}]$$

for some $\bar{q} \in \mathbb{P}^2(\mathbb{C})$. We remark here that T_{89} can be expressed in the form

$$(154) \quad T_{89} = w s_{89}, \quad w = s_{128} s_{348} s_{567} s_{348} s_{128} \in W_{3,8},$$

where $W_{3,8} = \langle s_0, s_1, \dots, s_7 \rangle$. Hence, by applying $s_{89} s_{9,10} \in W_{3,10}$ to (153) from the right, we obtain

$$(155) \quad [p_1, \dots, p_7, p_8, q, p_9].w = [p_1, \dots, p_7, \bar{p}_9, \bar{q}, \bar{p}_8] \quad \text{in } \mathbb{X}_{3,10}.$$

(Note that $s_{9,10}$ commute with $w \in W_{3,8}$.) Since $w \in W_{3,8}$, this formula projects to

$$(156) \quad [p_1, \dots, p_7, p_8, q].w = [p_1, \dots, p_7, \bar{p}_9, \bar{q}] \quad \text{in } \mathbb{X}_{3,9}.$$

This implies that \bar{q} does not depend on p_9 . (This fact can be seen clearly in formula (133) for T_{78} . In fact, none of the polynomials P_i, Q_i, R_i, S_i depends on the parameter ε_8 .) Hence, by considering the configuration $[p_1, \dots, p_8, \bar{p}_9, q]$ on C with \bar{p}_9 replaced for p_9 , we have

$$(157) \quad [p_1, \dots, p_7, p_8, \bar{p}_9, q].T_{89} = [p_1, \dots, p_7, p_8, \bar{p}_9, \bar{q}].$$

(The 8th and 9th components remain invariant since $p_1, \dots, p_8, \bar{p}_9$ are already the base points of a pencil of cubic curves containing C .) Then by applying Lemma 7.1 to C , we conclude that \bar{q} is determined by condition (3).

Geometric description of discrete time evolutions of type T_{ij} as described above is proved in [8] by a more geometric argument based on the results of [9] and [12, 13]. We remark that this geometric approach has been employed in the study of hypergeometric solutions to elliptic and multiplicative discrete Painlevé equations in [8], [7]. It is also used by [24] in order to clarify the relationship between the elliptic difference Painlevé equation and the integrable mapping of Quispel-Roberts-Thompson [18].

References

- [1] A. B. COBLE – *Algebraic geometry and theta functions*, American Mathematical Society Colloquium Publications, vol. 10, American Mathematical Society, Providence, R.I., 1982, Reprint of the 1929 edition.
- [2] I. DOLGACHEV & D. ORTLAND – *Point sets in projective spaces and theta functions*, Astérisque, vol. 165, Paris, 1988.

- [3] I. B. FRENKEL & V. G. TURAEV – Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions, in *The Arnold-Gelfand mathematical seminars*, Birkhäuser Boston, Boston, MA, 1997, p. 171–204.
- [4] B. GRAMMATICOS & A. RAMANI – Discrete Painlevé equations: a review, in *Discrete integrable systems*, Lecture Notes in Phys., vol. 644, Springer, Berlin, 2004, p. 245–321.
- [5] B. GRAMMATICOS, A. RAMANI & V. PAPAGEORGIOU – Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1991), no. 14, p. 1825–1828.
- [6] V. G. KAC – *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.
- [7] K. KAJIWARA, T. MASUDA, M. NOUMI, Y. OHTA & Y. YAMADA – Hypergeometric solutions to the q -Painlevé equations, *Int. Math. Res. Not.* **2004** (2004), no. 47, p. 2497–2521.
- [8] K. KAJIWARA, T. MASUDA, M. NOUMI, Y. OHTA & Y. YAMADA – ${}_{10}E_9$ solution to the elliptic Painlevé equation, *J. Phys. A* **36** (2003), no. 17, p. L263–L272.
- [9] J. I. MANIN – The Tate height of points on an Abelian variety, its variants and applications, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), p. 1363–1390.
- [10] S. MIZOGUCHI & Y. YAMADA – $W(E_{10})$ symmetry, M-theory and Painlevé equations, *Phys. Lett. B* **537** (2002), no. 1-2, p. 130–140.
- [11] M. MURATA – New expressions for discrete Painlevé equations, *Funkcial. Ekvac.* **47** (2004), no. 2, p. 291–305.
- [12] M. NAGATA – On rational surfaces. I. Irreducible curves of arithmetic genus 0 or 1, *Mem. Coll. Sci. Univ. Kyoto Ser. A Math.* **32** (1960), p. 351–370.
- [13] ———, On rational surfaces. II, *Mem. Coll. Sci. Univ. Kyoto Ser. A Math.* **33** (1960/1961), p. 271–293.
- [14] M. NOUMI – *Painlevé equations through symmetry*, Translations of Mathematical Monographs, vol. 223, American Mathematical Society, Providence, RI, 2004, Translated from the 2000 Japanese original by the author.
- [15] M. NOUMI & Y. YAMADA – Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Comm. Math. Phys.* **199** (1998), no. 2, p. 281–295.
- [16] Y. OHTA, A. RAMANI & B. GRAMMATICOS – An affine Weyl group approach to the eight-parameter discrete Painlevé equation, *J. Phys. A* **34** (2001), no. 48, p. 10523–10532, *Symmetries and integrability of difference equations* (Tokyo, 2000).
- [17] ———, Elliptic discrete Painlevé equations, *J. Phys. A* **35** (2002), no. 45, p. L653–L659.
- [18] G. R. W. QUISPÉL, J. A. G. ROBERTS & C. J. THOMPSON – Integrable mappings and soliton equations. II, *Phys. D* **34** (1989), no. 1-2, p. 183–192.
- [19] A. RAMANI, B. GRAMMATICOS & J. HIETARINTA – Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* **67** (1991), no. 14, p. 1829–1832.
- [20] A. RAMANI, B. GRAMMATICOS & V. PAPAGEORGIOU – Singularity confinement, in *Symmetries and integrability of difference equations (Estérel, PQ, 1994)*, CRM Proc. Lecture Notes, vol. 9, Amer. Math. Soc., Providence, RI, 1996, p. 303–318.
- [21] A. RAMANI, B. GRAMMATICOS & J. SATSUMA – Bilinear discrete Painlevé equations, *J. Phys. A* **28** (1995), no. 16, p. 4655–4665.
- [22] H. SAKAI – Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Comm. Math. Phys.* **220** (2001), no. 1, p. 165–229.
- [23] V. P. SPIRIDONOV & A. S. ZHEDANOV – Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, in *Special functions 2000: current perspective and future directions (Tempe, AZ)*, NATO Sci. Ser. II Math. Phys. Chem., vol. 30, Kluwer Acad. Publ., Dordrecht, 2001, p. 365–388.

- [24] T. TSUDA – Integrable mappings via rational elliptic surfaces, *J. Phys. A* **37** (2004), no. 7, p. 2721–2730.
- [25] M. YOSHIDA – *Hypergeometric functions, my love: modular interpretations of configuration spaces*, Aspects Math., vol. E32, Friedr. Vieweg & Sohn, Braunschweig, 1997.

K. KAJIWARA, Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8512, Japan • *E-mail* : kaji@math.kyushu-u.ac.jp

T. MASUDA, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan
E-mail : masuda@math.kobe-u.ac.jp

M. NOUMI, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan
E-mail : noumi@math.kobe-u.ac.jp

Y. OHTA, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan
E-mail : ohta@math.kobe-u.ac.jp

Y. YAMADA, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan
E-mail : yamaday@math.kobe-u.ac.jp