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FAMILIES OF LINEAR DIFFERENTIAL EQUATIONS ON THE PROJECTIVE LINE

by

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Abstract. — The aim is to extend results of M.F. Singer on the variation of differential Galois groups. Let C be an algebraically closed field of characteristic 0. One considers certain families of connections of rank n on the projective line parametrized by schemes X over C. Let $G \subset \mathrm{GL}_n$ be an algebraic subgroup. It is shown that $X(=G)$, the set of closed points with differential Galois group G, is constructible for all families if and only if G satisfies a condition introduced by M.F. Singer. For the proof, techniques for handling families of vector bundles and connections are developed.

Résumé **(Familles d'équations différentielles linéaires sur la droite projective)**

Le but est de compléter des résultats de M.F. Singer concernant la variation des groupes de Galois différentiels. Soit C un corps algébriquement clos, de caractéristique 0. On considère des familles de connections de rang n sur la droite projective, paramétrisées par des schémas X sur C. Soit $G \subset \mathrm{GL}_n$ un sous-groupe algébrique. On montre que $X(=G)$, l'ensemble des points fermés de X avec G comme groupe de Galois différentiel, est constructible pour toute famille si et seulement si le groupe G satisfait une condition introduite par M.F. Singer. Pour la démonstration, des techniques concernant des familles de fibrés vectoriels et des connections sont développées.

1. Introduction

C is an algebraically closed field of characteristic 0 and X denotes a scheme of finite type over C. We fix a vector space V of dimension n over C and an algebraic subgroup G of $\mathrm{GL}(V)$. We will define *families of linear differential equations* on the projective line C, parametrized by X. These families are of a more general nature than the moduli spaces, defined in [**Ber02**]. For each closed point x of X (i.e., $x \in X(C)$), the differential equation corresponding to x has a differential Galois group, denoted by $\mathrm{Gal}(x)$. It is shown that the condition "$\mathrm{Gal}(x) \subset G$" for closed points x of X defines a closed subset of X. This generalizes Theorem 4.2 of [**Ber02**], where this statement is proved for moduli spaces.

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The aim is to show that the set of closed points $x \in X$ for which the differential Galois group $\mathrm{Gal}(x)$ of the corresponding equation is equal to G is a *constructible* subset of X, i.e., of the form $\cup_{i=1}^{n}(O_i \cap F_i)$ for open sets O_i and closed sets F_i. This statement (and the earlier one) has to be made more precise by providing a suitable definition of "family of differential equations" and a meaning for the expression $\mathrm{Gal}(x) \subset G$. Moreover, a condition on the group G is essential.

In his paper [**Sin93**], M.F. Singer defines a set of differential operators, by giving some local data. He proves that under a certain condition on G, the subset of the differential equations with Galois group equal to (a conjugate of) G is constructible. This condition on G will be called the *Singer condition*. We consider the same problem, in our context of families of differential equations parametrized by a scheme X. We will construct for any group G that does not satisfy the "Singer condition" an example of a moduli family $\mathbb{M}$ such that $\{x \in \mathbb{M} \mid \mathrm{Gal}(x) = G\}$ is not constructible. Finally, from these constructions one deduces an alternative description of the Singer condition.

2. The Singer condition

Let G be a linear algebraic group over C. First we will recall the Singer condition on G, as given in [**Sin93**]. A character χ of G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$, where $\mathbb{G}_m$ stands for the multiplicative group C^*. The set $X(G)$ of all characters is a finitely generated abelian group. Let $\ker X(G)$ denote the intersection of the kernels of all $\chi \in X(G)$. This intersection is a characteristic (closed) subgroup of G. As usual, G^o denotes the connected component of the identity of G. The group $\ker X(G^o)$ is a normal, closed subgroup of G^o and of G. Let $\chi_1, \ldots, \chi_s$ generate $X(G^o)$. Then $\ker X(G^o)$ is equal to the intersection of the kernels of $\chi_1, \ldots, \chi_s$. In other words $\ker X(G^o)$ is the kernel of the morphism $G^o \to \mathbb{G}_m^s$, given by $g \mapsto (\chi_1(g), \ldots, \chi_s(g))$. The image is a connected subgroup of $\mathbb{G}_m^s$ and therefore a torus T. Hence $G^o/\ker X(G^o)$ is isomorphic to T. Moreover, by definition, T is the largest torus factor group of G^o. One considers the exact sequence:

$$1 \longrightarrow G^0/\ker X(G^0) \longrightarrow G/\ker X(G^0) \longrightarrow G/G^0 \longrightarrow 1.$$

Since $G^0/\ker X(G^0)$ is abelian, this sequence induces an action of G/G^0 on $G^0/\ker X(G^0)$ by conjugation.

Definition 2.1. — A linear algebraic group G satisfies the *Singer Condition* if the action of G/G^0 on $G^0/\ker X(G^0)$ is trivial.

The Singer condition can be stated somewhat simpler, using $U(G) \subset G$, the subgroup generated by all unipotent elements in G.

Lemma 2.2. — *$U(G) = U(G^o)$ is equal to* $\ker X(G^o)$ *and the Singer condition is equivalent to "$G^o/U(G)$ lies in the center of $G/U(G)$".*

Proof. — Fix an embedding $G \subset \mathrm{GL}(V)$, where V is a finite dimensional vector space over C. First we prove that $U(G)$ is a closed connected normal subgroup of G. Let $I+B,\ B \neq 0$ be a unipotent element of G. Then $I+B = e^D$, for some nilpotent element $D = \sum(-1)^{i-1}\frac{B^i}{i} \in \mathrm{End}(V)$. The Zariski closure $\overline{\{(I+B)^n | n \in \mathbb{Z}\}}$ of the group generated by $I+B$ lies in G and is equal to the group $\{e^{tD} | t \in C\}$, which is isomorphic to the additive group $\mathbb{G}_a$ over C. Hence $U(G)$ is generated by these connected subgroups of G and by Proposition 2.2.6 of [**Spr98**] the group $U(G)$ is closed and connected. Further $U(G)$ is a normal subgroup and even a characteristic subgroup, since the set of unipotent elements of G is stable under any automorphism of G. The connectedness of $U(G)$ implies $G^o \supset U(G) = U(G^o)$.

Now we will show that $G^o/U(G^o)$ is a torus. Since the *unipotent radical* $R_u(G^o)$ lies in $U(G^o)$, we may divide G^o by $R_u(G^o)$ and assume that G^o to be *reductive*. Then by [**Spr98**][corollary 8.1.6] we have $G^o = R(G^o)\cdot(G^o, G^o)$, where $R(G^o)$ is the radical of G^o, and where (G^o, G^o) is the *commutator subgroup* of G^o. The latter group is a semi-simple subgroup, according to the same corollary. By [**Spr98**][theorem 8.1.5] we get that (G^o, G^o) is generated by unipotent elements, so $(G^o, G^o) \subset U(G^o)$. Since $R(G^o)$ is a torus, its image $G^o/U(G^o)$ is a torus, too. This proves $U(G^o) \supset \ker X(G^o)$. The other inclusion follows from the observation that every unipotent element lies in the kernel of every character.

Finally, the triviality of the action of G/G^o on $G^o/U(G^o)$ is clearly equivalent to $G^o/U(G^o)$ lies in the center of $G/U(G^o)$. □

Remarks 2.3

(1) Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. *For the moment we admit the following items* (see 3.4, 3.5 (2), 4.1 and 4.2):

- The definition of a family of differential equations, parametrized by X.
- The meaning of $\mathrm{Gal}(x) \subset G$ for $x \in X(C)$.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset G\}$ is closed.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset hGh^{-1}$ for some $h \in \mathrm{GL}(V)\}$ is constructible.

Consider the following finiteness condition for the group G: $(*)$ G has finitely many proper closed subgroups $H_1, \ldots, H_s$, such that every proper closed subgroup is contained in a conjugate of one of the H_i. One easily deduces: *If G satisfies $(*)$, then $\{x \in X(C) \mid \mathrm{Gal}(x) = G\}$ is constructible.*

(2) *If G satisfies $(*)$, then $G/U(G)$ is a finite group and in particular G satisfies the Singer condition.* Indeed, $(*)$ also holds for $G/U(G)$. If $T := G^o/U(G) \neq \{1\}$, then one can produce infinitely many proper normal subgroups of $G/U(G)$. Namely, for any integer $m > 1$ the subgroup $T[m]$, consisting of the m-torsion elements of T, is a normal subgroup. One concludes that $G/U(G)$ is finite.

(3) Consider $G := \mathrm{SL}_2(C)$. The classification of the proper closed subgroups H of G states that H is either contained in a Borel subgroup or in a conjugate of the

infinite dihedral group $D_\infty^{\mathrm{SL}_2}$ or is conjugated to one of the special finite groups: the tetrahedral group, the octahedral group, the icosahedral group. Thus G satisfies $(*)$ and moreover, $G/U(G) = \{1\}$.

(4) The infinite dihedral group $G = D_\infty^{\mathrm{SL}_2}$ has the properties: $G^o = \mathbb{G}_m$, $U(G^o) = 1$ and G/G^o acts non-trivially on G^o. Thus G does not satisfy the Singer condition. For this group one can produce moduli spaces $\mathbb{M}$ such that $\{x \in \mathbb{M}(C) \mid \mathrm{Gal}(x) = G\}$ is not constructible (see example 2.6).

(5) For the following two examples, namely moduli spaces and the groups $\mathbb{G}_a^3$ and, $\mathbb{G}_m^n$, the Singer condition is valid, but $(*)$ does not hold. We will show explicitly that these groups define constructible subsets.

***Example 2.4* (A moduli space with differential Galois groups in $\mathbb{G}_a^3$)**

Moduli spaces of the type considered here are defined in [**Ber02**]. V is a 4-dimensional vector space over C with basis $e_1, \ldots, e_4$. The element $N \in \mathrm{End}(V)$ is given by $N(e_i) = 0$ for $i = 1, 2, 3$ and $N(e_4) = e_1$. The data for the moduli problem are:

– Three distinct singular points $a_1, a_2, a_3 \in C^*$. The point ∞ is allowed to have a, non prescribed, regular singularity.
– For each singular point a_i, the differential operator $\frac{d}{d(z-a_i)} + \frac{N}{z-a_i}$.

Some calculations lead to an identification $\mathrm{GL}(4, C) \times \mathrm{GL}(4, C) \to \mathbb{M}$, where $\mathbb{M}$ is the moduli space of the problem. Let $m := (\phi_2, \phi_3)$ denote a closed point of the first space, then the corresponding universal differential operator is

$$\frac{d}{dz} + \frac{N}{z - a_1} + \frac{\phi_2 N \phi_2^{-1}}{z - a_2} + \frac{\phi_3 N \phi_3^{-1}}{z - a_3}.$$

Let $G := \mathbb{G}_a^3$ the subgroup of $\mathrm{GL}(V)$ consisting of the maps of the form $I + B$, $Be_1 = 0$ and $Be_i \in Ce_1$ for $i = 2, 3, 4$. The condition $\mathrm{Gal}(m) \subset \mathbb{G}_a^3$ can be seen to be equivalent to $\phi_2(e_1), \phi_3(e_1) \in Ce_1$. This describes the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ completely. The above differential operator evaluated at a point of $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ has the form

$$\frac{d}{dz} + \begin{pmatrix} 0 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where}$$

$$(h_1, h_2, h_3) = \frac{1}{z - a_1}(0, 0, 1) + \frac{1}{z - a_2}(f_1, f_2, f_3) + \frac{1}{z - a_3}(g_1, g_2, g_3).$$

Moreover, f_1, f_2, f_3 are polynomials of degree ≤ 2 in the entries of ϕ_2 and the g_1, g_2, g_3 are polynomials of degree ≤ 2 in the entries of ϕ_3.

Now G has infinitely many (non-conjugated) maximal proper closed subgroups and there is no obvious reason why $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = G\}$ should be constructible. We continue the calculation. The differential Galois group $\mathrm{Gal}(m)$, with m such that

$\mathrm{Gal}(m) \subset G$, is in fact the differential Galois group for the three inhomogeneous equations $y_i' = h_i,\ i = 1, 2, 3$ over $C(z)$. Thus $\mathrm{Gal}(m)$ is a proper subgroup of G if and only if there is a non trivial linear combination $c_1h_1 + c_2h_2 + c_3h_3$ with $c_1, c_2, c_3 \in C$ such that $y' = c_1h_1 + c_2h_2 + c_3h_3$ has a solution in $C(z)$. Now y exists if and only if $c_1h_1 + c_2h_2 + c_3h_3$ has residue 0 at the points a_1, a_2, a_3. The existence of such a linear combination translates into a linear dependence and the explicit equation $f_1(a_2)g_2(a_3) - f_2(a_2)g_1(a_3) = 0$. This defines a closed subset of $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ and so $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = G\}$ is constructible. We note that every linear subspace of $\mathbb{G}_a^3 \cong C^3$, which contains $(0, 0, 1)$, occurs as differential Galois group.

Example 2.5 **(A moduli space with differential Galois groups in $\mathbb{G}_m^n$)**

The data for the moduli problem are:

- A vector space V of dimension n over C and basis $e_1, \ldots, e_n$.
- Singular points $a_1, \ldots, a_s$, different from 0 and ∞, We allow ∞ to have a non-prescribed regular singularity.
- Local differential operators $\frac{d}{d(z-a_i)} + \frac{M_i}{z-a_i}$, where $e_1, \ldots, e_n$ are eigenvectors for all $M_i \in \mathrm{End}(V)$.

The moduli space $\mathbb{M}$ can be identified with $\mathrm{GL}(V)^{s-1}$. At a closed point $m = (\phi_2, \ldots, \phi_s) \in \mathrm{GL}(V)^{s-1}$ the universal differential operator reads

$$\frac{d}{dz} + \sum_{i=1}^{s} \frac{\phi_i M_i \phi_i^{-1}}{z - a_i},$$

where $\phi_1 = I$. The group $\mathbb{G}_m^n \cong G \subset \mathrm{GL}(V)$ consists of the maps for which each e_i is an eigenvector. Above the closed subset $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ the differential operator has the form

$$L := \frac{d}{dz} + \sum_{i=1}^{s} \frac{N_i}{z - a_i},$$

with $N_1 = M_1$ and each N_i is a diagonal matrix w.r.t. the basis $e_1, \ldots, e_n$ and having the same eigenvalues as M_i. The space $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ has positive dimension and is rather large if there is at least one M_i with $i > 1$ having an eigenvalue with multiplicity > 1. However the number of differential operators L is finite! Thus only a finite number of algebraic subgroups of $G \cong \mathbb{G}_m^n$ occur as differential Galois group $\mathrm{Gal}(m)$. One concludes that for every algebraic subgroup $H \subset G$, the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = H\}$ is constructible.

This example is the general pattern for "families" with differential Galois groups contained in some torus T. Again, there are only finitely many distinct differential operators L possible and therefore only finitely many possibilities for the differential Galois group. This implies that for every algebraic subgroup $H \subset T$ the set of the points with differential Galois group equal to H is constructible.

Example 2.6 **(A moduli space with differential Galois groups in $D_\infty^{\mathrm{SL}_2}$)**

Let $V = Ce_1 + Ce_2$. By $D_\infty^{\mathrm{SL}_2}$ we will denote the subgroup of $\mathrm{SL}(V)$ consisting of the maps which permute the lines Ce_1, Ce_2. The data for the moduli problem are:

- Singular points $a_1, \dots, a_4 \in C$, and ∞ is supposed to be regular.
- For each i the differential operator $\frac{d}{d(z-a_i)} + \frac{1}{z-a_i}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{-1}{4} \end{pmatrix}$ with respect to the basis e_1, e_2.

The moduli space $\mathbb{M}$ for this problem can be made explicit. The universal differential equation has 4 regular singular points with local exponents $1/4$ and $-1/4$. This is essentially the Lamé equation. It has a closed subset $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset D_\infty^{\mathrm{SL}_2}\}$. Let $D_n^{\mathrm{SL}_2} \subset D_\infty^{\mathrm{SL}_2}$ denote the dihedral subgroup (of order $4n$). It turns out that $D_\infty^{\mathrm{SL}_2}$ and $D_n^{\mathrm{SL}_2}$ for $n \geq 2$ occur as differential Galois groups $\mathrm{Gal}(m)$ for closed points. The conclusion is that $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = D_\infty^{\mathrm{SL}_2}\}$ is not constructible (provided that C is uncountable). Indeed, this set is the complement in $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset D_\infty^{\mathrm{SL}_2}\}$ of a countable union of closed subsets (see the proof of Proposition 5.3 for more details).

One way to understand this is to consider the case where C is the field of the complex numbers. Since the a_i are regular singular points, the differential Galois group is the algebraic closure of the monodromy group. This monodromy group is generated by four elements $A_1, \dots, A_4 \in \mathrm{SL}_2(C)$ having product 1 and such that each $A_i^2 = -I$. Above the moduli space $\mathbb{M}$ essentially all groups with these generators and relations do occur. Therefore all $D_n^{\mathrm{SL}_2}$ and $D_\infty^{\mathrm{SL}_2}$ occur as differential Galois group.

A more algebraic approach is to consider the elliptic curve E, given as the covering of degree two of the projective line and ramified in the four points $a_1, \dots, a_4$. The points of the moduli space corresponding to the differential Galois groups $D_n^{\mathrm{SL}_2}$ (any n) correspond to the points of finite order on the elliptic curve. For an algebraically closed field (countable or not) C of characteristic 0, the complement in $E(C)$ of the points of finite order is not constructible. This implies that for any algebraically closed field C of characteristic 0, the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = D_\infty^{\mathrm{SL}_2}\}$ is not constructible. A detailed study of this moduli family, by F. Loray, M. van der Put and F. Ulmer, is in preparation.

3. Families of differential equations

We will now come to the definition of families of linear differential equations on the projective line, parametrized by a scheme X. We first recall some facts on formal differential modules.

3.1. Formal connections and semi-simple modules. — C is again an algebraically closed field of characteristic 0. The usual differentiation on the field of formal Laurent series $C((u))$ is given by the formula $\sum a_n u^n \mapsto \frac{d}{du}\Big(\sum a_n u^n\Big) :=$

$\sum a_n n u^{n-1}$. For notational convenience we will use (in this section) the differentiation $f \mapsto \delta(f) := u\frac{d}{du}f$. A differential module M over $C((u))$ is a finite dimensional vector space over $C((u))$ provided with an additive map $\delta = \delta_M : M \to M$ satisfying $\delta(fm) = f\delta(m) + \delta(f)m$. Put $\mathcal{Q} := \bigcup_{m\geq 1} z^{-1/m}C[z^{-1/m}]$. The Galois group of the algebraic closure of $C((u))$ acts on $\mathcal{Q}$. Take $q \in \mathcal{Q}$ and let $m \geq 1$ be minimal such that $q \in u^{-1/m}C[u^{-1/m}]$. The differential module $E(q)$ over $C((u^{1/m}))$ is defined by $E(q) = C((u^{1/m}))e$ and $\delta(e) = qe$. This module can also be viewed as a differential module of dimension m over $C((u))$. As such, it depends only on the Galois orbit oq of q in $\mathcal{Q}$. We write $E(oq)$ for $E(q)$ considered as a differential module over $C((u))$. We note that $E(oq)$ is an irreducible differential module. The classification of differential modules over $C((u))$ can be formulated as follows:

Every differential module M over $C((u))$ can be written uniquely in the form $\oplus_{i=1}^{s} E(oq_i) \otimes M_i$, where the $oq_1, \ldots, oq_s$ are distinct Galois orbits in $\mathcal{Q}$ and where the M_i are regular singular differential modules.

We recall that a differential module N is regular singular if there exists a basis $b_1, \ldots, b_r$ of N over $C((u))$ such that the free $C[[u]]$-module $\Lambda := C[[u]]b_1 + \cdots + C[[u]]b_r$ is invariant under δ. One associates to a regular singular N a semi-simple regular singular differential module N_{ss} by the following construction. (compare [**Lev75**]).

The operator δ leaves $u^m\Lambda$ invariant for each $m \geq 0$. Thus δ induces a C-linear endomorphism on δ_m on $\Lambda/u^m\Lambda$. The additive Jordan decomposition of δ_m is written as $\delta_m = \delta_{m,ss} + \delta_{m,nilp}$. Here ss denotes the semi-simple part and $nilp$ denotes the nilpotent part. It is easily seen that the families of endomorphisms $\{\delta_{m,ss}\}$ and $\{\delta_{m,nilp}\}$ form projective systems. Now we write δ_{ss} and δ_{nilp} the induced maps on Λ. One verifies that δ_{nilp} is $C[[u]]$-linear and that $\delta_{ss}(fm) = f\delta_{ss}(m) + \delta(f)m$ for $f \in C[[u]]$ and $m \in \Lambda$. Both operators are extended to N. The vector space N provided with δ_{ss} is denoted by N_{ss}. It is a differential module over $C((u))$ and it is semi-simple in the sense that every submodule of N_{ss} has a complement.

In terms of matrix differential equations this construction has an easy translation. One knows that N contains a basis such that the corresponding matrix differential equation has the form $u\frac{d}{du} + A$, where A is a constant matrix (i.e., has entries in C). Then N_{ss} corresponds (on the same basis) with the matrix differential equation $u\frac{d}{du} + A_{ss}$, where $A = A_{ss} + A_{nilp}$ is the usual Jordan decomposition of A. We note that the "classical" solution space for the matrix differential equation $u\frac{d}{du} + A$ contains logarithmic terms if $A_{nilp} \neq 0$.

For a differential module M over $C((u))$ with canonical decomposition $\oplus_{i=1}^{s} E(oq_i) \otimes M_i$ we define $M_{ss} := \oplus_{i=1}^{s} E(oq_i) \otimes M_{i,ss}$. Thus M_{ss} is equal to M as vector space over $C((u))$. One has $\delta_M = \delta_{M_{ss}} + E$ where E is a nilpotent endomorphism of M commuting with $\delta_{M_{ss}}$ and δ_M. In particular, every submodule of M is also a submodule of M_{ss}. Moreover, the differential module M_{ss} is semi-simple.

A formal connection is a connection $\nabla : N \to C[\![u]\!]u^{-k}du \otimes N$, where N is a free $C[\![u]\!]$-module of finite rank. One associates to N the differential module $M = C((u)) \otimes N$ (with δ_M induced by $\nabla_{u\frac{d}{du}}$). The formal connection N_{ss} is now defined as the connection on N induced by the $\delta_{M_{ss}}$ on M_{ss}. We will call N_{ss} and M_{ss} the *semi-simplifications* of N and M. Suppose that $R \subset N$ is a $C[\![u]\!]$-submodule such that N/R is free and $\nabla R \subset C[\![u]\!]u^{-k}du \otimes R$. Then also $\nabla_{ss}R \subset C[\![u]\!]u^{-k}du \otimes R$.

3.2. Defining families. — The statement that we want to prove concerns the closed points of X and therefore we may suppose that X is reduced. For the same reason we may suppose (at every stage of the proof) that X is irreducible and affine. Assume that $X = \mathrm{Spec}(R)$ with R reduced and finitely generated over C. We have not investigated the technical complications involved in moveable singularities and we will consider families for which the singular points (apparent or not) lie in a fixed subset $\{a_1, \ldots, a_s\}$ of $\mathbb{P}^1_C$. For convenience we suppose that $0, \infty \notin \{a_1, \ldots, a_s\}$.

A *first attempt* to define a family parametrized by $X = \mathrm{Spec}(R)$, is to consider a matrix differential equation $\frac{d}{dz} + A$ where A is an $R\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big]$-linear endomorphism of $R\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_C V$. More explicitely, A has the form $\sum_{j=1}^{s}\sum_{i=1}^{k} \frac{A(i,j)}{(z-a_j)^i}$ where each $A(i,j)$ is an R-linear endomorphism of $R \otimes V$. For every closed point x of X, i.e., $x \in X(C)$, one writes $A(x)$ for the $C\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big]$-linear endomorphism of $C\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes V$, obtained by applying $x : R \to C$ to A. In this way, $\frac{d}{dz} + A$ is a family of differential equations on the projective line over C. The equation $\frac{d}{dz} + A$ is regular at $z = 0$. One considers $R[\![z]\!] \otimes_C V$ and the canonical map

$$\mathrm{Mod}_z : R[\![z]\!] \otimes_C V \longrightarrow R[\![z]\!] \otimes_C V/(z) \xrightarrow{\cong} R \otimes_C V.$$

Lemma 3.1. — *Consider the kernels:*

$$S = \ker\Big(\frac{d}{dz} + A, R[\![z]\!] \otimes_C V\Big) \text{ and } S(x) = \ker\Big(\frac{d}{dz} + A(x), C[\![z]\!] \otimes_C V\Big).$$

The maps $\mathrm{Mod}_z : S \to R \otimes_C V$ *and* $\mathrm{Mod}_z : S(x) \to V$ *are bijections. Moreover, the image of* S *under the map* $R[\![z]\!] \otimes_C V \to C[\![z]\!] \otimes_C V$*, induced by* $x : R \to C$*, is equal to* $S(x)$*.*

Proof. — One considers an endomophism $F = F_0 + zF_1 + \cdots$ of $R[\![z]\!] \otimes_C V$ (i.e., each F_i is an endomorphism of $R \otimes_C V$) with $F_0 = 1$. One requires that F is a “fundamental matrix”, which means that $F' + AF = 0$. Put $A = A_0 + A_1 z + \cdots$. This leads to equations

$$(n+1)F_{n+1} + A_0F_n + A_1F_{n-1} + \cdots + A_nF_0 = 0 \text{ for all } n \geq 0.$$

Clearly F exists and is unique. This implies that $\mathrm{Mod}_z : S \to R \otimes_C V$ is a bijection. Let $F(x)$, for a closed point x, be obtained from F by the map $x : R \to C$; then $F(x)$

is a fundamental matrix for $\frac{d}{dz}+A(x)$. The other two statements of the lemma follow from this. □

$\frac{d}{dz}+A(x)$ is viewed as differential equation over the ring $C\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$. Let $PVR(x)$ denote the subring of $C[\![z]\!]$ generated over $C\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$ by all the entries of $F(x)$ and the inverse of the determinant of $F(x)$. Then $PVR(x)$ is a Picard–Vessiot ring for $\frac{d}{dz}+A(x)$. Let $\mathrm{Gal}(x)$ denote the group of the differential automorphisms of $PVR(x)$ over $C\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$. By construction $S(x)=\ker\big(\frac{d}{dz}+A(x),PVR(x)\otimes_C V\big)$ and $\mathrm{Gal}(x)$ acts faithfully on $S(x)$. Using the isomorphism $\mathrm{Mod}_z:S(x)\to V$, one finds a faithful action of $\mathrm{Gal}(x)$ on V. We conclude that the above contructions provide a canonical way to embed every $\mathrm{Gal}(x)$ into $\mathrm{GL}(V)$.

The next lemma will not be used in the proof of the main result. However, its contents and the ideas behind it are closely related to our main theme. In what follows we will prove a converse of this lemma.

***Lemma 3.2* (Specialization of the differential Galois group).** — *We use the above notation. Suppose that R is a domain with field of fractions K. Then $\frac{d}{dz}+A$ can be considered as a differential equation over $K\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$. Let $\overline{K}$ denote an algebraic closure of K. Then the following holds:*

(a) *the differential Galois group $H_{\overline{K}}$ over the field of constants $\overline{K}$ descends to an algebraic subgroup H of $\mathrm{GL}(K\otimes V)$,*
(b) *the schematic closure H_R of H as algebraic subgroup of $\mathrm{GL}(R\otimes V)$, has the property: for every closed point x, with corresponding maximal ideal m_x, there is an inclusion $\mathrm{Gal}(x)\subset(H_R\otimes R/x)$.*

We note that this lemma and its proof are rather close to a result of O. Gabber (see [**Kat90**, Thm 2.4.1 on page 39]).

Proof

(a) The solution space $\ker(\frac{d}{dz}+A,K[\![z]\!]\otimes_C V)$ is equal to $K\otimes_R S$. Let PVR denote the subring of $K[\![z]\!]$, generated over $K\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$ by the entries of F and the inverse of the determinant of F. Then $\overline{K}\otimes_K PVR$ is a Picard–Vessiot ring and we write $H_{\overline{K}}$ for its differential Galois group. The latter is characterized as the group of the $\overline{K}\big[z,\frac{1}{(z-a_1)\cdots(z-a_s)}\big]$-linear differential automorphisms of $\overline{K}\otimes PVR$. The group $H_{\overline{K}}$ acts faithfully on $\overline{K}\otimes_C V$. Choose a basis of V over C. The affine ring of $\mathrm{GL}(\overline{K}\otimes_C V)$ can be written as $\overline{K}\big[\{X_{i,j}\}_{i,j=1}^n,\frac{1}{\det}\big]$, where det denotes the determinant of the matrix $(X_{i,j})$. The ideal J defining $H_{\overline{K}}$ is the kernel of the $\overline{K}$-homomorphism $\phi:\overline{K}\big[\{X_{i,j}\}_{i,j=1}^n,\frac{1}{\det}\big]\to\overline{K}\otimes PVR$, given by $\phi(X_{i,j})$ is equal to $F_{i,j}$ (the (i,j)-entry of the matrix F). Since ϕ "descends" to K, the ideal J descends to an ideal I of $K\big[\{X_{i,j}\}_{i,j=1}^n,\frac{1}{\det}\big]$. The latter defines an algebraic subgroup H of $\mathrm{GL}(K\otimes_C V)$ satisfying $H\otimes_K\overline{K}$ coincides with $H_{\overline{K}}$.

(b) The schematic closure H_R of H is the group scheme over R given by the ideal $I_R := I \cap R\big[\{X_{i,j}\}_{i,j=1}^n, \frac{1}{\det}\big]$. This explains the terminology of the lemma. We will show that the inclusion $\mathrm{Gal}(x) \subset H_R \otimes R/m_x$ follows from a combination of Chevalley's theorem and some properties of matrix differential operators (or connections). The expression $\frac{d}{dz} + A$ is seen as a regular differential operator on $\mathrm{Spec}(R) \times (\mathbb{P}^1_C \setminus \{a_1, \dots, a_s\})$. Let V^a_b denote the tensor product $V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V$ (of a copies of the dual V^* of V and b copies of V.) There is a K-subspace W of some finite direct sum $K \otimes_C \oplus_i V^{a_i}_{b_i}$ such that H is the stabilizer of W. The differential operator $\frac{d}{dz} + A$ on $R\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_C V$ induces a differential operator $\frac{d}{dz} + B$ on $R\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_C \oplus_i V^{a_i}_{b_i}$. By differential Galois theory, $K\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_K W$ is invariant under $\frac{d}{dz} + B$.

Put $\tilde{W} := W \cap (R \otimes_C \oplus_i V^{a_i}_{b_i})$. Then $\tilde{W}$ is invariant under H_R and moreover $R\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_R \tilde{W}$ is invariant under $\frac{d}{dz} + B$. The regularity of this differential operator implies that $\tilde{W}$ is a projective R-module (see [**Kat90**, p. 39], for more details). Let $x \in X(C)$. The group $R/m_x \otimes H_R$ is defined by the invariance of the subspace $R/m_x \otimes_R \tilde{W}$ of $R/m_x \otimes_C \oplus_i V^{a_i}_{b_i} = \oplus_i V^{a_i}_{b_i}$. Furthermore, the space $C\big[z, \frac{1}{(z-a_1)\cdots(z-a_s)}\big] \otimes_C (R/m_x \otimes_R \tilde{W})$ is invariant under $\frac{d}{dz} + B(x)$. By differential Galois theory, the group $\mathrm{Gal}(x)$ leaves $R/m_x \otimes_R \tilde{W}$ invariant. Hence $\mathrm{Gal}(x) \subset (R/m_x \otimes H_R)$. ☐

In our present setup the result that we want to prove is not valid. This is illustrated by the rather obvious example: $R = C[t]$ and the differential operator $\frac{d}{dz} + \frac{t}{z-a_1}$. If the value of t is rational number of the form $\frac{p}{q}$ with $q \geq 1$ and $(p,q) = 1$, then the differential Galois is a cyclic group of order q. For other values of t in C, the differential Galois group is the multiplicative group $\mathbb{G}_m$. However, the group $\mathbb{G}_m$ satisfies the "Singer condition".

In order to avoid this and other examples of this sort we will suppose that there are only finitely many possibilities for the semi-simplification of the formal local structure of $\frac{d}{dz} + A(x)$ at any of the singular points $a_1, \dots, a_s$. Again this is not sufficient for our goal, namely the statement that the set of closed points x with $\mathrm{Gal}(x) = G$ is constructible. The new problem is that the formal isomorphism between $\frac{d}{dz} + A(x)$ at a_j and one of the prescribed formal connections can have a pole at a_j of arbitrarely high order. A remedy for this is the introduction of connections on the projective line over C. In order to work out this idea the following (probably known) result on vector bundles on $\mathbb{P}^1_X := X \times \mathbb{P}^1_C$ is needed. We introduce some notation. Let $pr_1 : X \times \mathbb{P}^1_C \to X$ and $pr_2 : X \times \mathbb{P}^1_C \to \mathbb{P}^1_C$ denote the two projections. For vector bundles $\mathcal{A}$ and $\mathcal{B}$ on X and $\mathbb{P}^1_C$ we write $\mathcal{A} \otimes \mathcal{B}$ for the vector bundle $pr_1^*\mathcal{A} \otimes pr_2^*\mathcal{B}$. The line bundle of degree d on $\mathbb{P}^1_C$ is denoted by $\mathcal{O}(d)$. For $\mathcal{O}_X \otimes \mathcal{O}(d) = pr_2^*\mathcal{O}(d)$ we also write $\mathcal{O}_X(d)$. We recall that any vector bundle of rank n on $\mathbb{P}^1_C$ has the form $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \oplus \cdots \oplus \mathcal{O}(k_n)$ with unique $k_1 \geq k_2 \geq \cdots \geq k_n$. We call the sequence $k_1 \geq \cdots \geq k_n$ the *type of the vector bundle*.

Proposition 3.3. — *Let X be a scheme of finite type over C and let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$ of rank n. Let $x \in X$ be a closed point. Suppose that the induced vector bundle $\mathcal{M}(x)$ on $\mathbb{P}_C$ is free. Then there exists an open neighbourhood U of x such that the restriction of $\mathcal{M}$ to $\mathbb{P}^1_U$ is free.*

Proof. — We remark that $\mathcal{M}(x)$ denotes the vector bundle on $\mathbb{P}^1_C$ obtained by evaluating $\mathcal{M}$ at x. More precisely, write $j_x : \mathrm{Spec}(C) \to X$ for the morphism corresponding to x and write $g_x = j_x \times id : \mathbb{P}^1_C = \mathrm{Spec}(C) \times \mathbb{P}^1_C \to X \times \mathbb{P}^1_C$. Then $\mathcal{M}(x)$ is defined as $g_x^* \mathcal{M}$.

One may suppose that X is affine. Let D_0 and D_∞ denote the divisors $X \times \{0\}$ and $X \times \{\infty\}$. Define the sheaf $\mathcal{N} = \mathcal{O}(-D_\infty) \otimes \mathcal{M}$ and consider the covering of $\mathbb{P}^1_X$ by the affine sets $U_0 = \mathbb{P}^1_X - D_\infty$ and $U_\infty = \mathbb{P}^1_X - D_0$. Put $U_{0,\infty} = U_0 \cap U_\infty$. The following sequence

$$0 \longrightarrow H^0(\mathcal{N}) \longrightarrow \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \longrightarrow H^1(\mathcal{N}) \longrightarrow 0$$

is exact. The two $O(X)$-modules $H^0(\mathcal{N})$ and $H^1(\mathcal{N})$ are finitely generated. Indeed, since the natural projection $pr : \mathbb{P}^1_X \to X$ is proper one has that $pr_*\mathcal{N}$ and $R^1 pr_*\mathcal{N}$ are coherent. Moreover, $H^0(\mathcal{N}) = H^0(X, pr_*\mathcal{N})$ and $H^1(\mathcal{N}) = H^0(X, R^1 pr_*\mathcal{N})$ (by Leray's spectral sequence). Let m_x denote the maximal ideal of $O(X)$ corresponding to the closed point x. The assumption that $\mathcal{M}(x)$ is free implies that $H^0(\mathcal{N}(x)) = H^1(\mathcal{N}(x)) = 0$. This implies that the map $\alpha \otimes_{O(X)} O(X)/m_x$ is a bijection. Hence x does not lie in the support of the $O(X)$-module $H^1(\mathcal{N})$. After shrinking X, we may assume that $H^1(\mathcal{N}) = 0$ and that α is surjective. The $O(U_{0,\infty})$-module $\mathcal{N}(U_{0,\infty})$ is projective. Therefore $\mathcal{N}(U_{0,\infty})$ is also a projective module over the ring $O(X)$. Hence the exact sequence of $O(X)$-modules

$$0 \longrightarrow H^0(\mathcal{N}) \longrightarrow \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \longrightarrow 0$$

splits. The bijectivity of the map $\alpha \otimes_{O(X)} O(X)/m_x$ implies that the module $H^0(\mathcal{N}) \otimes_{O(X)} O(X)/m_x = 0$. After shrinking X, we may suppose that $H^0(\mathcal{N}) = 0$. Define the sheaf $\mathcal{Q}$ by the exactness of

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Then $\mathcal{Q} = \mathcal{M}/(\mathcal{O}(-D_\infty) \otimes \mathcal{M})$ and therefore $\mathcal{Q}$ is a vector bundle on $X \cong X \times \{\infty\}$. The rank n of $\mathcal{Q}$ is the same as the rank of $\mathcal{M}$. After shrinking X, we may suppose that $\mathcal{Q}$ is a free vector bundle on X. The above exact sequence of sheaves yields: $H^0(\mathcal{M}) = H^0(X, \mathcal{Q}) = O(X)^n$ and $H^1(\mathcal{M}) = 0$. It suffices now to show that $\mathcal{M}$ is generated at every closed point w of $\mathbb{P}^1_X$ by its group of global sections $H^0(\mathcal{M})$. This property is equivalent to the surjectivity of the map $H^0(\mathcal{M}) \to \mathcal{M}_w/m_w\mathcal{M}_w$, where m_w denotes the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^1_X, w}$. The point w lies on a divisor $D = X \times \{p\}$ for some closed point p of $\mathbb{P}^1_C$. Put $w = (q, p)$. Define the sheaf S by the exact sequence

$$0 \longrightarrow \mathcal{O}(-D) \otimes \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow S \longrightarrow 0.$$

We note that $\mathcal{O}(-D)$ is isomorphic to $\mathcal{O}(-D_\infty)$. As before one concludes that S is a vector bundle on $X \cong X\times\{p\}$ and that $H^0(\mathcal{M}) \to H^0(X\times\{p\}, S)$ is surjective. Since $X\times\{p\}$ is affine, the map $H^0(X\times\{p\}, S) \to S_q/m_qS$ (where m_q denotes the maximal ideal corresponding to the point $w = (q,p)$) is surjective. Finally, $\mathcal{M}_w/m_w\mathcal{M}_w \to S_q/m_qS_q$ is an isomorphism. □

Remarks 3.4 **(More on vector bundles on $\mathbb{P}^1_X$)**

(1) We start with an example showing that the type of a vector bundle on $\mathbb{P}^1_X$ is not locally constant, i.e., the type of $\mathcal{M}(x)$ is not locally constant in X. Take $X = \mathrm{Spec}(C[t])$ and consider a vector bundle $\mathcal{M}$ of rank 2 on $\mathbb{P}^1_X$. Let z denote the usual global parameter on $\mathbb{P}^1_C$. Write again $D_0 = X\times\{0\}$, $D_\infty = X\times\{\infty\}$, $U_0 = \mathbb{P}^1_X - D_\infty$ and $U_\infty = \mathbb{P}^1_X - D_0$. The restriction of $\mathcal{M}$ to the two affine sets U_0, U_∞ is free (since every projective module over a polynomial ring is free). Hence $\mathcal{M}$ is given by a matrix $A \in \mathrm{GL}(2, C[t][z,z^{-1}])$. This matrix defines a unique double coset $\mathrm{GL}(2, C[t][z])\cdot A\cdot \mathrm{GL}(2, C[t][z^{-1}])$. On the other hand each double coset, as above, defines a vector bundle of rank 2 on $\mathbb{P}^1_X$. We consider now the vector bundle associated to

$$A = \begin{pmatrix} z & 0 \\ t & z^{-1} \end{pmatrix}.$$

For $t = 0$, this defines the vector bundle $\mathcal{O}(1)\oplus\mathcal{O}(-1)$ on $\mathbb{P}^1_C$. For $t\neq 0$, this defines the free vector bundle on $\mathbb{P}^1_C$. Indeed,

$$A = \begin{pmatrix} 1 & t^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -t^{-1} \\ t & z^{-1} \end{pmatrix}.$$

(2) Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$ of rank n. Then the set of closed points $x\in X(C)$, such that $\mathcal{M}(x)$ has type $a_1\geq a_2\geq\cdots\geq a_n$, is a constructible subset. We sketch the proof of this result. It suffices to consider the case where X is affine and connected. For a point $x\in X(C)$, the type $a_1\geq\cdots\geq a_n$ of the vector bundle $\mathcal{M}(x)$ is determined by the dimensions $h^i(k,x)$, $i=0,1$ of the cohomology groups $H^i(\mathbb{P}^1_C, \mathcal{M}(x)\otimes\mathcal{O}(k))$, for all $k\in\mathbb{Z}$. The degree D of $\mathcal{M}(x)$ is independent of $x\in X(C)$. By Riemann-Roch, $h^0(k,x) - h^1(k,x) = D+n$ for all k. There exists an integer N, depending on $\mathcal{M}$, such that for $k\geq N$ one has $h^1(k,x)=0$ and for $k\leq -N$ one has $h^0(k,x)=0$. Hence the type of $\mathcal{M}(x)$ is determined by the values of $h^1(k,x)$ for $-N<k<N$. Therefore we have to investigate the dependence of $h^1(k,x)$ on x. For convenience we consider $h^1(0,x)$. The proof of Proposition 3.3 asserts that $H^1(\mathbb{P}^1_C, \mathcal{M}(x)) = O(X)/m_x\otimes H^1(\mathcal{M})$, where m_x denotes the maximal ideal corresponding to x. This implies that for any integer $d\geq 0$ the set $\{x\in X(C)|\ h^1(0,x)\leq d\}$ is open. From this observation the above statement follows.

(3) The *defect* of a vector bundle on $\mathbb{P}^1_C$ of type $a_1\geq\cdots\geq a_n$ is defined as $a_1 - a_n$. The reasoning in (2) above implies that for any integer $d\geq 0$ the set

$\{x \in X(C) |$ the defect of $\mathcal{M}(x)$ is $\le d\}$ is open. This generalizes the statement of Proposition 3.3.

(4) *Suppose that X is a reduced, irreducible scheme of finite type over C. Let $\mathcal{M}$ be a vector bundle on $\mathbb{P}^1_X$. Suppose that there exists a closed point $x_0 \in X$ such that $\mathcal{M}(x_0)$ is free. Then the set of closed points x such that $\mathcal{M}(x)$ is not free is either empty or equal to a divisor on X.*

Sketch of the proof. — We may suppose that $X = \mathrm{Spec}(R)$ with R a finitely generated C-algebra having no zero-divisors. We will use the notation of the proof of Proposition 3.3. The statement that we want to prove is equivalent to: the R-module $H^1(\mathcal{N})$ is either 0, or its support is a divisor on X. Consider again the exact sequence

$$0 \longrightarrow H^0(\mathcal{N}) \longrightarrow \mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty) \xrightarrow{\alpha} \mathcal{N}(U_{0,\infty}) \longrightarrow H^1(\mathcal{N}) \longrightarrow 0$$

The assumption that $\mathcal{M}(x_0)$ is free implies that α becomes an isomorphism after localizing R at a suitable non-zero element. Thus $H^0(\mathcal{N}) = 0$ (since R has no zero-divisors) and $H^1(\mathcal{N})$ is a finitely generated torsion module over R. The modules $\mathcal{N}(U_0) \oplus \mathcal{N}(U_\infty)$ and $\mathcal{N}(U_{0,\infty})$ are projective R-modules of infinite rank. The above exact sequence is therefore a resolution of $H^1(\mathcal{N})$ by projective modules of infinite rank. Consider an exact sequence

$$0 \longrightarrow V_1 \xrightarrow{f} V_0 \longrightarrow H^1(\mathcal{N}) \longrightarrow 0$$

with V_0 a finitely generated free R-module. Then V_1 is a projective R-module (of finite rank). After replacing $\mathrm{Spec}(R)$ by the elements of an open affine covering, we may suppose that V_1 is a free R-module, too. Furthermore, V_1 and V_0 have the same rank. The support of $H^1(\mathcal{N})$ is equal to the closed subset defined by $\det(f) = 0$. This finishes the proof. □

The above result is also valid in the complex analytic case. A proof is given by B. Malgrange in [**Mal83**], Section 4.

(5) In trying to classify the vector bundles on $X \times \mathbb{P}^1_C$, one encounters the question whether a vector bundle $\mathcal{M}$ of rank n on $X \times \mathbb{P}^1_C$ has, at least locally with respect to X, the property that the restrictions of $\mathcal{M}$ to the affine sets $\mathrm{Spec}(R) \times (\mathbb{P}^1_C - \{\infty\})$ and $\mathrm{Spec}(R) \times (\mathbb{P}^1_C - \{0\})$ are free. If the answer is positive, then $\mathcal{M}$ is (locally with respect to X) defined by a double coset $\mathrm{GL}(n, R[z]) \cdot A \cdot \mathrm{GL}(n, R[z^{-1}])$ with $A \in \mathrm{GL}(n, R[z, z^{-1}])$. This seems a useful way to present $\mathcal{M}$. The above question is directly related to the following question posed by H. Bass and D. Quillen:

Let R be a regular noetherian ring. Does every finitely generated projective module over $R[z]$ come from a finitely generated projective module over R?

There are partial answers to this question (see [**Lin78**]). It seems that the general problem remains unsolved.

Definition 3.5 **(A family of differential equations on the projective line $\mathbb{P}^1$, parametrized by X)**

Distinct points $\{a_1, \ldots, a_s\} \subset \mathbb{P}^1_C \setminus \{0, \infty\}$ are given. Moreover, a finite set I of semi-simple formal connections $\nabla_i : C[\![u]\!]^n \to C[\![u]\!]u^{-k}du \otimes C[\![u]\!]^n$ (with $i \in I$) is given. This collection will be called the *local data*. The next items are $X, \mathcal{M}, \nabla, V$ where:

(i) X is a reduced scheme of finite type over C.
(ii) $\mathcal{M}$ is a vector bundle on $\mathbb{P}^1_X$ of the form $\mathcal{O}_X \otimes \mathcal{N}$, where $\mathcal{N}$ is a vector bundle on $\mathbb{P}^1_C$.
(iii) V is a C-vector space of dimension n and an isomorphism $\mathcal{N}_0/(z) \to V$ is given.
(iv) A connection $\nabla : \mathcal{M} \to \Omega(k[a_1] + \cdots + k[a_s]) \otimes \mathcal{M}$.

For every point $x \in X(C)$, one defines the vector bundle $\mathcal{M}(x)$ on $\mathbb{P}^1_C$ by $\mathcal{M}(x) = j_x^*(\mathcal{M})$ where $j_x : \mathbb{P}^1_C = \{x\} \times \mathbb{P}^1_C \to \mathbb{P}^1_X$. The above data induce a connection $\nabla(x) : \mathcal{M}(x) \to \Omega(k[a_1] + \cdots + k[a_s]) \otimes \mathcal{M}(x)$. For every point $x \in X(C)$ and every j, we write u for the local parameter $z - a_j$. We require that the semi-simplification of the formal connection $\widehat{\nabla(x)}_j : \widehat{\mathcal{M}(x)}_{a_j} \to C[\![u]\!]u^{-k}du \otimes \widehat{\mathcal{M}(x)}_{a_j}$ is isomorphic to ∇_i for some $i \in I$. More precisely, there exists a $C[\![u]\!]$-linear isomorphism $\widehat{\mathcal{M}(x)}_{a_j,ss} \to C[\![u]\!]^n$ that is compatible with the connections. □

Remarks 3.6

(1) A more precise formulation of part (iv) is:

$$\nabla : \mathcal{M} \longrightarrow \Omega_{\mathbb{P}^1_X/X}\Big(\sum k[X \times \{a_i\}]\Big) \otimes \mathcal{M},$$

where the $[X \times \{a_i\}]$ are divisors on $\mathbb{P}^1_X$. Moreover, the integer k occuring here can be replaced by any integer $\ell \geq k$ without changing the family.

(2) A moduli space, as defined in [**Ber02**], is a special case of a family. It is a family, parametrized by $X = \mathbb{M}$ and with $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1_X} \otimes V$ and ∇ such that $(\mathcal{M}, \nabla, \{\phi_i\})$ is the universal connection.

(3) Let a family, parametrized by X be given. For every $x \in X(C)$, there is a full solution space $W(x)$ of $\nabla(x)$ inside $\widehat{\mathcal{M}(x)}_0$. We want to identify $W(x)$ with V. The induced map $W(x) \to \widehat{\mathcal{M}(x)}_0/(z) = \mathcal{M}(x)_0/(z)$ is an isomorphism. By (ii), $\mathcal{M}(x)$ is canonically isomorphic to $\mathcal{N}$. Moreover, in (iii) an isomorphism $\mathcal{N}_0/(z) \to V$ is given. Combining these one obtains the isomorphism $W(x) \to V$. The differential Galois group $\mathrm{Gal}(x)$, which acts in a natural way on $W(x)$ is, via this isomorphism, embedded in $\mathrm{GL}(V)$.

(4) Let a family, parametrized by, say, $X = \mathrm{Spec}(R)$, be given. We will make some changes to this family. The given isomorphism $V \to \mathcal{N}_0/(z)$ can be lifted to an injective C-linear map $V \to \mathcal{N}_0$. One replaces $\mathcal{N}$ with $\mathcal{N}^1 := \mathcal{N}(\ell[b_1] + \cdots \ell[b_r])$, for

suitable $\ell > 0$ and points $b_1, \ldots, b_r$ (different from 0), such that $V \subset H^0(\mathcal{N}^1)$. Then we consider the free vector bundle $\mathcal{F} := \mathcal{O}_{\mathbb{P}^1_X} \otimes V$, subbundle of $\mathcal{N}^1$, and the free vector bundle $\mathcal{F}_X$.

In general, $\nabla(\mathcal{F})_X \subset \Omega\big(\sum_{i=1}^s k[a_i]\big) \otimes \mathcal{F}_X$ does not hold. At the cost of introducing some points $\{a_{s+1}, \ldots, a_t\}$ as new (apparent) singularities and adding finitely many new items to the local data, one obtains a new family, parametrized by X, with

$$\nabla : \mathcal{F}_X \longrightarrow \Omega\bigg(\sum_{i=1}^t k[a_i]\bigg) \otimes \mathcal{F}_X \text{ (for a suitable, large enough } k > 0).$$

One of the new singular points a_j might be the point ∞. An automorphism of $\mathbb{P}^1_C$, which fixes 0, takes care of that. The original family is closely related to this new family. In particular, for the constructibility result that we want to prove, we may replace the original family by the new one. In what follows we may therefore (at any stage of the proof) assume that the vector bundle $\mathcal{M}$ on $\mathbb{P}^1_X$ is equal to $\mathcal{O}_X \otimes \mathcal{N}$ with $\mathcal{N}$ a free vector bundle on $\mathbb{P}^1_C$. Moreover V is identified with $H^0(\mathcal{N})$. In other terms $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$.

(5) For an algebraic subgroup H of $\mathrm{GL}(V)$ we write $X(\subset H)$ (resp. $X(H \subset)$) for the set of closed points $x \in X$ such that $\mathrm{Gal}(x) \subset H$ (resp. $H \subset \mathrm{Gal}(x)$). For two algebraic subgroups $H_1 \subset H_2$ we will write $X(H_1 \subset, \subset H_2)$ for $X(H_1 \subset) \cap X(\subset H_2)$. Furthermore, $X(= H) := X(H \subset, \subset H)$. The main result of this paper is the following.

Theorem 3.7. — *Suppose that the linear algebraic subgroup $G \subset \mathrm{GL}(V)$ satisfies the "Singer condition". Let a family of linear differential equation, parametrized by X be given. Then $X(= G)$ is a constructible subset of X.*

In the proof we follow some of the steps of the proof given in [**Sin93**]. However, we like to point out some important differences. In our setup, the differential Galois group $\mathrm{Gal}(x)$ is given as a subgroup of $\mathrm{GL}(V)$, whereas in [**Sin93**] this group is only determined up to conjugacy in $\mathrm{GL}(V)$. The bounds B and real algebraic subspaces $\mathcal{L}(n, m, B)$ of $\mathcal{L}(n, m)$ are not present in our proof. The prescribed local connections and the type of the vector bundle $\mathcal{M}$ provide the necessary bounds on the degrees of ∇-invariant line bundles. The "constructions of linear algebra", needed in the proof, are rather involved for differential operators (especially when one has to produce another "cyclic vector"). Here the constructions are the natural ones, known for differential modules. Our proof can be adapted to the case where the singular points are not fixed. However we prefer to avoid the technical complications introduced by "moving singularities". Finally, Singer's result applies to certain sets of differential operators. It seems possible to make a translation between these sets of differential operators and our families of connections on $\mathbb{P}^1$, allowing this time moving singularities.

4. Proof of Singer's theorem for families

4.1. The set $X(\subset G)$ is closed. — As before, we denote by V_b^a the tensor product of a copies of the dual V^* of V and of b copies of V. One considers a subspace W of dimension d of a finite sum $\oplus_i V_{b_i}^{a_i}$. Then $G := \{g \in \mathrm{GL}(V) | \ gW = W\}$ is an algebraic subgroup of $\mathrm{GL}(V)$. According to Chevalley's theorem, every algebraic subgroup of $\mathrm{GL}(V)$ has this form. Put $Z := \Lambda^d(\oplus_i V_{b_i}^{a_i})$ and $L := \Lambda^d W$. Then G is equal to $\{g \in \mathrm{GL}(V) | \ gL = L\}$, too. The subgroups of $\mathrm{GL}(V)$, conjugated to G, are the stabilizers of the lines $hL \subset Z$ with $h \in \mathrm{GL}(V)$. This family of lines in Z is a constructible subset of $\mathbb{P}(Z)$. Write $L = Cz_0$. Then the set $\{hz_0 | \ h \in \mathrm{GL}(V)\}$ is also constructible. Indeed, the action of $\mathrm{GL}(V)$ on Z and $\mathbb{P}(Z)$ is algebraic.

Proposition 4.1. — *Let a family of differential equations on the projective line, parametrized by X, be given. Then $X(\subset G)$ is closed.*

Proof. — We have to extend the proofs of [**Ber02**] to the present more general situation. We suppose that X is reduced, irreducible and affine. Furthermore, we will suppose (as we may) that $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$. Let G be given as above as the stabilizer of a (special) line L in a construction Z. Each step in the construction of Z can be supplemented by a new family of differential equations parametrized by the same X. Indeed, for the dual V^* one constructs from the given family, a new family obtained by taking everywhere duals. This works well since the free vectorbundle $\mathcal{O}_{\mathbb{P}^1_C} \otimes V$ has an obvious dual $\mathcal{O}_{\mathbb{P}^1_C} \otimes V^*$. For a tensor product, like V_b^a, one can form the tensor product of the corresponding vector bundles (including their connections and the local data). Direct sums and exterior powers are treated in the obvious way. Thus we find a family, parametrized by X and corresponding to Z, consisting of a free vector bundle $\mathcal{N}$, identified with $\mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes Z)$, a connection ∇ on $\mathcal{N}$ and a new finite set of prescribed semi-simple formal connections over $C[\![u]\!]$. Then, according to [**Ber02**][Lemma 4.0.3], the set $X(\subset G)$ consists of the closed points x such that there is a line bundle $\mathcal{L}$, contained in $\mathcal{N}(x)$ and satisfying:

(i) $\mathcal{L}$ is invariant under $\nabla(x)$,
(ii) $\mathcal{N}(x)/\mathcal{L}$ is again a vector bundle,
(iii) $\mathcal{L}_0/z\mathcal{L}_0$ is equal to L.

We follow closely the proof of [**Ber02**][Theorem 4.2]. Write $L = Cv_0$. Let $-d \leq 0$ denote the degree of a putative $\mathcal{L}$. Then one finds an equation for the generator $v_0 + v_1 z + \cdots + v_d z^d$ of $\mathcal{L}(d \cdot [\infty])$ (see the proof of [**Ber02**, Theorem 4.2]). This equation has the form

$$\left(\frac{d}{dz} + \sum_{i=1}^{s}\sum_{j=1}^{k} \frac{B_{i,j}(x)}{(z-a_i)^j} - T\right)\left(\sum_{i=0}^{d} v_i z^i\right) = 0,$$

where the $B_{i,j}$ are endomorphisms of $O(X) \otimes Z$; $B_{i,j}(x)$ is the evaluation of $B_{i,j}$ at x, and $T := \sum \frac{g_{i,j}}{(z-a_i)^j}$ with $g_{i,j} \in C$. We note that T does not depend on $x \in X(C)$.

There are finitely many possibilities for T and each possibility yields at most one value for d (see [**Ber02**, Lemma 4.0.4] and Definition 3.4). Now we consider a fixed choice for the term T. The equation

$$\Big(\frac{d}{dz}+\sum_{i=1}^{s}\sum_{j=1}^{k}\frac{B_{i,j}}{(z-a_i)^j}-T\Big)\Big(\sum_{i\geq 0}v_iz^i\Big)=0,$$

with the prescribed $v_0\in Z$ and $v_i\in O(X)\otimes Z$ for $i\geq 1$ has a unique solution (which is denoted by the same symbols). One can see v_i, for $i\geq 1$, as a morphism from X to Z. This determines a closed subset, say $X(T)$ of X, defined by $v_i(x)=0$ for $i>d$. In other words, $X(T)$ is the intersection $\cap_{i>d}v_i^{-1}(0)$. Finally, $X(\subset G)$ is the union of the finitely many closed sets $X(T)$. □

Corollary 4.2. — *Let a family $(\mathcal{M},\nabla,V,\{\nabla_i\}_{i\in I})$ of differential equations, parametrized by X, be given.*

(1) *Consider a vector space Z of the form $\Lambda^d(\oplus_i V_{b_i}^{a_i})$ and a constructible subset S of $Z\setminus\{0\}$. The set of the closed points $x\in X(C)$ such that* $\mathrm{Gal}(x)\subset\mathrm{GL}(V)$ *fixes a line $Cs\subset Z$ with $s\in S$ (for the induces action of* $\mathrm{Gal}(x)$ *on Z), is constructible.*

(2) *Let G be an algebraic subgroup of* $\mathrm{GL}(V)$. *The set of the closed points $x\in X(C)$, such that* $\mathrm{Gal}(x)$ *lies in a conjugate of G, is constructible.*

Proof

(1) As in the proof of Proposition 4.1, one supposes that $\mathcal{M}$ is equal to $\mathcal{O}_X\otimes(\mathcal{O}_{\mathbb{P}^1_C}\otimes V)$. There is an induced family $(\mathcal{N},\nabla,Z,\text{local data})$. As in that proof, a fixed choice for the term T is made. The element v_0 is not fixed but lies in a given constructible subset S of $Z\setminus\{0\}$. The elements v_i with $i\geq 1$ are now viewed as morphisms $S\times X\to Z$. The set $\cap_{i>d}v_i^{-1}(0)$ is a closed subset of $S\times X$. Its image $X(T,S)$, under the projection $S\times X\to X$, is constructible. The union of the finitely many $X(T,S)$ is the set of the closed points x such that $\mathrm{Gal}(x)\subset\mathrm{GL}(V)$ fixes, for its action on Z, a line L of the form $L=Cs$ with $s\in S$.

(2) Take Z as in (1) and a line $L\subset Z$ such that $G=\{g\in\mathrm{GL}(V)|\ gL=L\}$. Write $L=Cv_0$. Then (1), applied to the constructible set $S=\{hv_0|\ h\in\mathrm{GL}(V)\}$, yields (2). □

4.2. Galois invariant subspaces and subbundles. — Let a family of differential equations $(\mathcal{M},\nabla,V,\{\nabla_i\})$, parametrized by a reduced, irreducible, affine X be given. Let W be a subspace of V such that W is invariant under all $\mathrm{Gal}(x)$. Our aim is to prove that $\mathcal{M}$ has a subbundle, invariant under ∇, corresponding to W. We start by discussing the special case where $W=Ce$ (with $e\neq 0$) and we suppose (as we may) that $\mathcal{M}$ is equal to $\mathcal{O}_X\otimes(\mathcal{O}_{\mathbb{P}^1_C}\otimes V)$. Then $\nabla_{\frac{d}{dz}}$ has the explicit form

$$\frac{d}{dz}+A=\frac{d}{dz}+\sum_{i,j}\frac{A_{i,j}}{(z-a_j)^i},$$

where the $A_{i,j}$ are $O(X)$-linear endomorphisms of $O(X)\otimes_C V$ and $\sum_j A_{1,j}=0$. We return to the proof and the terminology of Proposition 4.1 and [**Ber02**, Theorem 4.2].

For a fixed $x\in X(C)$, there is a term $T=\sum_{i,j}\frac{g_{i,j}}{(z-a_j)^i}$ with all $g_{i,j}\in C$ such that $\sum_j g_{1,j}$ is an integer $d\geq 0$ and there is a solution $v_0+v_1z+\cdots+v_dz^d$ of $\big(\frac{d}{dz}+A(x)\big)(v_0+v_1z+\cdots+v_dz^d)=T(v_0+v_1z+\cdots+v_dz^d)$, such that $v_0=e$ and $v_d\neq 0$. Moreover, there are only finitely many possibilities for T. Now we fix T and consider the equation $\big(\frac{d}{dz}+A\big)\Big(\sum_{i\geq 0}v_iz^i\Big)=T\Big(\sum_{i\geq 0}v_iz^i\Big)$ with $v_0=e$ and $v_i\in O(X)\otimes V$ for $i\geq 1$. This equation has a unique solution. The closed subset of X given by $v_i(x)=0$ for $i>d$, is denoted by $X(T)$. By assumption X is the union of the finitely many sets $X(T)$. Since X is irreducible, X is equal to a single $X(T)$. We continue with this T.

Let $v_0+v_1z+\cdots+v_dz^d$ denote the solution corresponding to this T (with again $v_0=e$ and $v_i\in O(X)\otimes_C V$ for $i\geq 1$). It is, a priori, possible that v_d is identical zero. Let ℓ be maximal such that v_ℓ is not identical zero. It is also possible that $v_0+v_1z+\cdots+v_\ell z^\ell$ is divisible by some $(z-a_j)$. We divide $v_0+v_1z+\cdots+v_\ell z^\ell$ by $(z-a_1)^{m_1}\cdots(z-a_s)^{m_s}$ with $m_1,\ldots,m_s\geq 0$ as large as possible (this changes the T as well). The result is a section, say $v_0+w_1z+\cdots+w_rz^r$, of $\mathcal{M}(r\cdot[\infty])$ such that none of the expressions w_r and $v_0+w_1a_j+\cdots+w_ra_j^r$ for $j=1,\ldots,s$, is identical zero. Let X' be the open, non-empty, subset of X given by $w_r(x)\neq 0$ and the $v_0+w_1(x)a_j+\cdots+w_r(x)a_j^r\neq 0$ for $j=1,\ldots,r$. We claim that the section $v_0+w_1z+\cdots+w_rz^r$ of $\mathcal{M}(r\cdot[\infty])$ does not vanish on $X'\times\mathbb{P}^1_C$. For points (x,∞) or (x,a_j) with x a closed point of X', this is obvious. For a point (x,a) with $a\notin\{a_1,\ldots,a_s,\infty\}$ and $x\in X'(C)$, the expression $v_0+w_1(x)z+\cdots+w_r(x)z^r$ is a solution of the differential operator $\frac{d}{dz}+A(x)-T$. Since this operator is regular at a, the vanishing of $v_0+w_1(x)a+\cdots+w_r(x)a^r$ implies that $v_0+w_1(x)z+\cdots+w_r(x)z^r$ is identical zero. This contradicts $w_r(x)\neq 0$.

In what follows, X is already replaced with a non-empty open subset of X'. In the next steps, we will shrink X even further. Let $\mathcal{F}=\mathcal{O}_X\otimes\mathcal{O}_{\mathbb{P}^1_C}$. The line bundle $\mathcal{F}$ is embedded into $\mathcal{M}(r\cdot[\infty])$ by sending the global section 1 of $\mathcal{O}_{\mathbb{P}^1_C}$ to $v_0+w_1z+\cdots+w_rz^r$. This induces a connection on $\mathcal{F}$ and local data for $\mathcal{L}$. Moreover, we identify $(\mathcal{O}_{\mathbb{P}^1_C})_0/(z)$ with Cv_0, by sending 1 to $v_0=e$. Now we consider $\mathcal{L}:=\mathcal{F}(-r\cdot[\infty])=\mathcal{O}_X\otimes\mathcal{O}_{\mathbb{P}^1_C}(-r\cdot[\infty])$. The above data make $(\mathcal{L},\nabla,Cv_0,\text{local data})$ into a family, parametrized by X.

The quotient $\mathcal{Q}:=\mathcal{M}/\mathcal{L}$ is a vector bundle on $\mathbb{P}^1_X$ with an induced connection and induced local data. After shrinking X, there exists a vector bundle $\mathcal{N}$ on $\mathbb{P}^1_C$ such that $\mathcal{Q}=\mathcal{O}_X\otimes\mathcal{N}$. A choice of an isomorphism $\lambda:\mathcal{N}_0/(z)\to V/Ce$ induces an isomorphism $\mathcal{Q}_0/(z)\to\mathcal{O}_X\otimes(V/Ce)$. We require that this map is induced by the given isomorphism $\mathcal{M}_0/(z)\to\mathcal{O}_X\otimes V$.

For every closed point x of X, there is an induced exact sequence of connections $0\to\mathcal{L}(x)\to\mathcal{M}(x)\to\mathcal{Q}(x)\to 0$ on $\mathbb{P}^1_C$. The action of $\mathrm{Gal}(x)$ on V induces the actions on Ce and V/Ce for the connections $\mathcal{L}(x)$ and $\mathcal{Q}(x)$. We come now to the general result.

Proposition 4.3. — *Let $(\mathcal{M}, \nabla, V, \{\nabla_i\})$ be a family, parametrized by a reduced, irreducible scheme X of finite type over X. Let $W \subset V$ be a proper subspace such that W is invariant under* $\mathrm{Gal}(x)$ *for all $x \in X(C)$.*

Then, after repacing X with a non-empty open subset, there exists family $(\mathcal{N}, \nabla^*, W,$ local data$)$, parametrized by X, such that:

(i) *$\mathcal{N}$ is a subbundle of $\mathcal{M}$, invariant under ∇. Moreover, ∇^*, the local data of $\mathcal{N}$ and the isomorphism $\mathcal{N}_0/(z) \to \mathcal{O}_X \otimes W$ are induced by those of $\mathcal{M}$.*

(ii) *The sheaf $\mathcal{Q} := \mathcal{M}/\mathcal{N}$ is a vector bundle on $\mathbb{P}^1_X$, isomorphic to $\mathcal{O}_X \otimes \mathcal{S}$ for a suitable vector bundle $\mathcal{S}$ on $\mathbb{P}^1_C$. Moreover, $\mathcal{Q}$ can be made into a family, parametrized by X, with connection, local data, and isomorphism $\mathcal{Q}_0/(z) \to \mathcal{O}_X \otimes (V/W)$, induced by those of the family $\mathcal{M}$.*

(iii) *For every closed point $x \in X(C)$, the exact sequence*

$$0 \longrightarrow \mathcal{N}(x) \longrightarrow \mathcal{M}(x) \longrightarrow \mathcal{Q}(x) \longrightarrow 0 \text{ of connections on } \mathbb{P}^1_C\ ,$$

has the property that the action of $\mathrm{Gal}(x)$ *on V induces the actions of the differential Galois groups on W and V/W, that are produced by $\mathcal{N}(x)$ and $\mathcal{Q}(x)$.*

Proof. — As before we may suppose that $\mathcal{M} = \mathcal{O}_X \otimes (\mathcal{O}_{\mathbb{P}^1_C} \otimes V)$. Put $d = \dim W$. The case $d = 1$ is discussed above. For the general case one considers $L = \Lambda^d W \subset \Lambda^d V$ and the family $(\Lambda^d \mathcal{M}, \dots)$ associated to $\Lambda^d V$. One finds a line bundle $\mathcal{L} \subset \Lambda^d \mathcal{M}$ (above a suitable open subset of X) with the required properties. This line bundle is decomposable since the line $L \subset \Lambda^d V$ is decomposable. Thus there exists a vector bundle $\mathcal{N} \subset \mathcal{M}$ (above a suitable open subset of X) with $\Lambda^d \mathcal{N} = \mathcal{L}$ and $\mathcal{N}$ has the required properties. In particular, $\mathcal{Q}$ is a connection on $\mathbb{P}^1_X$. It is not difficult to provide $\mathcal{N}$ and $\mathcal{Q}$ with the additional structure, which makes them into families, parametrized by X. This proves (i) and (ii). Part (iii) follows from the explicit construction. □

Proposition 4.3 is a sort of converse of Lemma 3.2. Indeed, let K denote the function field of X. The assumption that W is invariant under all $\mathrm{Gal}(x)$ implies that the differential Galois group $H \subset \mathrm{GL}(K \otimes V)$ of the generic differential equation on $\mathrm{Spec}(K) \otimes \mathbb{P}^1_C$ leaves the subspace $K \otimes_C W$ invariant.

4.3. Constructions of linear algebra. — Let H be an algebraic subgroup of $\mathrm{GL}(V)$. In other words, V is a faithful H-module. Let W be another H-module. It is well known that W can be obtained by a "construction of linear algebra" from V. Explicitly, $W \cong W_2/W_1$, where $W_1 \subset W_2$ are H-invariant subspaces of a finite direct sum $\oplus_i V_{n_i}^{m_i}$.

Proposition 4.4. — *Let a family* $(\mathcal{M}, \nabla, V, \{\nabla_i\})$, *parametrized by a reduced, irreducible scheme* X *of finite type over* C, *be given. Let* H *be an algebraic subgroup of* $\mathrm{GL}(V)$ *and suppose that* $\mathrm{Gal}(x) \subset H$ *for every closed point* $x \in X$. *For any construction of linear algebra* $W := W_2/W_1$, *as above, there exists a family* $(\mathcal{N}, \nabla, W, \textit{local data})$, *parametrized by a non-empty open subset* U *of* X *such that:*

(i) *For every closed point* $x \in U(C)$, *the connection* $(\mathcal{N}(x), \nabla(x))$ *on* $\mathbb{P}^1_C$ *is obtained by the same construction.*
(ii) *The action of* $\mathrm{Gal}(x)$ *on* W, *induced by the construction of linear algebra, coincides with the action of the differential Galois group of the connection* $\mathcal{N}(x)$ *on* W.

Proof. — We may suppose that $\mathcal{M}$ is free (at the cost of enlarging the set of singular points and the local data). For an H-module of the form $\tilde{V} = \oplus_i V_{b_i}^{a_i}$ the construction of the new family, parametrized by X, is discussed before. For a H-submodule W_2 we apply Proposition 4.3 and we have to replace X with an open subset of X. For a H-submodule W_1 of W_2 one applies Proposition 4.3 again. The result is a family, parametrized by an open subset of X, corresponding to the H-module W_2/W_1. The construction of $(\mathcal{N}, W, \dots)$ implies at once the properties (i) and (ii). □

4.4. The set $X(U(G^o) \subset)$ is constructible. — We introduce some notation. Let H be a linear algebraic group over C acting upon a finite dimensional vector space W over C. For every character $\chi : H \to \mathbb{G}_m = C^*$ one defines $W_\chi := \{w \in W \mid hw = \chi(h)w \text{ for all } h \in H\}$. This is a subspace of W. Let $\chi_1, \dots, \chi_r$ denote the distinct characters of H such that $W_{\chi_i} \neq 0$. Then $\sum_{i=1}^r W_{\chi_i} \subset W$ is in fact a direct sum $\oplus_{i=1}^r W_{\chi_i}$. This space is denoted by $\mathrm{Ch}_H(W)$.

As before, an algebraic subgroup $G \subset \mathrm{GL}(V)$ is given. The group $U(G^o) = U(G)$ denotes, as before, the algebraic subgroup of G generated by all the unipotent elements of G. Any character of G^o is trivial on $U(G^o)$ and $G^o/U(G^o)$ is a torus. It easily follows that for any G-module W one has $\mathrm{Ch}_{G^o}(W) = W^{U(G^o)}$ (i.e., the set of $U(G^o)$-invariant elements $w \in W$). An essential result is the following.

***Theorem 4.5* (M.F. Singer)**. — *There exists a faithful* G*-module* W *such that for every algebraic subgroup* H *of* G *the following statements are equivalent.*

(1) $U(G^o) \subset H$.
(2) $\mathrm{Ch}_{G^o}(W) = \mathrm{Ch}_{H \cap G^o}(W)$.

We note that $\mathrm{Ch}_{G^o}(W) \subset \mathrm{Ch}_{H\cap G^o}(W)$ is valid for any G-module W. Moreover, for any G-module W, the implication (1)⇒(2) holds. Indeed, $U(G^o) \subset H$ implies that $U(G^o) \subset H^o \subset H \cap G^o$. Since $G^o/U(G^o)$ is a torus, one has $U(G^o) = U(H^o)$. Hence

$$\mathrm{Ch}_{H\cap G^o}(W) \subset \mathrm{Ch}_{H^o}(W) = W^{U(H^o)} = W^{U(G^o)} = \mathrm{Ch}_{G^o}(W).$$

For the rather involved proof of the existence of a faithful G-module W for which the implication (2)⇒(1) holds, we refer to [**Sin93**].

Corollary 4.6. — *Put $m := [G : G^o]$. For the faithful G-module W of Theorem 4.5 the following statements for any algebraic subgroup H of G are equivalent.*

(i) $U(G^o) \subset H$.

(ii) *For every $r \leq m^m$ and for every H-invariant decomposable line $L = Cu_1 \otimes u_2 \otimes \cdots \otimes u_r \subset \mathrm{Sym}^r W$, the elements $u_1, \ldots, u_r$ belong to $\mathrm{Ch}_{G^o}(W)$.*

Proof. — (i)⇒(ii). As remarked above, the implication (1)⇒(2) in Theorem 4.5 holds for every G-module. This implication, applied to the symmetric power $\mathrm{Sym}^r W$, yields that $u_1 \otimes \cdots \otimes u_r$ lies in $(\mathrm{Sym}^r W)^{U(G^o)}$. Let $x_1, \ldots, x_n$ denote a basis of W over C. The algebra $\oplus_{m\geq 0}\mathrm{Sym}^m W$ is identified with $C[x_1, \ldots, x_n]$. The group G acts linearly on $C[x_1, \ldots, x_n]$ and the element $u := u_1 \otimes \cdots \otimes u_r$ is a homogeneous polynomial which is a product of homogeneous linear terms. From the $U(G^o)$-invariance of u, the connectedness of $U(G^o)$ and the unicity of the decomposition of u (up to scalars and order) , one deduces that $g(u_i)$ is a C^*-multiple of u_i for every $g \in U(G^o)$ and every i. Then all u_i are invariant under $U(G^o)$, since $U(G^o)$ is generated by unipotent elements.

(ii)⇒(i). We will show that (ii) implies condition (2) of Theorem 4.5. It suffices to show that any $H \cap G^o$-invariant line $Cu \subset W$ belongs to $\mathrm{Ch}_{G^o}(W)$. The group $H \cap G^o$ is a subgroup of H of index at most $m := [G : G^o]$. There is a normal subgroup $\tilde{H}$ of H contained in $H \cap G^o$, such that $[H : \tilde{H}] \leq m^m$. Let $1 = h_1, \ldots, h_r$ denote representatives of $H/\tilde{H}$. Then the line $h_1 u \otimes h_2 u \otimes \cdots \otimes h_r u \in \mathrm{Sym}^r W$ is decomposable and invariant under H. By (ii), $u \in \mathrm{Ch}_{G^o}(W)$. □

Proposition 4.7. — *Let a family $(\mathcal{M}, \nabla, V, \{\nabla_i\})$, parametrized by an irreducible, reduced X, be given. Let G be an algebraic subgroup of $\mathrm{GL}(V)$. Suppose that $\mathrm{Gal}(x) \subset G$ holds for every closed point x of X. There exists an open non-empty subset X' such that the set $X'(U(G^o) \subset)$ is constructible.*

Proof. — As always, we may suppose that $\mathcal{M}$ is free. Let W be the G-module having the properties of Theorem 4.5 and Corollary 4.6. By Proposition 4.4, there corresponds to W a family $(\mathcal{N}, \nabla, W, \ldots)$, parametrized by an open non-empty subset X' of X. Again we may suppose that $\mathcal{N}$ is free. Consider some integer r with $1 \leq r \leq m^m$, where $m := [G : G^o]$. The set $S(r)$ of elements $u = u_1 \otimes \cdots \otimes u_r \in \mathrm{Sym}^r W$ with all $u_i \neq 0$, and not all u_i belonging to $\mathrm{Ch}_{G^o}(W)$, is constructible. By part (1) of Corollary 4.2, the set $X'(r)$, consisting of the closed points $x \in X'(C)$ such that $\mathrm{Gal}(x)$ fixes a line $Cu \subset \mathrm{Sym}^r W$ with $u \in S(r)$, is constructible. $X'(U(G^o) \subset)$ is constructible since it is, by Corollary 4.6, the complement in X' of $\bigcup_{1\leq r\leq m^m} X'(r)$. □

4.5. The final step, involving the Singer condition. — As before, an algebraic subgroup $G \subset \mathrm{GL}(V)$ is given. We suppose that G satisfies the "Singer condition". Let a family $\mathcal{F} := (\mathcal{M}, \nabla, V, \text{local data})$, parametrized by an irreducible, reduced X, be given. We will show, by induction on the dimension of X, that $X(= G)$ is constructible.

We have shown that there exists an open non-empty $X' \subset X$ such that $X'(U(G^o) \subset, \subset G)$ is constructible. By induction, $\{x \in X \setminus X' \mid \mathrm{Gal}(x) = G\}$ is constructible. After replacing X with an irreducible component of $X'(U(G^o) \subset, \subset G)$, one has $U(G^o) \subset \mathrm{Gal}(x) \subset G$ for all $x \in X$.

Consider a faithful $G/U(G^o)$-module W. The family $\mathcal{F}$ induces a family $\mathcal{G} := (\mathcal{N}, \nabla, W, \text{ local data})$, parametrized by X. For every $x \in X(C)$, one has $\mathrm{Gal}(x) \subset G/U(G^o)$. For the family $\mathcal{G}$, we have to prove that $X(= G/U(G^o))$ is constructible. We change the notation and write G for $G/U(G^o)$ and V for W. If G is finite, then an application of Proposition 4.1 finishes the proof. If G is infinite, then G^o is a torus and G^o lies in the center of G (this is precisely the Singer condition).

We continue the proof. For a closed point x and a singular point a_j one obtains a differential module $\mathcal{M}(x, a_j) := C((z - a_j)) \otimes \widehat{\mathcal{M}(x)}_{a_j}$ over the differential field $C((z-a_j))$. Let $PVF(x, a_j)$ denote a Picard–Vessiot field for this differential module. The formal local Galois group $\mathrm{Gal}(x, a_j)$ is the group of the differential automorphisms of $PVF(x, a_j)/C((z - a_j))$. Let $PVF \supset C(z)$ denote the Picard-Vessiot field for the generic differential module $\mathcal{M}(x)_\xi$ over $C(z)$. The differential Galois group $\mathrm{Gal}(x)$ is the group of the differential automorphisms of $PVF/C(z)$. This group is canonical embedded into $\mathrm{GL}(V)$ by our constructions. There exists a $C(z)$-linear embedding $PVF \subset PVF(x, a_j)$. This induces an injective algebraic homomorphism $\mathrm{Gal}(x, a_j) \to \mathrm{Gal}(x)$. Another embedding changes this homomorphism by conjugation (with an element in $\mathrm{Gal}(x)$). The connected component of the identity $\mathrm{Gal}(x, a_j)^o$ is mapped to a subgroup of $\mathrm{Gal}(x)^o \subset G^o$, and lies therefore in the center of G and $\mathrm{Gal}(x)$. In particular, the image of $\mathrm{Gal}(x, a_j)^o$ in G does not depend on the chosen embedding $PVF \to PVF(x, a_j)$.

We note that the local connection $\mathcal{M}(x, a_j)$ is semi-simple since the formal local differential Galois group does not contain $\mathbb{G}_a$. Therefore there are finitely many possibilities for the equivalence class of $\mathcal{M}(x, a_j)$. It is easily seen that this equivalence class depends in a constructible way on x. Therefore there exists an open non-empty subset of X, where the equivalence classes of $\mathcal{M}(x, a_j)$ does not depend on x. After restricting to this open subset, all the differential modules $\mathcal{M}(x, a_j)$ are isomorphic. In particular, $PVF(x, a_j)$ and $\mathrm{Gal}(x, a_j)$ do not depend on x. We will write $PVF(a_j)$ and $\mathrm{Gal}(a_j)$ for these objects. For a fixed embedding $PVF \to PVF(a_j)$, one has a fixed image of the groups $\mathrm{Gal}(x, a_j) = \mathrm{Gal}(a_j)$ into $\mathrm{Gal}(x)$. Moreover, the image of $\mathrm{Gal}(x, a_j)^o$ into $\mathrm{Gal}(x)$ does not depend on any choice and is independent of x.

Let $H \subset G^o$ denote the subgroup, generated by the images of all $\mathrm{Gal}(a_j)^o$. Then H does not depend on x and H is a connected normal subgroup of G. Now we take a faithful G/H-module W and its corresponding family, parametrized by a non-empty open subset X' of X. For notational convenience, we replace G with G/H. For this new family, parametrized by X', one has:

(i) the differential Galois groups are contained in G,
(ii) the formal local differential Galois groups are finite,
(iii) the singularities are regular singular,
(iv) the group $\mathrm{Gal}(x)$ is generated (as an algebraic group) by the finite local differential Galois groups.

We have to show that $X'(=G)$ is constructible.

By [BS64] Lemme 5.11 (also known as Platonov's Theorem), there is a finite subgroup $E \subset G$ that maps surjectively to G/G^o. The surjective map $\tilde{G} := G^o \times E \to G$ has a finite kernel. The group $\tilde{G}$ has the property: any subgroup generated by s subgroups, each one of order bounded by some D, is finite (and in fact contained in $G^o[m] \times E$ for a suitable m depending in D). Thus the same statement holds for G. It follows that all $\mathrm{Gal}(x)$ are finite. If $G^o \neq \{1\}$, then $X'(=G) = \varnothing$. If $G^o = \{1\}$, then G is finite and therefore $X'(=G)$ is constructible.

5. Non-constructible sets $X(=G)$

The aim of this section is to produce for any linear algebraic G that does not satisfy the "Singer condition", a family of differential equations, parametrized by some X, such that $X(=G)$ is not constructible. We start by investigating a rather special case namely, G is a semi-direct product $G = T \rtimes E$. Here E is a finite group and T is a torus. Furthermore, there is given a homomorphism of groups $\psi : E \to \mathrm{Aut}(T)$. The group structure of G is then defined by the formula $ete^{-1} = \psi(e)(t)$. The induced action ϕ of E on the character group $X(T)$ of T, is given by the formula $(\phi(e)\chi)(t) = \chi(e^{-1}te)$.

Lemma 5.1. — *The following properties of $G = T \rtimes E$ are equivalent.*

(i) $\sum_{e \in E} \mathrm{im}(\phi(e) - 1)$ *has finite index in* $X(T)$.
(ii) $\bigcap_{e \in E} \ker(\phi(e) - 1) = 0$.
(iii) *The E-module $X(T) \otimes \mathbb{Q}$ does not contain the trivial representation.*
(iv) *The center of G is finite.*

Proof. — The vector space $X(T) \otimes \mathbb{Q}$ is an E-module and can be written as a direct sum of irreducible E-modules $I_1, \ldots, I_r$. Consider a non-trivial irreducible representation $\rho : E \to \mathrm{GL}(W)$ over $\mathbb{Q}$. Then the submodule $\sum_{e \in E} \mathrm{im}(\rho(e) - 1)$ of W is not zero and hence equal to W. Moreover, $\bigcap_{e \in E} \ker(\rho(e) - 1)$ is a proper submodule of W and hence equal to $\{0\}$. For the trivial, 1-dimensional representation $\rho : E \to \mathrm{GL}(\mathbb{Q})$, one has that $\sum_{e\in E}\operatorname{im}(\rho(e)-1)=0$ and $\bigcap_{e\in E}\ker(\rho(e)-1)=\mathbb{Q}$. This proves the equivalence of (i),(ii) and (iii).

The elements of T can be considered as group homomorphisms $t : X(T) \to C^*$. If t lies in the center of G then $\chi(e^{-1}te) = \chi(t)$ for every χ and every $e \in E$. This translates into: t is equal to 1 on the submodule $\sum_{e\in E}\operatorname{im}(\phi(e)-1)$. This proves the equivalence of (i) and (iv). □

Lemma 5.2. — *As above $G = T \rtimes E$. Suppose that $X(T)\otimes\mathbb{Q}$ is a non-trivial irreducible E-module. Let H be an algebraic subgroup of G which maps surjectively to E. Then:*

(i) *If $H \neq G$, then there exists an integer $n \geq 1$ such that $H \subset T[n] \rtimes E$. Here $T[n]$ denotes the subgroup of T consisting of the elements with order dividing n.*

(ii) *Let $e \in E$ have order $m > 1$ and let $t \in T$ be given as a homomorphism $t : X(T) \to X(T)/\ker(\phi(e)-1) \to C^*$. Then $(te)^m = 1$.*

(iii) *There exist integers $N, M \geq 1$ and subgroups $G_n \subset T[n] \rtimes E$ for all $n \geq 1$ such that the following holds.*
   (a) *The index of G_n in $T[n] \rtimes E$ is bounded by a constant independent of n.*
   (b) *G and every G_n is generated, as an algebraic subgroup, by N elements of order $\leq M$.*

Proof

(i) Since $H \to E$ is surjective, $H \cap T$ and the subtorus $(H\cap T)^o$ of T are invariant under the action of E on T. There exists a unique submodule $N \subset X(T)$ such that $X(T)/N$ has no torsion and $(H\cap T)^o$ consists of the homomorphisms $t : X(T) \to C^*$ which are 1 on N. If $N = X(T)$, then H is finite and clearly contained in $T[n] \rtimes E$ for some $n \geq 1$. If $N \neq X(T)$, then $N = 0$ and $H = G$.

(ii) One verifies that

$$(te)^m = t\cdot\psi(e)(t)\cdot\psi(e^2)(t)\cdots\psi(e^{m-1})(t).$$

For any character χ one finds

$$\chi((te)^m) = \chi(t)\cdot(\phi(e^{-1})\chi)(t)\cdots(\phi(e^{-m+1}\chi)(t).$$

Therefore we have to show that t has value 1 on the submodule $(1+\phi(e^{-1})+\cdots+\phi(e^{-m+1}))X(T)$ of $X(T)$. Since this submodule is contained in $\ker(\phi(e^{-1})-1) = \ker(\phi(e)-1)$, one concludes that $(te)^m = 1$.

(iii) For G one takes as set of generators E and an element te, with $e \in E$ of order m, $t \in T$ of infinite order and te of order m. It follows from (i) that G is generated, as an algebraic subgroup, by this set.

Consider an integer $n > 1$. Let G_n be the subgroup of $T[n] \rtimes E$ generated by E and for every $e \in E$ a collection of products te, with $t \in T$, that we now describe. Let $e \in E$ have order $m > 1$. Take a $\mathbb{Z}$-basis $b_1, \dots, b_r$ of $X(T)/\ker(\phi(e)-1)$ and define the homomorphisms $h_1, \dots, h_r : X(T)/\ker(\phi(e)-1) \to C^*$ by $h_i(b_j) = 1$ if $i \neq j$ and $h_i(b_i) = \zeta_n$ for $i = 1, \dots, r$ and with ζ_n a fixed nth root of unity. The t_ie

that we use as generators of G_n are $t_i : X(T) \to X(T)/\ker(\phi(e)-1) \xrightarrow{h_i} C^*$. Part (b) is clear. For the proof of part (a) we consider the obvious map $\alpha : X(T) \to M := \oplus_{e\in E} X(T)/\ker(\phi(e)-1)$. This map is injective by Lemma 5.1. For every homomorphism $h : M \to \mu_n$, (here μ_n denotes the group of the nth roots of unity), the element $t = h \circ \alpha$ belongs to G_n. Let N denote the smallest submodule of M such that im $\alpha \subset N$ and M/N has no torsion. The image of $\mathrm{Hom}(M,\mu_n) \to \mathrm{Hom}(X(T),\mu_n) = T[n]$ is contained in G_n. Further, $\mathrm{Hom}(M,\mu_n) \to \mathrm{Hom}(N,\mu_n)$ is surjective. Now (a) follows from $[N : \text{im } \alpha] < \infty$. □

Proposition 5.3. — *Suppose that C is the field of the complex numbers $\mathbb{C}$. Let $G = T \rtimes E$ and suppose that $X(T)\otimes\mathbb{Q}$ is a non-trivial irreducible E-module. There is a moduli space $\mathbb{M}$ such that $\mathbb{M}(=G)$ is not constructible.*

Proof. — Let $G \subset \mathrm{GL}(V)$ be a faithful irreducible representation. Fix a finite subset $\{a_1,\dots,a_s\}$ of $\mathbb{C}^*$ and integers $d_i > 1$ for $i = 1,\dots,s$. Let π_1 denote the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \{a_1,\dots,a_s\}$ with base point 0. Take loops $\lambda_1,\dots,\lambda_s \in \pi_1$ around the s points such that π_1 is generated by $\lambda_1,\dots,\lambda_s$ and such that the only relation between these generators is $\lambda_1\cdots\lambda_s = 1$. From Lemma 5.2 it follows that for a suitable choice of s and the d_i, there exist homomorphisms $\rho, \rho_n : \pi_1 \to G \subset \mathrm{GL}(V)$ with the following properties:

(a) $\rho(\lambda_i)$ and the $\rho_n(\lambda_i)$ have order d_i (for $i = 1,\dots,s$),
(b) the image of ρ is Zariski dense in G and $G_n = \text{im } \rho_n$ for every n.

The Riemann-Hilbert correspondence attaches to each ρ_n a differential module $M_n \cong \mathbb{C}(z)\otimes V$ over $\mathbb{C}(z)$ (unique up to conjugation). For each M_n and each i one chooses the local data at a_i of the form $\frac{d}{d(z-a_i)} + \frac{A_i}{z-a_i}$, where A_i is a diagonal map with diagonal entries in $[0,1)\cap\mathbb{Q}$ (independent of n) and such that the local monodromy has order d_i. This defines a unique connection $(\mathcal{M}_n,\nabla)$ with generic differential module M_n. Now $\mathcal{M}_n$ is in general not free, but has the form $\mathcal{O}(k_1)\oplus\cdots\oplus\mathcal{O}(k_v)$ with $k_1 \geq \cdots \geq k_v$ and $v := \dim V$. The sum $k_1+\cdots+k_v$ is fixed since the local exponents of $\Lambda^v\mathcal{M}_n$ are given. Since ρ_n is irreducible the defect of $\mathcal{M}_n$ is uniformely bounded (see [**MvdP03**, Proposition 6.21]). It follows that there is an infinite subset $I \subset \mathbf{N}$ such that $\mathcal{M}_n$ type $k_1 \geq \cdots \geq k_v$ for all $n \in I$. The embedding of V in M_n and the regularity of M_n at the point $z = 0$ yield a canonical isomorphism $\mathbb{C}[z]_{(z)}\otimes V \to (\mathcal{M}_n)_0$. One defines now a moduli problem by fixing the type of the vector bundle $\mathcal{M}$ (namely $k_1 \geq \cdots \geq k_v$), an identification $\mathbb{C}[z]_{(z)} \to \mathcal{M}_0$ and the above local data. There is a universal family, parametrized by a variety $\mathbb{M}$. Then $\mathbb{M}(=G_n)$ is not empty for $n \in I$. We remove from $\mathbb{M}(\subset G)$ the union of the finitely many closed subsets $\mathbb{M}(\subset T \rtimes E')$ with E' a proper subgroup of E. For notational convenience we call the result again $\mathbb{M}(\subset G)$. The set $\mathbb{M}(=G)$ is the complement in $\mathbb{M}(\subset G)$ of the sets $Z_n := \mathbb{M}(\subset T[n!]\rtimes E)$ for $n \geq 1$. It suffices now to show that

$\cup_{n\geq 1} Z_n$ is not constructible. Indeed, $\mathbb{M}(=G)$ is the complement in the closed set $\mathbb{M}(\subset G)$ of the non-constructible set $\cup_{n\geq 1} Z_n$.

By construction, $\{Z_n\}$ is an increasing (not stationary) sequence of closed sets. Suppose that this union is equal to $\cup_{i=1}^d O_i \cap F_i$ with open sets O_i and closed sets F_i's. For some i the sets $Z_n \cap (O_i \cap F_i)$ from again an increasing (not stationary) sequence of closed subsets. After replacing $O_i \cap F_i$ by a suitable irreducible component, say Y, we have an increasing sequence of closed subsets $Y_n = Z_n \cap Y$ with union Y and such that each $Y_n \neq Y$. We may suppose that Y is affine of dimension $d > 0$ and consider a finite morphism $\mathbb{A}^d_{\mathbb{C}} \to Y$. It follows that $\mathbb{A}^d(\mathbb{C})$ is a countable union of proper Zariski closed subsets. This is not possible because the field $\mathbb{C}$ is uncountable. □

Remarks 5.4

(1) The moduli space $\mathbb{M}$ occuring in the proof of Proposition 5.3 is in general not the one studied in detail in [**Ber02**], since the vector bundle $\mathcal{M}$ need not be free. Suppose that one of the local data $\frac{d}{d(z-a_i)} + \frac{A_i}{z-a_i}$ is such that the eigenvalues of A_i have multiplicity 1, then one can change each $\mathcal{M}_n$ (with $n \in I$) into a free vector bundle by shifting the eigenvalues of A_i over integers. There are only finitely many ways to do this. Thus for some infinite subset $I' \subset I$ one single change of A_i will make all $\mathcal{M}_n$ with $n \in I'$ into a free vector bundle. Now one can define the moduli space $\mathbb{M}$ by a free vector bundle $\mathcal{M}$ with $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{M})$ identified with V and with the prescribed local data.

(2) Proposition 5.3 remains valid for the case where C is any uncountable algebraically closed field of characteristic 0. Indeed, one may replace C be a subfield, still uncountable and algebraically closed, of cardinality less than or equal to that of $\mathbb{C}$. Then C is embedded into $\mathbb{C}$. The moduli space $\mathbb{M}$ of the proof descends to C, i.e., $\mathbb{M} = \mathbf{N} \otimes_C \mathbb{C}$ for a suitable space $\mathbf{N}$. The group G is given as an algebraic subgroup of $\mathrm{GL}(V)$ where V is a vector space over C. One easily verifies that $\mathbb{M}(\subset G \otimes_C \mathbb{C}) = \mathbf{N}(\subset G) \otimes_C \mathbb{C}$. The same statement is valid for the groups G_n. It follows that $\mathbf{N}(=G)$ is not constructible.

Suppose that the algebraically closed field C is countable. Then any algebraic variety Z, of finite type over C, is the countable union of its finite (closed) subsets. It seems possible that Proposition 5.3 and Theorem 5.5 do not hold for C. However, Example 2.6 remains valid for this C.

We now give the proof of the general result, omitting some of the more obvious details.

Theorem 5.5. — *Let C be the field of the complex numbers $\mathbb{C}$. Suppose that the linear algebraic group G does not satisfy the Singer condition. Then there is a moduli space $\mathbb{M}$ such that $\mathbb{M}(=G)$ is not constructible.*

Proof. — As we will show, it suffices to prove this theorem for a linear algebraic group G' for which there exists a surjective morphism $G' \to G$ with finite kernel. According to a result of Platonov, there exists a finite subgroup E of G such that $E \to G/G^o$ is surjective. Thus we may replace G with $G^o \rtimes E$. The group $G^o/U(G^o)$ is a torus. We will need the following lemma.

Lemma 5.6. — *(We use the above notations.) There is a torus $T \subset G^o$, invariant under conjugation by the elements of E, such that $T \to G^o/U(G^o)$ is surjective and has finite kernel.*

Proof. — First we will assume that G^o is *reductive*. Then, by [Sp98] Corollary 8.1.6 (G^o, G^o) is semi-simple and $G^o = (G^o, G^o) \cdot R(G^o)$, where $R(G^o)$ is the radical of G^o. By [Sp98] Proposition 7.3.1, $R(G^o)$ is a central torus and $R(G^o) \cap (G^o, G^o)$ is finite. Furthermore, by [Sp98] Theorem 8.1.5, we have $(G^o, G^o) \subset U(G^o)$. The surjectivity of the morphism $R(G^o) \to G^o/(G^o, G^o)$ implies that $G^o/(G^o, G^o)$ is a torus and thus $(G^o, G^o) = U(G^o)$. Now $R(G^o)$ is a characteristic subgroup of G^o and in particular $eR(G^o)e^{-1} = R(G^o)$ for all $e \in E$. Thus we can take $T = R(G^o)$ in this case.

We now consider the general case. We take T to be a maximal torus in $R(G^o)$. We have $R(G^o) = R_u(G^o) \rtimes T$, where $R_u(G^o)$ is the unipotent radical of G^o. The image of $R(G^o)$ under the map $\pi : G^o \to G^o/R_u(G^o)$ is the radical of $G^o/R_u(G^o)$. Thus $\pi(T) = \pi(R(G^o))$ is the radical of $G^o/R_u(G^o)$. It follows that $T \to G^o/U(G^o)$ is surjective and has a finite kernel.

We are left with showing that there exists a maximal torus T'' which is invariant under conjugation by the elements of E. We use the following notation: $H_1 = R(G^o)$, $U_1 = R_u(G^o)$; let $U_1 \supset \cdots \supset U_a = 0$ be a decreasing family of closed characteristic subgroups such that each U_i/U_{i+1} is an abelian group and hence isomorphic to a $\mathbb{C}$-vector space. Since the U_i are characteristic subgroups, they are invariant under conjugation with the elements of E. In particular, U_i/U_{i+1} has a linear action of the group E. In other words, U_i/U_{i+1} is an E-module.

As above, we fix a maximal torus T. Every maximal torus is conjugated to T by an element which can be chosen in U_1. Thus for $e \in E$ there is an element $c(e) \in U_1$ such that $eTe^{-1} = c(e)Tc(e)^{-1}$. Let N denote the linear subspace of U_1/U_2 consistsing of the elements n such that nTn^{-1} and T are equal modulo U_2. We claim that N is an E-submodule of U_1/U_2. Indeed, $e^{-1}neT(e^{-1}ne)^{-1} = e^{-1}nc(e)Tc(e)^{-1}n^{-1}e = e^{-1}c(e)nTn^{-1}c(e)^{-1}e = e^{-1}c(e)Tc(e)^{-1}e = T$. Let $C(e) \in (U_1/U_2)/N$ denote the image of $c(e)$. By construction, $C(e)$ does not depend on the choice of $c(e)$. The map $e \mapsto C(e)$ is a 1-cocycle with values in the E-module $(U_1/U_2)/N$, i.e., $C(e_1e_2) = C(e_1) \cdot e_1C(e_2)e_1^{-1}$. This 1-cocycle is trivial because $(U_1/U_2)/N$ is a vector space over a field of characteristic 0. We conclude that there exists a conjugate T' of T such that for every $e \in E$ the two tori $eT'e^{-1}$ and T' are equal modulo U_2. Now one considers the subgroup $U_2 \rtimes T'$ with its E-action. By induction (with respect to a)

one concludes that $U_2 \rtimes T'$ contains a maximal torus T'' invariant under the action of E. □

We continue the proof of the theorem. T is chosen as in the above Lemma. We may replace G^o with $U(G^o) \rtimes T$. Since G does not satisfy the Singer condition, the character group $X(T)$ of T contains a, non-trivial, irreducible E-submodule N such that $X(T)/N$ has no torsion. After replacing T with a torus T' such that $T' \to T$ is surjective and has a finite kernel, one can write T as a product of two tori T_1 and T_2, both invariant under conjugation by E and such that the group $T_2 \rtimes E$ satisfies the assumptions of Lemma 5.2.

The result after these changes is a group G' of the form

$$(U(G^o) \rtimes T_1) \rtimes (T_2 \rtimes E)$$

which maps surjectively to G and has a finite kernel. We will produce a moduli space $\mathbb{M}$ such that $\mathbb{M}(= G)$ is not constructible.

One takes a finite subset $\{b_1, \ldots, b_t, a_1, \ldots, a_s\}$ in $\mathbb{C}^*$. The fundamental group π_1 of the complement of this set in $\mathbf{P}^1_{\mathbb{C}}$, with base point 0, is given generators $\mu_1, \ldots, \mu_t, \lambda_1, \ldots, \lambda_s$ according to loops around these points. The only relation is $\mu_1 \cdots \mu_t \lambda_1 \cdots \lambda_s = 1$. We will consider homomorphisms $\rho : \pi_1 \to G'$ by assigning images for these $t + s$ generators. For notational convenience we will ignore the relation between the generators of π_1. The trick which allows us to do so is the following. One doubles the finite set by adding new points $a_s^*, \ldots, a_1^*, b_t^*, \ldots, b_1^*$. The fundamental group has now generators $\mu_1, \ldots, \mu_t, \lambda_1, \ldots, \lambda_s, \lambda_s^*, \ldots, \lambda_1^*, \mu_t^*, \ldots, \mu_1^*$. The only relation is their product being 1. Suppose that we want to assign elements $g_1, \ldots, g_t, h_1, \ldots, h_s \in G'$ to $\mu_1, \ldots, \lambda_s$. Then for the larger fundamental group, we complete this by assigning $h_s^{-1}, \ldots, h_1^{-1}, g_t^{-1}, \ldots, g_1^{-1}$ to the generators $\lambda_s^*, \ldots, \mu_1^*$.

The homomorphisms $\rho_n' : \pi_1 \to G'$ that interest us are given by:

(a) $\rho_n'(\mu_1), \ldots, \rho_n(\mu_{t-1}) \in U(G^o)$; these elements are unipotent, $\neq 1$ and they generate $U(G^o)$ as an algebraic group. Moreover, these elements will not depend on n.

(b) $\rho_n'(\mu_t) \in T_1$ which generates T_1 as an algebraic group. Moreover, this element will not depend on n.

(c) $\rho_n'(\lambda_1), \ldots, \rho_n'(\lambda_s) \in T_2 \rtimes E$ are chosen as in the proof of Proposition 5.3.

As indicated above, this is completed by assigning values to $\mu_t^*, \ldots, \lambda_1^*$. The homomorphism $\rho_n : \pi_1 \to G$ are obtained by composing ρ_n' with $G' \to G$. We take an irreducible faithful G-module V. Riemann-Hilbert produces a differential module $M_n = \mathbb{C}(z) \otimes V$ with singularities in the set $\{b_1, \ldots, a_1, \ldots, a_s^*, \ldots, b_t^*, \ldots, b_1^*\}$.

The local monodromies at the points $b_1, \ldots, b_t$ are fixed and we choose local connections for these singular points. For the local connections at the regular singular points $a_1, \ldots, a_s$ we make a choice which fits infinitely many of the ρ_n. The local data at the other points $a_s^*, \ldots, b_1^*$ are just the negatives of the corresponding points in $\{b_1, \dots, a_s\}$. As in the proof of Proposition 5.3, there exists an infinite subset I of $\mathbf{N}$, such that the corresponding vector bundles $\mathcal{M}_n$ have the same type. This defines the moduli problem and the moduli family, parametrized by some space $\mathbb{M}$. According to Proposition 4.7, $\mathbb{M}(U(G^o) \subset, \subset G)$ is constructible. Let H denote the image of the group $U(G^o) \rtimes T_1$ in G. Then it can be seen that $\mathbb{M}(H \subset, \subset G)$ is also constructible. The final part of the proof of Proposition 5.3 applies here as well and the result is that $\mathbb{M}(= G)$ is not constructible. □

Remarks 5.7. — Another formulation of the Singer condition.

(1) The constructions in Lemma 5.1, Lemma 5.2, Proposition 5.3 and Theorem 5.5 lead to the following observation:

A linear algebraic group G does not satisfy the Singer condition if and only if it has a factor group H of dim ≥ 1 with the following properties: There exist integers $N, M, I > 1$ such that every algebraic subgroup $K \subset H$ which is mapped surjectively to H/H^o contains an algebraic subgroup of index $\leq I$ which is, as algebraic group, generated by N elements of order $\leq M$.

(2) Theorem 5.5 remains valid for an algebraically closed field C that is not algebraic over $\mathbb{Q}$ (See Remarks 5.4).

(3) For Theorem 3.6 to hold, it is essential to consider families of differential equations on $\mathbb{P}^1$. Consider for example an elliptic curve E over $\mathbb{C}$ and a family of connections of rank 1 above this curve, parametrized by a suitable X. For every closed point x, one has $\mathrm{Gal}(x) \subset \mathrm{G}_{m,\mathbb{C}} = \mathbb{C}^*$. In [S93, p.384], a family of this type is given such that $X(= \mathbb{C}^*)$ is not constructible. The reason is the following. The family can be pushed down, by the canonical morphism $E \to \mathbb{P}^1$, to a family of rank two connections on $\mathbb{P}^1$ parametrized by, say, Y. This produces essentially the Lamé family of Example 2.6. As we have seen, the set $Y(= D_\infty^{\mathrm{SL}_2})$, which coincides with $X(= \mathbb{C}^*)$, is not constructible.

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