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RESIDUES OF CHERN CLASSES ON SINGULAR VARIETIES

by

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Abstract. — For a collection of sections of a holomorphic vector bundle over a complete intersection variety, we give three expressions for its residues at an isolated singular point. They consist of an analytic expression in terms of a Grothendieck residue on the variety, an algebraic one as the dimension of a certain complex vector space and a topological one as a mapping degree. Some examples are also given.

***Résumé* (Résidus de classes de Chern sur les variétés singulières).** — Étant donnée une famille de sections d'un fibré vectoriel complexe sur une variété intersection complète, on donne trois expressions pour le résidu en un point singulier isolé. Elles consistent en une expression analytique en termes d'un résidu de Grothendieck sur la variété, une expression algébrique comme dimension d'un certain espace vectoriel complexe et une expression topologique comme degré d'une application. Quelques exemples sont aussi donnés.

This is a partially expository article, in which we give various expressions for the residues of Chern classes of vector bundles, mainly over complete intersection varieties.

Let E be a complex vector bundle of rank r over some reasonable space X of real dimension m. For an ℓ-tuple of sections $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of E, we denote by $S(\boldsymbol{s})$ its singular set, *i.e.*, the set of points where the s_i's fail to be linealy independent. Let $c^i(E)$ denote the i-the Chern class of E, which is in $H^{2i}(X)$. For $i \geqslant r - \ell + 1$, there is a natural lifting $c^i_S(E, \boldsymbol{s})$ in $H^{2i}(X, X \smallsetminus S)$ of $c^i(E)$, $S = S(\boldsymbol{s})$. We call $c^i_S(E, \boldsymbol{s})$ the localization of $c^i(E)$ at S with respect to $\boldsymbol{s}$. Suppose S is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then, by the Alexander homomorphism $H^{2i}(X, X \smallsetminus S) \to H_{m-2i}(S) = \oplus_\lambda H_{m-2i}(S_\lambda)$, the class $c^i_S(E, \boldsymbol{s})$ determines, for

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each λ, the "residue" $\mathrm{Res}_{c^i}(\boldsymbol{s}, E; S_\lambda)$ in $H_{m-2i}(S_\lambda)$. If X is compact, we have the "residue formula"

$$\sum_\lambda (\iota_\lambda)_* \mathrm{Res}_{c^i}(\boldsymbol{s}, E; S_\lambda) = c^i(E) \frown [X],$$

where $\iota_\lambda \hookrightarrow X$ denotes the inclusion and $[X]$ the fundamental class of X. The formula itself is of rather trivial nature. However, everytime we have an explicit expression for the residues, it becomes really an interesting one.

In this article, we consider the case where X is a complex manifold M or a (locally) complete intersection variety V of dimension n. We also assume that $r-\ell+1=n$ and look at $c^n(E)$ so that the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$ under consideration is a number. In tha case S_λ consists of an isolated point p, we give analytic, algebraic and topological expressions for $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p)$. As a consequence we have the fact that these three expressions are the same, which is rather well-known in some cases, in particular in the case $X = M$, $r = n$ and $\ell = 1$ (see, *e.g.*, [**DA**], [**GH**], [**O**]). For the analytic expression, we quote results of [**Su4**] and for the algebraic one we try to give a complete proof. The proof for the topological one is not so difficult and we only state the outline.

In Section 1, we recall the residues and describe them in the case we consider. This is done in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology. In Section 2, we give fundamental properties of residues at isolated singularities. In particular, we show that they are positive integers and satisfy the "conservation law" under perturbations of sections. In Section 3, we give an analytic expression of the residue as a Grothendieck residue (on a variety), quoting the results in [**Su4**]. After we recall some commutative algebra in Section 4, we give an algebraic expression of the residue as the dimension of some complex vector space in Section 5. The proof is done by showing that this algebraic invariant also satisfies the conservation law. It should be noted that the idea of proof is inspired by [**EG1**] and [**Lo**, Ch. 4]. In Section 6, we give a topological expression as the degree of some map of the link of the singularity to the Stiefel manifold. This is also done by noting that the degree satisfies the conservation law. Finally in Section 7, we give some examples and applications.

After the preparation of the manuscript, the author's attention was drawn to a recent preprint of W. Ebeling and S.M. Gusein-Zade [**EG2**]. They consider also characteristic numbers (not only Chern classes) and define the index of a collection of sections topologically. Their algebraic formula in Theorem 2 is more general than the one in Theorem 5.5 below. They also give a formula (Theorem 4), which corresponds to the one in Theorem 5.8 below, for collections of 1-forms.

1. Residues of Chern classes

We refer to [**Su2**, Ch. IV, 2, Ch. VI, 4] and [**Su4**] for details of the material in this section.

1a. Non-singular base spaces. — Let M be a complex manifold of dimension n and E a (C^∞, for the moment) complex vector bundle of rank r over M. Then, for $i = 1, \dots, r$, we have the i-th Chern class $c^i(E)$ in $H^{2i}(M)$. If we use the obstruction theory, it is the primary obstruction to constructing $r - i + 1$ sections linearly independent everywhere (see, *e.g.*, [**St**]). The Chern-Weil theory provides us with a canonical way of constructiong a closed $2i$-form representing the class $c^i(E)$ in the de Rham cohomology. To be a little more precise, let ∇ be a connection for E. For the i-th Chern polynomial c^i, we have a closed $2i$-form $c^i(\nabla)$ on M. Moreover, for two connections ∇ and ∇', we have the "Bott difference form" $c^i(\nabla, \nabla')$, which is a $(2i-1)$-form satisfying

$$c^i(\nabla', \nabla) = -c^i(\nabla, \nabla') \quad \text{and} \quad d\, c^i(\nabla, \nabla') = c^i(\nabla') - c^i(\nabla).$$

Then the class of $c^i(\nabla)$ is independent of the choice of ∇ and is equal to $c^i(E)$. Hereafter we assume that $r \geqslant n$ and look at the class $c^n(E)$, which is in the cohomology of M of the top dimension.

For an ℓ-tuple of sections $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of E, we denote by $S(\boldsymbol{s})$ its singular set, *i.e.*, the set of points where $s_1, \dots, s_\ell$ fail to be linearly independent. Suppose we have such an $\boldsymbol{s}$ with $\ell = r - n + 1$ and set $S = S(\boldsymbol{s})$. Then there is the "localization" $c^n_S(E, \boldsymbol{s})$ in $H^{2n}(M, M \smallsetminus S; \mathbb{C})$, with respect to $\boldsymbol{s}$, of the n-th Chern class $c^n(E)$, which is described as follows.

Letting $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall that, in the Čech-de Rham cohomology for the covering $\mathcal{U}$, the class $c^n(E)$ is represented by a cocycle of the form

$$c^n(\nabla_\star) = (c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)), \tag{1.1}$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an $\boldsymbol{s}$-trivial connection (*i.e.*, a connection ∇_0 with $\nabla_0(s_i) = 0$ for $i = 1, \dots, \ell$), then $c^n(\nabla_0) = 0$ and the cocycle naturally defines a class in the relative cohomology $H^{2n}(M, M \smallsetminus S; \mathbb{C})$, which we denote by $c^n_S(E, \boldsymbol{s})$. It is sent to $c^n(E)$ by the canonical homomorphism $j^* : H^{2n}(M, M \smallsetminus S; \mathbb{C}) \to H^{2n}(M, \mathbb{C})$.

Suppose now that $S = S(\boldsymbol{s})$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then for each λ, the class $c^n_S(E, \boldsymbol{s})$ defines a number, which we call the residue of $\boldsymbol{s}$ at S_λ with respect to c^n and denote by $\operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$. It is also briefly called a residue of $c^n(E)$. For each λ, we choose a neighborhood U_λ of S_λ in U_1 so that the U_λ's are mutually disjoint, and let R_λ be a real $2n$-dimensional manifold with C^∞ boundary ∂R_λ in U_λ containing S_λ in its interior. Then the residue is given by

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda) = \int_{R_\lambda} c^n(\nabla_1) - \int_{\partial R_\lambda} c^n(\nabla_0, \nabla_1). \tag{1.2}$$

We have the "residue formula" (*cf.* [**Su2**, Ch. III, Theorem 3.5]):

Proposition 1.3. — *If R is a compact real $2n$-dimensional manifold with C^∞ boundary containing S in its interior, then*

$$\sum_\lambda \operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda) = \int_R c_R^n(E, \boldsymbol{s}),$$

where the right hand side is defined as that of (1.2) with ∇_0 an $\boldsymbol{s}$-trivial connection for E on a neighborhood of ∂R, ∇_1 a connection for E on a neighborhood of R and R_λ replaced by R.

In particular, if M is compact, the right hand side is equal to $\int_M c^n(E)$.

Remark 1.4. — Comparing with the obstruction theoretic definition of Chern classes, we see that the residue $\operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$ is in fact an integer. However, in the sequel we prove this fact more directly in the pertinent cases.

1b. Singular base spaces. — Let V be an analytic variety of pure dimension n in a complex manifold W of dimension $n+k$. We denote by $\operatorname{Sing}(V)$ the singular set of V and let $V' = V \smallsetminus \operatorname{Sing}(V)$ be the non-singular part.

Let S be a compact set in V (V may not be compact). We assume that S has a finite number of connected components, $S \supset \operatorname{Sing}(V)$ and that S admits a regular neighborhood in W. Let $\widetilde{U}_1$ be a regular neighborhood of S in W and $\widetilde{U}_0$ a tubular neighborhood of $U_0 = V \smallsetminus S$ in W. We consider the covering $\mathcal{U} = \{\widetilde{U}_0, \widetilde{U}_1\}$ of the union $\widetilde{U} = \widetilde{U}_0 \cup \widetilde{U}_1$, which may be assumed to have the same homotopy type as V.

For a complex vector bundle E over $\widetilde{U}$ of rank r ($\geqslant n$), the n-th Chern class $c^n(E)$ is in $H^{2n}(\widetilde{U}) \simeq H^{2n}(V)$. The corresponding class in $H^{2n}(V)$ is denoted by $c^n(E|_V)$. The class $c^n(E)$ is represented by a Čech-de Rham cocycle $c^n(\nabla_\star)$ on $\mathcal{U}$ given as (1.1) with ∇_0 and ∇_1 connections for E on $\widetilde{U}_0$ and $\widetilde{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on U_0, since there is a C^∞ retraction of $\widetilde{U}_0$ onto U_0. Suppose we have an ℓ-tuple $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of C^∞ sections linearly independent everywhere on U_0, $\ell = r - n + 1$, and let ∇_0 be $\boldsymbol{s}$-trivial. Then we have the vanishing $c^n(\nabla_0) = 0$ and the above cocycle $c^n(\nabla_\star)$ defines a class $c_S^n(E|_V, \boldsymbol{s})$ in $H^{2n}(V, V \smallsetminus S; \mathbb{C})$. It is sent to $c^n(E|_V)$ by the canonical homomorphism $j^* : H^{2n}(V, V \smallsetminus S; \mathbb{C}) \to H^{2n}(V, \mathbb{C})$.

Let $(S_\lambda)_\lambda$ be the connected components of S. Then, for each λ, $c_S^n(E|_V, \boldsymbol{s})$ defines the residue $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$. For each λ, we choose a neighborhood $\widetilde{U}_\lambda$ of S_λ in $\widetilde{U}_1$, so that the $\widetilde{U}_\lambda$'s are mutually disjoint. Let $\widetilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary $\partial\widetilde{R}_\lambda$ in $\widetilde{U}_\lambda$ containing S_λ in its interior such that $\partial\widetilde{R}_\lambda$ is transverse to V. We set $R_\lambda = \widetilde{R}_\lambda \cap V$. Then the residue is a number given by a formula as (1.2). We also have the residue formula:

Proposition 1.5. — *If $\widetilde{R}$ is a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\widetilde{U}$ containing S in its interior such that $\partial\widetilde{R}$ is transverse to V,*

$$\sum_\lambda \operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda) = \int_R c_R^n(E|_V, \boldsymbol{s}),$$

where the right hand side is defined as that of (1.2) with ∇_0 an $\boldsymbol{s}$-trivial connection for E on a neighborhood of ∂R in V, ∇_1 a connection for E on a neighborhood of $\widetilde{R}$ in W and R_λ replaced by R, $R = \widetilde{R} \cap V$.

In particular, if V is compact, the right hand side is equal to $\int_V c^n(E)$.

Remarks

(1) If S_λ is in the non-singular part V', $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$ coincides with the one defined in (1a) and if V itself is non-singular, Proposition 1.5 reduces to Proposition 1.3.

(2) If $\boldsymbol{s}$ extends to an ℓ-tuple $\widetilde{\boldsymbol{s}}$ of sections of E linearly independent everywhere on $\widetilde{U}_\lambda$, we may let both ∇_0 and ∇_1 equal to an $\widetilde{\boldsymbol{s}}$-trivial connection so that we have $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda) = 0$.

(3) As in the case of non-singular base spaces (*cf.* Remark 1.4), the residue $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$ is in fact an integer. In the sequel we prove this fact more directly in the pertinent cases.

1c. Residues at an isolated singularity. — Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$, as before. We do not exclude the case $k = 0$, where $V = W$ is a complex manifold of dimension n.

Suppose now that V has at most an isolated singularity at p and let E be a holomorphic vector bundle of rank r ($\geqslant n$) on a small coordinate neighborhood $\widetilde{U}$ of p in W. Sometimes we identify $\widetilde{U}$ with a neighborhood of 0 in $\mathbb{C}^{n+k}$ and p with 0. We may assume that E is trivial and let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be a holomorphic frame of E on $\widetilde{U}$. Let $\ell = r-n+1$ and suppose we have an ℓ-tuple of holomorphic sections $\widetilde{\boldsymbol{s}}$ of E on $\widetilde{U}$. Suppose that $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$. Then we have $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; p)$ with $\boldsymbol{s} = \widetilde{\boldsymbol{s}}|_V$. Let $\widetilde{R}$ be a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\widetilde{U}$ containing p in its interior such that $\partial\widetilde{R}$ is transverse to V and set $R = \widetilde{R} \cap V$. We also set $U = \widetilde{U} \cap V$ and let ∇_0 be an $\boldsymbol{s}$-trivial connection for E on $U \smallsetminus \{p\}$. We choose ∇_1 to be $\boldsymbol{e}$-trivial. Then we have $c^n(\nabla_1) = 0$ and

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; p) = -\int_{\partial R} c^n(\nabla_0, \nabla_1). \tag{1.6}$$

In the subsequent sections, we give various expressions of this number.

2. Fundamental properties of the residues

2a. Non-singular base spaces. — In the situation of (1c), suppose $V = W = M$ is a complex manifold of dimension n and write $\widetilde{U}$ and $\widetilde{\boldsymbol{s}}$ by U and $\boldsymbol{s}$, respectively. Thus our assumption is $S(\boldsymbol{s}) = \{p\}$.

Let us first assume that $r = n$. Thus $\ell = 1$ and we have only one section s. We write $s = \sum_{i=1}^n f_i\, e_i$ with f_i holomorphic functions on U.

Lemma 2.1. — *If $r=n$ and $\ell=1$, we have*

$$\mathrm{Res}_{c^n}(s,E;p)=\int_{\partial R} f^*\beta_n,$$

where β_n denotes the Bochner-Martinelli kernel on $\mathbb{C}^n$ and $f=(f_1,\dots,f_n)$.

Proof. — Recall that the residue is given by (1.7). Let $\{U^{(i)}\}$ be the covering of $U\smallsetminus\{p\}$ given by $U^{(i)}=\{\,q\in U\mid f_i(q)\neq 0\,\}$. For each i, let $\boldsymbol{e}_i$ be the frame of E on $U^{(i)}$ obtained from $\boldsymbol{e}$ replacing e_i by s and let $\nabla^{(i)}$ be the connection for E on $U^{(i)}$ trivial with respect to $\boldsymbol{e}_i$. Also, let $\rho_i=|z_i|^2/\|z\|^2$ and let ∇_0 be the connection for E on $U_0=U\smallsetminus\{p\}$ given by $\nabla_0=\sum_{i=1}^n\rho_i\nabla^{(i)}$. Then ∇_0 is s-trivial, since each $\nabla^{(i)}$ is. If we compute $c^n(\nabla_0,\nabla_1)$ using this connection, we get $c^n(\nabla_0,\nabla_1)=-f^*\beta_n$, as in the proof of [**Su2**, Ch. III, Theorem 4.4]. □

We may think of s (or f) as a map from ∂R to $\mathbb{C}^n\smallsetminus\{0\}$, which has the homotopy type of S^{2n-1}.

Corollary 2.2. — *If $r=n$ and $\ell=1$,*

$$\mathrm{Res}_{c^n}(s,E;p)=\deg s|_{\partial R},$$

the mapping degree of $s|_{\partial R}$. Thus $\mathrm{Res}_{c^n}(s,E;p)$ is a positive integer.

We say that p is a non-degenerate zero of s if $\det\frac{\partial(f_1,\dots,f_n)}{\partial(z_1,\dots,z_n)}(p)\neq 0$. In this case, $(f_1,\dots,f_n)$ form a coordinate system around 0. Hence we have

Corollary 2.3. — *If p is a non-degenerate zero of s,*

$$\mathrm{Res}_{c^n}(s,E;p)=1.$$

Now we go back to the general case of vector bundle E of rank $r\geqslant n$ with an ℓ-tuple $\boldsymbol{s}=(s_1,\dots,s_\ell)$ of sections, $\ell=r-n+1$. We consider the bundle $E^*=E\times T$ over $U^*=U\times T$, where T is a small neighborhood of 0 in $\mathbb{C}=\{t\}$. Suppose we have an ℓ-tuple of holomorphic sections $\boldsymbol{s}^*$ of E^* on U^* such that $\boldsymbol{s}^*(z,0)=\boldsymbol{s}(z)$. For t in T, we set $E_t=E^*|_{U\times\{t\}}$ and $\boldsymbol{s}_t(z)=\boldsymbol{s}^*(z,t)$. We call such an $\boldsymbol{s}^*$ (or $\boldsymbol{s}_t$) a perturbation of $\boldsymbol{s}$. Sometimes we identify $U\times\{t\}$ with U and $E|_t$ with E. Since we assumed that $S(\boldsymbol{s})=\{p\}$, by the upper semi-continuity of $\dim S(\boldsymbol{s}_t)$, $S(\boldsymbol{s}_t)$ consists at most of a finite number of points.

Lemma 2.4. — *The sum $\sum_{q\in S(\boldsymbol{s}_t)}\mathrm{Res}_{c^n}(\boldsymbol{s}_t,E_t;q)$ is continuous in t.*

Proof. — Let ∇_0^* be an $\boldsymbol{s}^*$-trivial connection for E^* on $U_0^*=U^*\smallsetminus S(\boldsymbol{s}^*)$ and ∇_1^* a connection for E^* on U^*. The statement follows computing the residues taking the restrictions of ∇_0^* and ∇_1^* and using (1.2) and Proposition 1.3. □

Next we consider the case where $s_1(p) \neq 0$ so that we have an exact sequence of vector bundles on a neighborhood of p:

$$(2.5) \qquad 0 \longrightarrow I \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

where I denotes the trivial line bundle determined by s_1 and E' is a vector bundle (still trivial) of rank $r-1$. Let $\boldsymbol{s}' = (s'_2, \dots, s'_\ell)$ denote the $(\ell-1)$-tuple of sections of E' determined by $(s_2, \dots, s_\ell)$.

Lemma 2.6. — *In the above situation, we have*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p) = \mathrm{Res}_{c^n}(\boldsymbol{s}', E'; p).$$

Proof. — Let ∇ be the connection for I trivial with respect to s_1. Let ∇'_0 be an $\boldsymbol{s}'$-trivial connection for E' on U_0 and take an $\boldsymbol{s}$-trivial connection ∇_0 for E so that $(\nabla, \nabla_0, \nabla'_0)$ is compatible (*cf.* [**BB**]) with (2.5). Also, let ∇'_1 be a connection for E' on U and take a connection ∇_1 for E so that $(\nabla, \nabla_1, \nabla'_1)$ is compatible with (2.5). Then we have

$$c^n(\nabla_1) = c^n(\nabla'_1) \quad \text{and} \quad c^n(\nabla_0, \nabla_1) = c^n(\nabla'_0, \nabla'_1).$$

The identity follows from (1.2). □

Lemma 2.7. — *The residue* $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p)$ *is a non-negative integer.*

Proof. — We proceed by induction on ℓ. By Corollary 2.2, it is true if $\ell = 1$. Suppose it is true for arbitrary $\ell - 1$ sections with isolated singularity. Take a perturbation $s_{1,t}$ of s_1 such that $s_{1,t}(p) \neq 0$ and set $\boldsymbol{s}_t = (s_{1,t}, s_2, \dots, s_\ell)$. Recalling that none of the s_i's vanish on $U \smallsetminus \{p\}$, we see that, for $t \neq 0$, at each point of $S(\boldsymbol{s}_t)$, at least one of the sections of $\boldsymbol{s}_t$ does not vanish. Hence the lemma follows from Lemmas 2.4 and 2.6. □

Corollary 2.8. — *In the situation of Lemma 2.4, the sum* $\sum_{q \in S(s_t)} \mathrm{Res}_{c^n}(\boldsymbol{s}_t, E_t; q)$ *is constant. In particular,*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p) = \sum_{q \in S(\boldsymbol{s}_t)} \mathrm{Res}_{c^n}(\boldsymbol{s}_t, E_t; q).$$

Remarks

(1) If $\ell = 1$, there exists always a "good perturbation" of s, *i.e.*, a holomorphic sections s^* of E^* near 0 such that $s^*(z, 0) = s(z)$ and that s_t has only non-degenerate zeros, for $t \neq 0$ ([**GH**, Ch. 5]).

(2) By Lemma 5.1 below, $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p)$ is in fact a positive integer.

2b. Singular base spaces. — Now we consider the situation of (1c) with $k > 0$. Let p be an isolated singular point in V and suppose that V is a complete intersection defined by $h = (h_1, \ldots, h_k) : (\widetilde{U}, p) \to (\mathbb{C}^k, 0)$. Let T be a small neighborhood of 0 in $\mathbb{C}^k$. For a point t in T, we set $V_t = h^{-1}(t)$. Let $C(h)$ denote the critical set of h and $D(h) = h(C(h))$ the discriminant, which is a hypersurface in T (see, *e.g.*, [**Lo**]). We have $\mathrm{Sing}(V_t) = C(h) \cap V_t$, which consists of at most a finite number of points. We set $\boldsymbol{s}_t = \widetilde{\boldsymbol{s}}|_{V_t}$ and $S(\boldsymbol{s}_t) = S(\widetilde{\boldsymbol{s}}) \cap V_t$. By the assumption $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$, we have $\dim S(\widetilde{\boldsymbol{s}}) \leqslant k$. Hence $S(\boldsymbol{s}_t)$ also consists of at most a finite number of points. Note that even if q is in $\mathrm{Sing}(V_t)$, if $q \notin S(\boldsymbol{s}_t)$, then $\mathrm{Res}_{c^n}(\boldsymbol{s}_t, E|_{V_t}; q) = 0$ (*cf.* Remark 1.6.2).

Lemma 2.9. — *The sum* $\sum_{q \in S(\boldsymbol{s}_t)} \mathrm{Res}_{c^n}(\boldsymbol{s}_t, E|_{V_t}; q)$ *is continuous in* t.

Proof. — Let ∇_0 be an $\widetilde{\boldsymbol{s}}$-trivial connection for E on $\widetilde{U} \smallsetminus S(\widetilde{\boldsymbol{s}})$ and ∇_1 a connection for E on $\widetilde{U}$. Then by Proposition 1.5, the above sum is equal to an integral over $R_t = \widetilde{R} \cap V_t$, which is continuous in t. □

Since $T \smallsetminus D(h)$ is dense in T, by Lemma 2.7, we have

Corollary 2.10. — *The sum* $\sum_{q \in S(\boldsymbol{s}_t)} \mathrm{Res}_{c^n}(\boldsymbol{s}_t, E|_{V_t}; q)$ *is constant. In particular,*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p) = \sum_{q \in S(\boldsymbol{s}_t)} \mathrm{Res}_{c^n}(\boldsymbol{s}_t, E|_{V_t}; q),$$

which is a non-negative integer.

Remark 2.11. — By Lemma 5.6 below, $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p)$ is in fact a positive integer.

3. Analytic expression

In this section, we review [**Su4**], see also [**Su3**].

3a. Grothendieck residues relative to a subvariety. — Let $\widetilde{U}$ be a neighborhood of 0 in $\mathbb{C}^{n+k}$ and V a subvariety of dimension n in $\widetilde{U}$ which contains 0 as at most an isolated singular point. Also, let $f_1, \ldots, f_n$ be holomorphic functions on $\widetilde{U}$ and $V(f_1, \ldots, f_n)$ the variety defined by them. We assume that $V(f_1, \ldots, f_n) \cap V = \{0\}$. For a holomorphic n-from ω on $\widetilde{U}$, the Grothendieck residue relative to V is defined by (*e.g.*, [**Su2**, Ch. IV, 8])

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ is the n-cycle in V given by

$$\Gamma = \{\, q \in \widetilde{U} \cap V \mid |f_i(q)| = \varepsilon_i, \quad i = 1, \ldots, n \,\}$$

for small positive numbers ε_i. It is oriented so that $d\arg(f_1) \wedge \cdots \wedge d\arg(f_n) \geqslant 0$.

If $k = 0$, it reduces to the usual Grothendieck residue (*e.g.*, [**GH**, Ch. 5]), in which case we omit the suffix V.

If V is a complete intersection defined by $h_1 = \cdots = h_k = 0$ in $\widetilde{U}$, we have

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \operatorname{Res}_0 \begin{bmatrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \dots, f_n, h_1, \dots, h_k \end{bmatrix}.$$

3b. The analytic expression. — We consider the situation of (1c). We write $\widetilde{s}_i = \sum_{j=1}^{r} f_{ij}\, e_j$, $i = 1, \dots, \ell$, with f_{ij} holomorphic functions on $\widetilde{U}$. Let F be the $\ell \times r$ matrix whose (i,j)-entry is f_{ij}. We set

$$\mathcal{I} = \{\, (i_1, \dots, i_\ell) \mid 1 \leqslant i_1 < \cdots < i_\ell \leqslant r \,\}.$$

For an element $I = (i_1, \dots, i_\ell)$ in $\mathcal{I}$, let F_I denote the $\ell \times \ell$ matrix consisting of the columns of F corresponding to I and set $\varphi_I = \det F_I$. If we write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, we have

$$\widetilde{s}_1 \wedge \cdots \wedge \widetilde{s}_\ell = \sum_{I \in \mathcal{I}} \varphi_I\, e_I.$$

Note that $S(\widetilde{\boldsymbol{s}})$ is the set of common zeros of the φ_I's. From the assumption $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$, we have ([**Su4**, Lemma 5.6]):

Lemma 3.1. — *We may choose a holomorphic frame $\boldsymbol{e} = (e_1, \dots, e_r)$ of E so that there exist n elements $I^{(1)}, \dots, I^{(n)}$ in $\mathcal{I}$ with $V(\varphi_{I^{(1)}}, \dots, \varphi_{I^{(n)}}) \cap V = \{p\}$.*

In general, let $\Omega = (\omega_{ij})$ be an $r \times r$ matrix with differential forms ω_{ij} in its entries. We define the determinant of Ω by

$$\det \Omega = \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}\, \sigma \cdot \omega_{\sigma(1)1} \cdots \omega_{\sigma(r)r},$$

where $\mathcal{S}_r$ denotes the symmetric group of degree r and the products of forms are exterior products.

Let $\boldsymbol{e}$ be a frame of E as in Lemma 3.1. We write $I^{(\alpha)} = (i_1^{(\alpha)}, \dots, i_\ell^{(\alpha)})$, $\alpha = 1, \dots, n$, and let $F^{(\alpha)}$ be the $r \times r$ matrix obtained by replacing the $i_j^{(\alpha)}$-th row of the $r \times r$ identity matrix by the j-th row of F, $j = 1, \dots, \ell$. Note that $\det F^{(\alpha)} = \varphi_{I^{(\alpha)}}$. Let $\check{F}^{(\alpha)}$ denote the adjoint matrix of $F^{(\alpha)}$ and set

$$\Theta^{(\alpha)} = \check{F}^{(\alpha)} \cdot dF^{(\alpha)},$$

which is an $r \times r$-matrix whose entries are holomorphic 1-forms. Let $\mathcal{A}$ denote the set of n-tuples of integers $(a_1, \dots, a_n)$ with $1 \leqslant a_1 < \cdots < a_n \leqslant r$. For an element $A = (a_1, \dots, a_n)$ in $\mathcal{A}$, we denote by $\Theta_A^{(\alpha)}$ the $n \times n$ matrix whose (i,j)-entry is the (a_i, a_j)-entry of $\Theta^{(\alpha)}$. For a permutation ρ of degree n, we denote by $\Theta_A(\rho)$ the $n \times n$-matrix whose i-th column is that of $\Theta_A^{(\rho(i))}$ and, for the collection $\Theta = \{\Theta^{(\alpha)}\}_\alpha$, we set

$$\sigma_n(\Theta) = \frac{1}{n!} \sum_{A \in \mathcal{A}} \sum_{\rho \in \mathcal{S}_n} \operatorname{sgn} \rho \cdot \det \Theta_A(\rho),$$

which is a holomorphic n-form on $\widetilde{U}$. With these we have ([**Su4**, Theorem 5.7]):

Theorem 3.2. — *In the above notation,*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p) = \mathrm{Res}_p \begin{bmatrix} \sigma_n(\Theta) \\ \varphi_{I^{(1)}}, \dots, \varphi_{I^{(n)}} \end{bmatrix}_V .$$

3c. Special cases

(1) The case $\ell = 1$ and $r = n$. Let $\boldsymbol{e} = (e_1, \dots, e_n)$ be an arbitrary frame of E and write $s = \sum_{i=1}^n f_i e_i$. Then we may set $\varphi_{I^{(i)}} = f_i$, $i = 1, \dots, n$, and we have $\sigma_n(\Theta) = df_1 \wedge \dots \wedge df_n$.

(2) The case $n = 1$ and $\ell = r$. Let $\boldsymbol{e} = (e_1, \dots, e_r)$ be an arbitrary frame of E and write $s_i = \sum_{j=1}^r f_{ij}\, e_j$, $i = 1, \dots, r$. Let $F = (f_{ij})$ and set $\varphi = \det F$. Then we may set $\varphi_{I^{(1)}} = \varphi$ and we have $\sigma_n(\Theta) = d\varphi$.

See [**Su4**] for more cases where the form $\sigma_n(\Theta)$ is computed explicitly.

4. Algebraic preliminaries

In this section, we recall some commutative algebra which we use subsequently. We list [**E**], [**Mat**] and [**Se**] as general references.

In this section, we denote by R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and by M a finitely generated R-module.

The height of a proper ideal I in R is denoted by ht I. The (Krull) dimension of M is denoted by $\dim_R M$, or simply by $\dim M$. Let I be an ideal in R with $IM \neq M$. The *depth* of I on M, denoted by $\mathrm{depth}(I; M)$, is the length of a maximal M-regular sequence in I. The depth of I on R is simply called the depth of I and is denoted by depth I. Let $(S, \mathfrak{n})$ be another Noetherian local ring and $\varphi : R \to S$ a local homomorphism. Then S has a natural R-module structure. We say that φ is *finite* if S is finitely generated over R. In this case, we have (*e.g.*, [**Se**, IV, Proposition 12])

(4.1) $$\dim_S S = \dim_R S, \qquad \mathrm{depth}(\mathfrak{n}; S) = \mathrm{depth}(\mathfrak{m}; S).$$

An R-module M is said to be *Cohen-Macaulay* (simply *CM*), if $M = 0$, or if $M \neq 0$ and $\mathrm{depth}(\mathfrak{m}; M) = \dim M$. The ring R is a CM ring if it is CM as an R-module. Note that a regular local ring is CM. From (4.1), we have:

(4.2) If $\varphi : R \longrightarrow S$ is finite and if S is a CM ring, then S is a CM R-module.

We need another fact about CM rings, which says that, if R is a CM ring, then for every proper ideal I of R,

(4.3) $$\mathrm{ht}\ I = \mathrm{depth}(I; R), \qquad \mathrm{ht}\ I + \dim R/I = \dim R,$$

The *projective dimension* of M, denoted by $\mathrm{pd}_R M$, is the minimum of the lengths of projective resolution of M. We quote the following Auslander-Buchsbaum formula

([**Mat**, p. 114], [**E**, p. 475]), which says that if pd M is finite,

$$\text{depth}(\mathfrak{m}; M) + \text{pd } M = \text{depth } \mathfrak{m}. \tag{4.4}$$

We also need some facts about determinantal ideals. Let $f : R^m \to R^n$ be an R-homomorphism, which may be represented by an $n \times m$ matrix. We assume that $m \geqslant n$ and denote by $I(f)$ the ideal generated by all the $n \times n$ minors of f. We assume $I(f) \neq R$. Then we have ([**Mac**]):

$$\text{ht } I(f) \leqslant m - n + 1. \tag{4.5}$$

We also have (see, *e.g.*, [**E**, Theorem 18.18]):

$$\text{If } R \text{ is CM and if } \text{ht } I(f) = m - n + 1, \text{ then } R/I(f) \text{ is CM.} \tag{4.6}$$

Let $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ denote the ring of covergent power series in n variables. A ring R is an *analytic ring* if $R \simeq \mathcal{O}_n/I$ for some proper ideal I in $\mathcal{O}_n$ (for some n). In this case R is a Noetherian local ring for which the maximal ideal $\mathfrak{m}$ is generated by the images of $z_1, \dots, z_n$. For an ideal I in $\mathcal{O}_n$, we denote by $V(I)$ the germ at 0 (in $\mathbb{C}^n$) of the variety defined by I. If $R = \mathcal{O}_n/I$, then $\dim_R R = \dim V(I)$. We denote by $\dim_{\mathbb{C}}$ the dimension of a complex vector space. By the Hilbert Nullstellensatz,

$$\dim_R R = 0 \text{ if and only if } \dim_{\mathbb{C}} R \text{ is finite.} \tag{4.7}$$

Let $\varphi : R \to S$ be a local homomorphism of analytic rings. The homomorphism φ induces $\mathbb{C} = R \otimes_R R/\mathfrak{m} \to S \otimes_R R/\mathfrak{m}$. We say that φ is *quasi-finite* if this homomorphism makes $S \otimes_R R/\mathfrak{m}$ a finite dimensional complex vector space. Clearly a finite homomorphism is quasi-finite. The coverse is also true (see, *e.g.*, [**N**, Ch. II, Theorem 1]):

$$\varphi \text{ is finite if and only if it is quasi-finite.} \tag{4.8}$$

Let $\pi : (X, \mathcal{O}_X) \to (T, \mathcal{O}_T)$ be a morphism of analytic spaces. For each point x of X, π induces a local homomorphism

$$\pi_x^* : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{X,x}, \quad t = \pi(x).$$

For a point t in T, the fiber X_t of π over t is the analytic space with support $\pi^{-1}(t)$ and structure sheaf $\mathcal{O}_{X_t} = \mathcal{O}_X/\mathfrak{m}_t\mathcal{O}_X$, where $\mathfrak{m}_t$ is the maximal ideal of $\mathcal{O}_{T,t}$. Thus, for a point x in $\pi^{-1}(t)$,

$$\mathcal{O}_{X_t,x} = \mathcal{O}_{X,x}/\mathfrak{m}_t\mathcal{O}_{X,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,t}} \mathcal{O}_{T,t}/\mathfrak{m}_t = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,t}} \mathbb{C}.$$

Hence by (4.7) and (4.8), we see that x is an isolated point in $\pi^{-1}(t)$ if and only if π_x^* is finite.

Suppose now that π is a finite morphism (*i.e.*, proper with finite fibers) of analytic spaces. Let t be a point in T. For a point x in $\pi^{-1}(t)$, we set $\nu(x) = \dim_{\mathbb{C}} \mathcal{O}_{X_t,x}$ and $\nu(t) = \sum_{x \in \pi^{-1}(t)} \nu(x)$. Recall that π is flat if, for every x in X, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{T,t}$-module, $t = \pi(x)$. If T is reduced, we have ([**Do**]):

$$\pi \text{ is flat if and only if } \nu(t) \text{ is locally constant.} \tag{4.9}$$

5. Algebraic expression

5a. Non-singular base spaces. — We consider the situation of (1c). We assume that $k = 0$ and set $\widetilde{U} = U$ and $\widetilde{\boldsymbol{s}} = \boldsymbol{s}$. In general, by (4.5), $\operatorname{codim} S(\boldsymbol{s}) \leqslant r - \ell + 1 = n$. Here we assume that $S(\boldsymbol{s}) = \{p\}$ so that $S(\boldsymbol{s})$ attains its maximum codimension. Let F and φ_I be defined as in (3b). We denote by $\mathcal{O}_U$ the sheaf of germs of holomorphic functions on U and by $\mathcal{F}$ the ideal sheaf in $\mathcal{O}_U$ generated by the (germs of) φ_I's. Note that $\mathcal{F}$ does not depend on the choice of the frame $\boldsymbol{e}$ of E.

Let $\boldsymbol{s}^* = (s_1^*, \dots, s_\ell^*)$ be a perturbation of $\boldsymbol{s}$ as in Lemma 2.4. We define F_I^* and φ_I^* as above, using the s_i^*'s. Let T be a small neighborhood of 0 in $\mathbb{C}$ and $\mathcal{F}^*$ the ideal sheaf generated by the φ_I^*'s in $\mathcal{O}_{U^*}$, $U^* = U \times T$. Also, let $\mathcal{F}_t$ be the ideal sheaf generated by the $\varphi_{I,t}$'s in $\mathcal{O}_{U_t}$, $U_t = U \times \{t\}$.

Lemma 5.1. — *We have* $\dim S(\boldsymbol{s}^*) = 1$ *and* $S(\boldsymbol{s}_t)$ *is a non-empty finite set.*

Proof. — By the upper semicontinuity of $\dim S(\boldsymbol{s}_t)$, we have $\dim S(\boldsymbol{s}^*) \leqslant 1$. On the other hand, by (4.5) we have $\operatorname{codim} S(\boldsymbol{s}^*) \leqslant r - \ell + 1 = n$. ☐

Lemma 5.2. — *In the above situation,*

$$\dim_{\mathbb{C}} \mathcal{O}_{U,p}/\mathcal{F}_p = \sum_{q \in S(\boldsymbol{s}_t)} \dim_{\mathbb{C}} \mathcal{O}_{U_t,q}/\mathcal{F}_{t,q}.$$

Proof. — Let X be the analytic space in U^* with support $S(\boldsymbol{s}^*)$ and structure sheaf $\mathcal{O}_X = \mathcal{O}_{U^*}/\mathcal{F}^*$. By Lemma 5.1, $\dim X = 1$ and the restriction π to X of the projection $U^* \to T$ is a finite morphism. We claim that π is flat. Let x be a point in X and set $t = \pi(x)$. In the following, we set $\mathcal{O}'_x = \mathcal{O}_{U^*,x}$, $\mathcal{O}_x = \mathcal{O}_{X,x}$ and $\mathcal{O}_t = \mathcal{O}_{T,t}$. Note that $\mathcal{O}'_x$ and $\mathcal{O}_t$ are regular local rings of dimensions $n+1$ and 1, respectively. We have $\operatorname{ht} \mathcal{F}_x^* = n = r - \ell + 1$. Hence by (4.6), the ring $\mathcal{O}_x$ is CM. Since the homomorphism $\pi^* : \mathcal{O}_t \to \mathcal{O}_x$ is finite, by (4.2), $\mathcal{O}_x$ is a CM $\mathcal{O}_t$-module. By (4.4), denoting by $\mathfrak{m}_t$ the maximal ideal in $\mathcal{O}_t$,

$$\operatorname{depth}(\mathfrak{m}_t; \mathcal{O}_x) + \operatorname{pd}_{\mathcal{O}_t} \mathcal{O}_x = \operatorname{depth} \mathfrak{m}_t.$$

We have $\operatorname{depth}(\mathfrak{m}_t; \mathcal{O}_x) = \dim_{\mathcal{O}_t} \mathcal{O}_x = \dim_{\mathcal{O}_x} \mathcal{O}_x = 1$ and $\operatorname{depth} \mathfrak{m}_t = \dim \mathcal{O}_t = 1$. Therefore, $\operatorname{pd}_{\mathcal{O}_t} \mathcal{O}_x = 0$ and π is flat.

Set $X_t = \pi^{-1}(t)$, which has a natural structure of (discrete) analytic space and is supported by $S(\boldsymbol{s}_t)$. For x in X_t, we have $\mathcal{O}_{X_t,x} = \mathcal{O}_{U_t,x}/\mathcal{F}_{t,x}$. Hence the lemma follows from (4.9). ☐

Suppose $r = n$ and $\ell = 1$. Then we have one section $s = \sum_{i=1}^n f_i e_i$ and $\mathcal{F}_p = (f_1, \dots, f_n)$. If p is a non-degenerate singularity of s, we have

$$\dim_{\mathbb{C}} \mathcal{O}_n/(f_1, \dots, f_n) = 1.$$

From Corollaries 2.3 and 2.8, Remark 2.9.1 and Lemma 5.2, we have

Corollary 5.3. — *In the case $r = n$ and $\ell = 1$,*

$$\operatorname{Res}_{c^n}(s, E; p) = \dim_{\mathbb{C}} \mathcal{O}_{U,p}/\mathcal{F}_p = \dim_{\mathbb{C}} \mathcal{O}_n/(f_1, \dots, f_n).$$

Now we go back to the general situation with $r \geqslant n$. We assume that $s_1(p) \neq 0$ as in the situation of Lemma 2.6. Then we may write $s'_i = \sum_{j=2}^{r} f'_{ij}\, e'_j$, $i = 2, \dots, \ell$, with f'_{ij} holomorphic functions on U and $\boldsymbol{e}' = (e'_2, \dots, e'_\ell)$ a frame of E' (*cf.* (2.5)). Let F' be the $(\ell - 1) \times (r - 1)$ matrix whose (i, j)-entry is f'_{ij}. We set

$$\mathcal{I}' = \{\, (i_2, \dots, i_\ell) \mid 2 \leqslant i_2 < \cdots < i_\ell \leqslant r \,\}.$$

For an element $I' = (i_2, \dots, i_\ell)$ in $\mathcal{I}'$, let $F'_{I'}$ denote the $(\ell - 1) \times (\ell - 1)$ matrix consisting of the columns of F' corresponding to I' and set $\varphi'_{I'} = \det F'_{I'}$.

Note that the set of common zeros of the $\varphi'_{I'}$'s consists only of p. Let $\mathcal{F}'_p$ denote the ideal of $\mathcal{O}_{U,p}$ generated by the $\varphi'_{I'}$'s.

Lemma 5.4. — *We have $\mathcal{F}_p = \mathcal{F}'_p$, and thus $\dim_{\mathbb{C}} \mathcal{O}_{U,p}/\mathcal{F}_p = \dim_{\mathbb{C}} \mathcal{O}_{U,p}/\mathcal{F}'_p$.*

Proof. — We may assume, without loss of generality, that $f_{11}(p) \neq 0$. Then, we may take as $(e'_2, \dots, e'_\ell)$ the sections determined by $(e_2, \dots, e_\ell)$. For $i \geqslant 2$, we have

$$s_i = \frac{1}{f_{11}} \biggl(f_{i1} s_1 + \sum_{j=2}^{r} \begin{vmatrix} f_{11} & f_{1j} \\ f_{i1} & f_{ij} \end{vmatrix} e_j \biggr).$$

Hence

$$f'_{ij} = \frac{1}{f_{11}} \begin{vmatrix} f_{11} & f_{1j} \\ f_{i1} & f_{ij} \end{vmatrix}.$$

For $I' = (i_2, \dots, i_\ell)$, we compute $\varphi_{(1,I')} = f_{11} \cdot \varphi'_{I'}$. Thus the ideal $\mathcal{F}'_p$ is generated by $\{\, \varphi_{(1,I')} \mid I' \in \mathcal{I}' \,\}$. On the other hand, for any $I = (i_1, \dots, i_\ell)$, considering the determinant of the $(\ell + 1) \times (\ell + 1)$ matrix whose first and second rows are $(f_{11}, f_{1i_1}, \dots, f_{1i_\ell})$ and whose k-th row is $(f_{k-1,1}, f_{k-1,i_1}, \dots, f_{k-1,i_\ell})$, $k \geqslant 3$, we have

$$f_{11} \cdot \varphi_I = \sum_{j=1}^{\ell} (-1)^{j-1} f_{1i_j} \cdot \varphi_{(1,i_1,\dots,\widehat{i_j},\dots,i_\ell)}.$$

Hence we have $\mathcal{F}'_p = \mathcal{F}_p$. □

Theorem 5.5. — *We have*

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E; p) = \dim_{\mathbb{C}} \mathcal{O}_{U,p}/\mathcal{F}_p.$$

Proof. — We prove this by induction on ℓ. The case $\ell = 1$ is Corollary 5.3. Suppose that the statement is true for $\ell - 1$ sections (with isolated singularity). Take a perturbation $s_{1,t}$ of s_1 so that $s_{1,t}(p) \neq 0$. For $t \neq 0$, the support of S_t consists of p, zeros of $s_{1,t}$ and the zeros of $s_{1,t} \wedge \cdots \wedge s_\ell$. However, at any one of these points, at least one of the sections is non-zero. The theorem follows from Lemmas 2.6 and 5.4 and the induction hypothesis. □

5b. Singular base spaces. — Now we consider the situation of (1c) with $k>0$. As in (2b), we suppose that V is a complete intersection defined by $h:(\widetilde{U},p)\to(\mathbb{C}^k,0)$. Let T be a small neighborhood of 0 in $\mathbb{C}^k$ and, for a point t in T, we set $V_t=h^{-1}(t)$. Also let $S(\boldsymbol{s}_t)=S(\widetilde{\boldsymbol{s}})\cap V_t$, as before. From the assumption $S(\widetilde{\boldsymbol{s}})\cap V=\{p\}$ we have

Lemma 5.6. — $\dim S(\widetilde{\boldsymbol{s}})=k$ *and* $S(\boldsymbol{s}_t)$ *is a non-empty finite set.*

Proof. — By the assumtion, we have $\dim S(\widetilde{\boldsymbol{s}})\leqslant k$. On the other hand, if by (4.5), $\operatorname{codim} S(\widetilde{\boldsymbol{s}})\leqslant r-\ell+1=n$. □

Let F and φ_I be defined as in (3b). We denote by $\mathcal{O}_{\widetilde{U}}$ the sheaf of germs of holomorphic functions on $\widetilde{U}$, by $\mathcal{F}$ the ideal sheaf in $\mathcal{O}_{\widetilde{U}}$ generated by the φ_I's, by $\mathcal{I}(V)=(h_1,\ldots,h_k)$ the ideal sheaf of V in $\mathcal{O}_{\widetilde{U}}$ and by $\mathcal{F}(V)$ the ideal sheaf generated by $\mathcal{F}$ and $\mathcal{I}(V)$. Also, for $t=(t_1,\ldots,t_k)\in T$, we denote by $\mathcal{F}(V_t)$ the ideal sheaf generated by $\mathcal{F}$ and $\mathcal{I}(V_t)=(h_1-t_1,\ldots,h_k-t_k)$.

Lemma 5.7. — *In the above situation,*

$$\dim_{\mathbb{C}}\mathcal{O}_{\widetilde{U},p}/\mathcal{F}(V)_p=\sum_{q\in S(\boldsymbol{s}_t)}\dim_{\mathbb{C}}\mathcal{O}_{\widetilde{U},q}/\mathcal{F}(V_t)_q.$$

Proof. — This is proved as Lemma 5.2. Let X be the analytic space in $\widetilde{U}$ with structure sheaf $\mathcal{O}_X=\mathcal{O}_{\widetilde{U}}/\mathcal{F}$. The support of X is $S(\widetilde{\boldsymbol{s}})$ and is k-dimensional, by Lemma 5.6. Thus the restriction π to X of the map $h:\widetilde{U}\to\mathbb{C}^k$ is a finite morphism. We claim that π is flat. Let x be a point in X and set $t=\pi(x)$. In the following, we set $\mathcal{O}'_x=\mathcal{O}_{\widetilde{U},x}$, $\mathcal{O}_x=\mathcal{O}_{X,x}$ and $\mathcal{O}_t=\mathcal{O}_{\mathbb{C}^k,t}$. Note that $\mathcal{O}'_x$ and $\mathcal{O}_t$ are regular local rings of dimensions $n+k$ and k, respectively. We have $\operatorname{ht}\mathcal{F}_x=n+k-k=n=r-\ell+1$. Hence by (4.6), the ring $\mathcal{O}_x$ is CM. Since the homomorphism $\pi^*:\mathcal{O}_t\to\mathcal{O}_x$ is finite, $\mathcal{O}_x$ is a CM $\mathcal{O}_t$-module. By (4.4), denoting by $\mathfrak{m}_t$ the maximal ideal in $\mathcal{O}_t$,

$$\operatorname{depth}(\mathfrak{m}_t;\mathcal{O}_x)+\operatorname{pd}_{\mathcal{O}_t}\mathcal{O}_x=\operatorname{depth}\mathfrak{m}_t.$$

We have $\operatorname{depth}(\mathfrak{m}_t;\mathcal{O}_x)=\dim_{\mathcal{O}_t}\mathcal{O}_x=\dim_{\mathcal{O}_x}\mathcal{O}_x=k$ and $\operatorname{depth}\mathfrak{m}_t=\dim\mathcal{O}_t=k$. Therefore, $\operatorname{pd}_{\mathcal{O}_t}\mathcal{O}_x=0$ and π is flat.

Set $X_t=\pi^{-1}(t)$, which has a natural structure of (discrete) analytic space and is supported by $S(\boldsymbol{s}_t)$. For x in X_t, we have $\mathcal{O}_{X_t,x}=\mathcal{O}_{\widetilde{U},x}/\mathcal{F}(V_t)_x$. Hence the lemma follows from (4.9) □

Since the regular values of h are dense, by Corollary 2.11, Theorem 5.5 and Lemma 5.7, we have the following theorem.

Theorem 5.8. — *We have*

$$\operatorname{Res}_{c^n}(\boldsymbol{s},E|_V;p)=\dim_{\mathbb{C}}\mathcal{O}_{\widetilde{U},p}/\mathcal{F}(V)_p.$$

Remark 5.9. — As we can see from the above proofs, the assumption that V is a complete intersection is necessary only to ensure that V admits a "smoothing" in $\widetilde{U}$.

6. Topological expression

Let V, $\widetilde{U}$, E and $\boldsymbol{s}$ be as in (1c). We assume that V is a complete intersection in $\widetilde{U}$ with at most an isolated singularity at p. Let $W_\ell(\mathbb{C}^r)$ denote the Stiefel manifold of ℓ-frames in $\mathbb{C}^r$. It is known that the space $W_\ell(\mathbb{C}^r)$ is $2(r-\ell)$-connected and $\pi_{2n-1}(W_\ell(\mathbb{C}^r)) \simeq \mathbb{Z}$ (recall $2r-2\ell+1=2n-1$). Let L denote the link of (V,p). Note that both of $W_\ell(\mathbb{C}^r)$ and L have a natural generator for the $(2n-1)$-st homology. Thus the degree of the map

$$\boldsymbol{s}|_L : L \longrightarrow W_\ell(\mathbb{C}^r)$$

is well-defined.

As for the algebraic expression in the previous section, Theorem 6.1 below is proved by the following steps, noting that the mapping degree satisfies the conservation law under perturbations of sections:

(1) reducing to the case of non-singular base space (as Corollary 2.11 or Lemma 5.7),

(2) reducing the number of sections (as Lemma 2.6 or Lemma 5.4), and going to the case of one section,

(3) applying Corollary 2.2 (or further reducing to the case of non-degenerate singularities, where everything is 1).

Theorem 6.1. — *We have*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p) = \deg \boldsymbol{s}|_L.$$

7. Examples and applications

7a. Index of a 1-form and multiplicity of a function. — Let M be a complex manifold of dimension n. The holomorphic cotangent bundle T^*M of M is naturally identified with its real cotangent bundle. Thus a C^∞ 1-form θ on M may be thought of as a section of T^*M. For a compact connected component S of the zero set $S(\theta)$ of θ having a neighborhood disjoint from the other components, we define the *index* $\mathrm{Ind}(\theta, S)$ of θ at S by

$$\mathrm{Ind}(\theta, S) = \mathrm{Res}_{c^n}(\theta, T^*M; S).$$

If M is compact and if $S(\theta)$ admits only a finite number of connected components (S_λ), by Proposition 1.3, we have

$$\sum_\lambda \mathrm{Ind}(\theta, S_\lambda) = (-1)^n \chi(M).$$

If θ is holomorphic and if S_λ consists of a point p, $\mathrm{Ind}(\theta, p)$ has the analytic, algebraic and topological expressions as given in the previous sections.

If we do similarly for a vector field v, we have the Poincaré-Hopf theorem for v.

For a C^∞ function f on M, its differential df is a section of T^*M and we have $S(df) = C(f)$, the critical set of f. For a compact connected component S of $C(f)$ as above, we define the *multiplicity* $m(f,S)$ of f at S by

$$m(f,S) = \mathrm{Ind}(df,S) = \mathrm{Res}_{c^n}(df, T^*M; S).$$

Note that, if f is holomorphic and if S consists of a point p, it coinsides with the usual multiplicity of f at p (*cf.* (3c) 1).

Now we consider the global situation. Let $f : M \to C$ be a holomorphic map of M onto a complex curve (Riemann surface) C. The differential $df : TM \to f^*TC$ of f determines a section of the bundle $T^*M \otimes f^*TC$, which is also denoted by df. The set of zeros of df is the critical set $C(f)$ of f. Suppose $C(f)$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then we have the residue $\mathrm{Res}_{c^n}(df, T^*M \otimes f^*TC; S_\lambda)$ for each λ. If M is compact, by Proposition 1.3,

$$\sum_\lambda \mathrm{Res}_{c^n}(df, T^*M \otimes f^*TC; S_\lambda) = \int_M c^n(T^*M \otimes f^*TC). \tag{7.1}$$

If the critical value set $D(f)$ of f consists of only isolated points, we have

$$\mathrm{Res}_{c^n}(df, T^*M \otimes f^*TC; S_\lambda) = \mathrm{Res}_{c^n}(df, T^*M; S_\lambda) = m(f, S_\lambda).$$

and, if moreover M is compact,

$$\int_M c^n(T^*M \otimes f^*TC) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

where F denotes a general fiber of f (*cf.* [**IS**, 2]). Thus in this situation, (7.1) becomes

$$\sum_\lambda m(f, S_\lambda) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

In particular, if $C(f)$ consists of isolated points, we recover a formula of [**I**] (see also [**F**, Example 14.1.5] and [**HL**, VI 3]):

$$\sum_{p \in C(f)} m(f,p) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)). \tag{7.2}$$

7b. Index of a holomorphic 1-form of Ebeling and Gusein-Zade

Let V be a complete intersection in $\widetilde{U}$ with an isolated singularity at p and defined by $(h_1, \dots, h_k)$, as before. Also, let L be the link of (V,p). For a holomorphic 1-form θ on $\widetilde{U}$, we consider the $(k+1)$-tuple $\widetilde{\boldsymbol{s}} = (\theta, dh_1, \dots, dh_k)$ of sections of $T^*\widetilde{U}$, which is of rank $n+k$. Thus $r - \ell + 1 = n + k - (k+1) + 1 = n$. We assume that $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$. Let $\boldsymbol{s} = \widetilde{\boldsymbol{s}}|_V$, which defines a map of $V \smallsetminus \{p\}$ to $W_\ell(\mathbb{C}^r)$. It should be emphasized that here we take the restrictions of components of $\widetilde{\boldsymbol{s}}$ as sections and not as differential forms.

Following [**EG1**], with different naming and notation, we define the *V-index* $\mathrm{Ind}_V(\theta,p)$ of θ at p by

$$\mathrm{Ind}_V(\theta,p) = \deg \boldsymbol{s}|_L.$$

Then by Theorem 6.1, it coincides with $\mathrm{Res}_{c^n}(\boldsymbol{s}, T^*\widetilde{U}|_V; p)$ and by Theorems 3.2 and 5.8, it has analytic and algebraic expressions. In fact the algebraic one is already given in [**EG1**].

Remark 7.3. — For a vector field, there is a similar index, which is called the GSV-index ([**GSV**], [**SS1**]). Namely, in the above situation let v be a holomorphic vector field on $\widetilde{U}$. Assume that v is tangent to $V \smallsetminus \{p\}$ and non-vanishing there. Set $\widetilde{\boldsymbol{s}} = (v, \overline{\mathrm{grad}\ h_1}, \dots, \overline{\mathrm{grad}\ h_k})$ and $\boldsymbol{s} = \widetilde{\boldsymbol{s}}|_V$. Then the GSV-index of v at p is defined by

$$\mathrm{GSV}(v, p) = \deg \boldsymbol{s}|_L.$$

Since $\boldsymbol{s}$ involves anti-holomorphic objects, we cannot directly apply our previous results. Note that it coincides with the "virtual index" of v ([**LSS**], [**SS2**]) and that there is an algebraic formula for it as a homological index, when $k = 1$ ([**Go**]).

7c. Multiplicity of a function on a local complete intersection. — We refer to [**IS**] for details of this subsection. Let V be a subvariety of dimension n in a complex manifold W of dimension $n + k$. We assume that there exist a holomorphic vector bundle N of rank k and a holomorphic section σ of N, generically transverse to the zero section, with $V = \sigma^{-1}(0)$. Thus V is a local complete intersection defined by the local components of σ. Note that the restriction of N to the non-singular part V' coincides with the normal bundle of V' in W. We denote the virtual bundle $(T^*W - N^*)|_V$ by τ_V^* and call it the virtual cotangent bundle of V. Let g be a C^∞ function on W and let f and f' be its restrictions to V and V', respectively. We define the singular set $S(f)$ of f by $S(f) = \mathrm{Sing}(V) \cup C(f')$. As in the case of vector bundles, we may define the localization of the n-th Chern class of τ_V^* by df, which in turn defines the residue $\mathrm{Res}_{c^n}(df, \tau_V^*; S)$ at each compact connected component S of $S(f)$. We define the *virtual multiplicity* $\widetilde{m}(f, S)$ of f at S by

$$\widetilde{m}(f, S) = \mathrm{Res}_{c^n}(df, \tau_V^*; S). \tag{7.4}$$

The *multiplicity* of f at S is then defined by

$$m(f, S) = \widetilde{m}(f, S) - \mu(V, S), \tag{7.5}$$

where, $\mu(V, S)$ denotes the (generalized) Milnor number of V at S as defined in [**BLSS**] (*cf.* [**A**], [**P**], [**PP**] in the case $k = 1$). Note that if S consists of a point p, it is the usual Milnor number $\mu(V, p)$ of the isolated complete intersection singularity (V, p) ([**Mi**], [**H**], see also [**Lo**]).

Note that, if S is in V', we have $\mathrm{Res}_{c^n}(df, \tau_V^*; S) = \mathrm{Res}_{c^n}(df, T^*V'; S)$. On the other hand, in this case we have $\mu(V, S) = 0$ so that $m(f, S)$ coincides with the one in (7a).

Let $g : W \to C$ be a holomorphic map onto a complex curve C and set $f = g|_V$, $f' = g|_{V'}$ and $S(f) = \mathrm{Sing}(V) \cup C(f')$. We assume that $S(f)$ is compact. We further set $V_0 = V \smallsetminus S(f)$ and $f_0 = g|_{V_0}$. Thus df_0 is a non-vanishing section of the bundle

$T^*V_0 \otimes f_0^*TC$, which is of rank n. If we look at $c^n(\varepsilon)$, $\varepsilon = \tau_V^* \otimes f^*TC$ and we see that there is a canonical localization $c_S^n(\varepsilon, df)$ in $H^{2n}(V, V \smallsetminus S; \mathbb{C})$ of $c^n(\varepsilon)$.

Let $(S_\lambda)_\lambda$ be the connected components of S and let $(R_\lambda)_\lambda$ be as in (1b). Then $c_S^n(\varepsilon, df)$ defines, for each λ, the residue $\mathrm{Res}_{c^n}(df, \tau_V^* \otimes f^*TC; S_\lambda)$, which is given by a formula similar to (1.2). Note that, if S_λ is in the non-singular part V', it coincides with the one in (7a). If V is compact, by Proposition 1.5, we have

$$\sum_\lambda \mathrm{Res}_{c^n}(df, \tau_V^* \otimes f^*TC; S_\lambda) = \int_V c^n(\tau_V^* \otimes f^*TC).$$

The both sides in the above are reduced as follows. If $f(S(f))$ consists of isolated points, we may write

$$\mathrm{Res}_{c^n}(df, \tau_V^* \otimes f^*TC; S_\lambda) = \widetilde{m}(f, S_\lambda) = m(f, S_\lambda) - \mu(V, S_\lambda)$$

and, if moreover, V is compact, then we have

$$\int_V c^n(\tau_V^* \otimes f^*TC) = (-1)^n\left(\chi(V) - \chi(F)\,\chi(C)\right) + \sum_\lambda \mu(V, S_\lambda),$$

where F is a general fiber of f ([**IS**, Lemma 5.2]). Thus, in the above situation, we have ([**IS**, Theorem 5.5]):

$$\sum_\lambda m(f, S_\lambda) = (-1)^n\left(\chi(V) - \chi(F)\,\chi(C)\right).$$

In particular, if $S(f)$ consists only of isolated points,

$$\sum_{p \in S(f)} m(f, p) = (-1)^n\left(\chi(V) - \chi(F)\,\chi(C)\right), \tag{7.6}$$

which generalizes (7.2) for a singular variety V.

If S_λ consists of a single point p, the residue $\mathrm{Res}_{c^n}(df, \tau_V^*; p)$ is given as follows. Let $\widetilde{U}$ be a small neighborhood of p in W so that the bundle N admits a frame $(\nu_1, \ldots, \nu_k)$ on $\widetilde{U}$. We write $\sigma = \sum_{i=1}^k h_i\,\nu_i$ with h_i holomorphic functions on $\widetilde{U}$. Then V is defined by $(h_1, \ldots, h_k)$ in $\widetilde{U}$. Consider the $(k+1)$-tuple of sections

$$\widetilde{\boldsymbol{s}} = (dg, dh_1, \ldots, dh_k)$$

of $T^*\widetilde{U}$. By the assumption, we have $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$. Since the rank of $T^*\widetilde{U}$ is $n+k$, we have the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, T^*\widetilde{U}|_V; p)$, $\boldsymbol{s} = \widetilde{\boldsymbol{s}}|_V$. Then we have ([**IS**, Theorem 4.6])

$$\widetilde{m}(f, p) = \mathrm{Res}_{c^n}(\boldsymbol{s}, T^*\widetilde{U}|_V; p). \tag{7.7}$$

The virtual multiplicity $\widetilde{m}(f, p)$ was defined as the residue of df on the virtual bundle τ_V^* (*cf.* (7.4)) and this definition led us to a global formula as (7.6). The identity (7.7) shows that it coincides with the residue of $\boldsymbol{s} = (dg|_V, dh_1|_V, \ldots, dh_k|_V)$ on the vector bundle $T^*\widetilde{U}|_V$. Thus we have various expressions for $\widetilde{m}(f, p)$ as given in the previous sections; by Theorem 3.2 we have a way to compute $\widetilde{m}(f,p)$ explicitly, by Theorem 5.8 we may express

$$\widetilde{m}(f,p)=\dim_{\mathbb{C}}\mathcal{O}_{n+k}/(J(g,h_1,\dots,h_k),h_1,\dots,h_k), \tag{7.8}$$

where $J(g,h_1,\dots,h_k)$ denotes the Jacobian ideal of the map $(g,h_1,\dots,h_k)$, *i.e.*, the ideal generated by the $(k+1)\times(k+1)$ minors of the Jacobian matrix $\frac{\partial(g,h_1,\dots,h_k)}{\partial(z_1,\dots,z_{n+k})}$, and by Theorem 6.1,

$$\widetilde{m}(f,p)=\mathrm{Ind}_V(dg,p). \tag{7.9}$$

From (7.5), (7.8) and the identity (*cf.* [**Gr**], [**Le**])

$$\mu(V,p)+\mu(V_g,p)=\dim_{\mathbb{C}}\mathcal{O}_{n+k}/(J(g,h_1,\dots,h_k),h_1,\dots,h_k),$$

where V_g denotes the complete intersection defined by $(g,h_1,\dots,h_k)$, assuming $g(p)=0$, we get

$$m(f,p)=\mu(V_g,p). \tag{7.10}$$

7d. Some others. — Let V be a complete intersection defined by $(h_1,\dots,h_k)$ in $\widetilde{U}$ and p an isolated singularity of V, as before.

The n-the polar multiplicity $m_n(V,p)$ of Gaffney ([**Ga**]) is defined by

$$m_n(V,p)=\dim_{\mathbb{C}}\mathcal{O}_{n+k}/(J(\ell,h_1,\dots,h_k),h_1,\dots,h_k),$$

where ℓ is a general linear function. By (7.8) and (7.9), we may write

$$m_n(V,p)=\mathrm{Ind}_V(d\ell,p)=\widetilde{m}(\ell|_V,p).$$

Also, in the expression

$$\mathrm{Eu}(V,p)=1+(-1)^{n+1}\mu(V_\ell,p)$$

for the Euler obstruction $\mathrm{Eu}(V,p)$ of V at p (*cf.* [**Du**], [**K**], see also [**BLS**]), we have by (7.10),

$$\mu(V_\ell,p)=m(\ell|_V,p).$$

Note that these local invariants appear in the comparison of the Schwartz-MacPherson, Mather and Fulton-Johnson classes of a local complete intersection with isolated singularities (*cf.* [**OSY**], [**Su1**]).

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