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ARC-ANALYTICITY IS AN OPEN PROPERTY

by

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***Abstract.* —** We prove that the locus of the points where a bounded continuous subanalytic function is not arc-analytic, is a closed nowhere dense subanalytic set. This shows that the property of being arc-analytic at a point, is an open property.

***Résumé* (L'arc-analyticité est une propriété ouverte). —** Nous montrons que l'ensemble de points où une fonction sous-analytique, bornée et continue n'est pas arc-analytique est un ensemble sous-analytique fermé. Autrement dit : la propriété d'être arc-analytique en un point est une propriété ouverte.

1. Introduction

Let U be an open subset of $\mathbb{R}^n$. Following [**9**] we say that a map $f : U \to \mathbb{R}^k$ is *arc-analytic* if for any analytic arc $\alpha : (-\varepsilon, \varepsilon) \to U$, $f \circ \alpha$ is also analytic. In general arc-analytic maps are very far from being analytic, in particular there are arc-analytic functions which are not subanalytic [**11**], not continuous [**3**], with a non-discrete singular set [**12**]. Hence it is natural to consider only arc-analytic maps with subanalytic graphs. Earlier T.-C. Kuo, motivated by equisingularity problems, introduced in [**8**] the notion of *blow-analytic* functions, *i.e.*, functions which become analytic after a composition with appropriate proper bimeromorphic maps (e.g. a composition of blowings up with smooth centers). Clearly any blow-analytic mapping is arc-analytic and subanalytic. The converse holds in a slightly weaker form [**2**] (see also [**16**]). Blow-analytic maps have been studied by several authors (see the survey [**4**]). It is known that in general subanalytic and arc-analytic functions are continuous [**9**], but not necessarily (locally) Lipschitz [**4**], [**17**].

The main result of this note is Theorem 3.1, which claims that the locus of the points at which a bounded, continuous, subanalytic function $f : U \to \mathbb{R}$ is not arc-analytic, is a closed subanalytic subset of U. In other words, if f is analytic on any germ of analytic arc at a given point $a \in U$, then f is arc-analytic in a neighbourhood of a.

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This property is of interest if we deal with germs of arc-analytic functions. For instance let us recall the main result of [**13**]. It states the following: if g is an arc-analytic function, such that for some natural r the function $f = g^r$ is analytic, then g is locally Lipschitz. Moreover, if r is less than the multiplicity of f, then g is C^1. Now, if we are interested in a local version of this result, thanks to our Theorem 3.1, it is enough to check the arc-analyticity of g only on analytic arcs passing through a given point.

The main tool in the proof of Theorem 3.1 is Parusiński's *Rectilinearization of subanalytic function* [**16**]. We thank the referee for careful reading and valuable remarks.

2. Definitions – Notations

2.1. Locally blow-analytic functions. — We recall some of the notions used in this paper (for more information see for instance [**3**], [**4**], [**5**], [**8**], [**11**], [**12**], [**18**]).

We recall first a definition of a local blowing up. Let M be an analytic manifold and $\Omega \subset M$ an open set. Assume that X is an analytic submanifold of M, closed in Ω. Then we can define the mapping $\tau : \widetilde{\Omega} \to \Omega$, the blowing up of Ω with the centre X, see for instance [**7**] or [**14**]. A restriction of τ to an open subset of $\widetilde{\Omega}$ is called *a local blowing up with a smooth (nowhere dense) centre*. Local blowings up have the important arc lifting property. We state it precisely below:

Lemma 2.1 **(Arc lifting property)**. — *Let M be an analytic manifold and let $\sigma : W \to M$ be a finite composition of local blowings up with smooth centres. Assume that $\gamma : (-\varepsilon, \varepsilon) \to M$ is an analytic arc, $\gamma((-\varepsilon, \varepsilon)) \subset \sigma(W)$. Then there exists an analytic arc $\widetilde{\gamma} : (-\varepsilon, \varepsilon) \to W$ such that $\sigma \circ \widetilde{\gamma} = \gamma$.*

Let U be a neighbourhood of the origin of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ denote a map defined on U except possibly some thin subset of U. We say that f is *locally blow-analytic* via a locally finite collection of analytic modifications $\sigma_\alpha : W_\alpha \to \mathbb{R}^n$, if for each α we have

i) W_α is isomorphic to $\mathbb{R}^n$ and σ_α is the composition of finitely many local blowings up with smooth nowhere dense centres, and $f \circ \sigma_\alpha$ has an analytic extension on W_α.

ii) There are subanalytic compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup \sigma_\alpha(K_\alpha)$ is a neighbourhood of $\overline{U}$.

The notion of *(locally) blow-analytic* functions (or maps) is very much related to the notion of *arc-analytic* functions, *i.e.*, functions $f : U \to \mathbb{R}$ such that $f \circ \alpha$ is analytic for any analytic arc $\alpha : I \to U$, here U is an open subset of $\mathbb{R}^n$ and I is an open interval. Indeed in [**2**], see also [**16**], it is proved that an *arc-analytic* function has *subanalytic* graph if and only if it is *locally blow-analytic*.

Let $f : U \to \mathbb{R}$ be a subanalytic function defined in an open subset of $\mathbb{R}^n$. We will say that f *is not arc-analytic at a point* $x \in U$, if there exists an analytic arc $\gamma : (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = x$ and the composed function $f \circ \gamma$ is not analytic at $t = 0$.

3. Main Results

Our main result is the following theorem.

Theorem 3.1. — *Let $f : U \to \mathbb{R}$ be a bounded continuous subanalytic function defined in an open subset of $\mathbb{R}^n$. Then the locus of the points in U at which f is not arc-analytic, is a closed, nowhere dense, subanalytic subset of U.*

Remark 3.2. — If f is semialgebraic, then the locus of the points in U at which f is not arc-analytic, is a closed, nowhere dense, semialgebraic subset of U.

Proof. — Let us denote

$$S_{\mathrm{naa}}(f) = \{x \in U \mid f \text{ is not arc-analytic at } x\}.$$

Clearly the set $S_{\mathrm{naa}}(f)$ is contained in the singular set of f:

$$S_{\mathrm{na}}(f) = \{x \in U \mid f \text{ is not analytic at } x\}.$$

It is known ([**19**], [**10**], [**1**]), that the set $S_{\mathrm{na}}(f)$ is subanalytic, closed and nowhere dense in U (*i.e.*, $\dim S_{\mathrm{na}}(f) \leqslant n-1$). However, in general, the set $S_{\mathrm{na}}(f)$ is larger than the set $S_{\mathrm{naa}}(f)$. Our proof follows an idea from [**10**] and it uses some facts on subanalytic functions of one variable.

Lemma 3.3. — *A subanalytic (and continuous) function in one variable $f \circ \gamma$ is not analytic at $0 \in \mathbb{R}$ for one of the following two reasons:*

i) *Puiseux expansion $f \circ \gamma(t) = \sum_{\nu=0}^{\infty} a_\nu t^{\nu/r}$, $t > 0$ contains a nonzero term with a fractional exponent. Hence $f \circ \gamma(t)$, $t > 0$ cannot be extended analytically through $0 \in \mathbb{R}$. Clearly, the same obstruction may come from extending of $f \circ \gamma(t)$, $t < 0$.*

ii) *Both functions $g_+ = f \circ \gamma(t)$, $t > 0$ and $g_- = f \circ \gamma(t)$, $t < 0$ have analytic extensions through 0, but the extensions of g_+ and g_- are not equal.*

Proof. — Immediate from the existence of Puiseux expansions for g_+ and g_-. □

The main tool in the proof of our theorem is the *Rectilinearization of subanalytic functions* due to Parusiński [**16**], [**15**]. In fact, this is a stronger version of Hironaka's *Rectilinearization Theorem* ([**7**], see also [**1**]). For the reader's convenience we recall it here.

***Theorem 3.4* (Parusiński [16])**. — *Let $f : U \to \mathbb{R}$ be a bounded continuous subanalytic function defined in an open subset of $\mathbb{R}^n$. Then there exists a locally finite collection Ψ of real analytic morphisms $\phi_\alpha : W_\alpha \to \mathbb{R}^n$ such that:*

i) *each* W_α *is isomorphic to* $\mathbb{R}^n$ *and there are compact subsets* $K_\alpha \subset W_\alpha$ *such that*

$$\bigcup \phi_\alpha(K_\alpha) = \overline{U}$$

· ii) *for each* α*, there exists* $r_i \in \mathbb{N}, i = 1, \ldots, n$*, such that*

$$\phi_\alpha = \sigma_\alpha \circ \psi_\alpha,$$

where $\sigma_\alpha : V_\alpha \to \mathbb{R}^n, V_\alpha$ *isomorphic to* $\mathbb{R}^n$*, is the composition of finite sequence of local blowings up with smooth centres and*

(3.1) $$\psi_\alpha = (\varepsilon_1 x_1^{r_1}, \varepsilon_2 x_2^{r_2}, \ldots, \varepsilon_n x_n^{r_n}), \textit{for some } \varepsilon_i = -1 \textit{ or } 1.$$

iii) *for each* α*,* $\phi_\alpha(W_\alpha) \subset \overline{U}$*, and* $f \circ \phi_\alpha$ *extends from* $\phi_\alpha^{-1}(U)$ *on* W_α *to one of the following functions:*

a) the function identically equal to zero,

b) a normal crossings.

iv) *if* $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha \in \Psi$ *and* $\phi_\alpha(0) \in U$*, then* $\phi_\alpha(W_\alpha) \subset U$ *and for each* ψ *as in* (3.1) *(i.e. with all possible* ε_i*, but fixed* r_i*), the composition* $\sigma_\alpha \circ \psi \in \Psi$*.*

Remark 3.5. — The original statement of Theorem 2.7 in [**16**] contains an inaccuracy: at (i) it is claimed that $\bigcup \phi_\alpha(K_\alpha)$ is a neighbourhood of $\overline{U}$, but in fact the family $\phi_\alpha(K_\alpha)$ is only a covering of $\overline{U}$. However the set $\bigcup \sigma_\alpha(K_\alpha)$ is actually a neighbourhood of $\overline{U}$. Note that, as stated in theorem 2.7 in [**16**], in the claim (iii) we have also the third possibility, namely that $f \circ \phi_\alpha$ extends to an inverse of normal crossing. But this will not happen in our case since we consider only bounded functions.

We consider now a composed function $g_\alpha = f \circ \sigma_\alpha : \sigma_\alpha^{-1}(U) \to \mathbb{R}$. Let Q_α be an open quadrant in $V_\alpha = \mathbb{R}^n$. Note that by (iii) in the above theorem the function $g_\alpha = f \circ \sigma_\alpha$ extends analytically on Q_α. For simplicity we denote this extension again by g_α, observe that this extension is subanalytic.

We will study the arc-analyticity of our subanalytic function $g_\alpha : Q_\alpha \to \mathbb{R}$ also at the points of the boundary of Q_α. To this end we denote by $S^+_{\mathrm{naa}}(g_\alpha)$ the set of points $x \in V_\alpha$, such that there exists an analytic arc

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow V_\alpha, \quad \gamma(0) = x, \ \gamma(0, \varepsilon) \subset Q_\alpha,$$

and such that $g_\alpha \circ \gamma(t)$, $t > 0$, cannot be extended analytically on $(-\varepsilon', \varepsilon)$, for any $\varepsilon' > 0$.

We have the following lemma.

Lemma 3.6. — *The set* $S^+_{\mathrm{naa}}(g_\alpha)$ *is a closed subanalytic, nowhere dense, subset of* V_α*.*

Proof. — Clearly $S^+_{\mathrm{naa}}(g_\alpha) \subset \overline{Q_\alpha} \setminus Q_\alpha$. We may assume that Q_α is the set $\{x_i > 0 \mid i = 1, \ldots, n\}$. Recall that g_α is analytic on this quadrant, hence $S^+_{\mathrm{naa}}(g_\alpha)$ will be contained in its boundary.

By Theorem 3.4, there are integers $r_i \in \mathbb{N}$, $i = 1, \ldots, n$, such that

(3.2) $$h_\alpha = g_\alpha(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n}),$$

extends to an analytic function on $W_\alpha = \mathbb{R}^n$. Let us denote by H_i the hyperplane $\{x_i = 0\}$. Since our function g_α is analytic on the first quadrant $\{x_i > 0 \mid i = 1, \ldots, n\}$, then clearly we have $S^+_{\mathrm{naa}}(g_\alpha) \subset \bigcup_{i=1}^n \overline{H}{}^+_i$, where $H^+_i = \{x \in H_i;\, x_j > 0,\, j \in \{1, \ldots, n\} \smallsetminus \{i\}\}$ is an open quadrant in H_i. Let us consider fixed quadrant $\overline{H}{}^+_1 = \{x_1 = 0;\, x_j \geqslant 0,\, j \in \{2, \ldots, n\}\}$. Now we have a Puiseux expansion (which follows from (3.2))

$$g_\alpha(x_1, x') = \sum_{\nu=0}^{\infty} a_\nu(x') x_1^{\nu/r_1}, \quad \nu, r_1 \in \mathbb{N}, \tag{3.3}$$

for $x' = (x_2, \ldots, x_n)$ and x_1 such that $x_i > 0$, $i = 1, \ldots, n$. Moreover, a_ν are analytic functions in H^+_1 such that $a_\nu(x_2^{r_2}, \ldots, x_n^{r_n})$ extend to analytic functions.

Let $(0, x') \in H^+_1$. The following observations are immediate consequences of (3.3):

i) if there is an open (in H^+_1) neighbourhood Ω of x' such that $a_\nu = 0$ in Ω, for all $\nu \in \mathbb{N} \smallsetminus r_1\mathbb{N}$, then $x' \notin S^+_{\mathrm{naa}}(g_\alpha)$, (more precisely $\Omega \cap S^+_{\mathrm{naa}}(g_\alpha) = \varnothing$).

ii) if there exists $\nu_0 \in \mathbb{N} \smallsetminus r_1\mathbb{N}$ such that $a_{\nu_0}(x') \neq 0$, then g_α cannot be extended, through $x_1 = 0$, on the arc (linear segment) $x_1 \to (x_1, x')$. Therefore then $(0, x') \in S^+_{\mathrm{naa}}(g_\alpha)$.

Observe that in the first case we may assume that $\Omega = H^+_1$, since all a_ν are analytic functions in H^+_1. So in this case $H^+_1 \cap S^+_{\mathrm{naa}}(g_\alpha) = \varnothing$.

So we are left with the second case. We shall prove that

$$H_1 \cap S^+_{\mathrm{naa}}(g_\alpha) = H_1 \cap \overline{Q_\alpha}. \tag{$*$}$$

Note that here we are in the hyperplane H_1 and not in the open quadrant H^+_1. By i), ii) and $(*)$ it follows that $S^+_{\mathrm{naa}}(g_\alpha)$ is closed and subanalytic.

To prove $(*)$ we denote by ν_0 the smallest $\nu \in \mathbb{N} \smallsetminus r_1\mathbb{N}$ such that $a_\nu \not\equiv 0$ in H^+_1. Let $(0, x') \in \overline{H}{}^+_1$, if $a_{\nu_0}(x') \neq 0$, then by ii), $(0, x') \in S^+_{\mathrm{naa}}(g_\alpha)$. Assume that $a_{\nu_0}(x') = 0$. Let $\eta(t)$, $t \in (-\varepsilon, \varepsilon)$ be an analytic arc in H_1 such that $\eta(0) = x'$ and $\eta(t) \in H^+_1$, $a_{\nu_0}(\eta(t)) \neq 0$ for $t \in (0, \varepsilon)$. Let r be the smallest common multiple of $r_2, \ldots, r_n$. By (ii) of Theorem 3.4 it follows that $a_\nu(\eta(t^r))$ is analytic at $0 \in \mathbb{R}$, for any $\nu \in \mathbb{N}$. For simplicity we denote again $\eta(t^r)$ by $\eta(t)$.

We are going to choose a suitable exponent $N \in \mathbb{N}$ such that on the arc

$$\gamma(t) = (t^N, \eta(t)), \quad t > 0,$$

the function g_α cannot be analytically extended through 0. Note that, if we substract in (3.3), all terms $a_\nu(x') x_1^{\nu/r_1}$ with $\nu < \nu_0$, the set $S^+_{\mathrm{naa}}(g_\alpha)$ remains the same (indeed all these terms are analytic in V_α). So we may assume that in (3.3) we have only terms for $\nu \geqslant \nu_0$. Hence we obtain the Puiseux expansion

$$g_\alpha(t^N, \eta(t)) = \sum_{\nu=\nu_0}^{\infty} a_\nu(\eta(t)) t^{\nu N/r_1}, \quad t > 0. \tag{3.4}$$

Denote by k_0 the order of $a_{\nu_0}(\eta(t))$, and take $N \in \mathbb{N}$ such that $\nu_0 N$ is not divisible by r_1 and

$$N \geqslant k_0 r_1.$$

Thus, for any $\nu > \nu_0$, the order of $a_\nu(\eta(t))t^{\nu N/r_1}$ is strictly greater than the order of $a_{\nu_0}(\eta(t))t^{\nu_0 N/r_1}$. So in the expansion (3.4) there is a nonzero term with fractional exponent. Hence the function $g_\alpha(t^N, \eta(t))$ cannot be extended analytically through 0. This ends the proof of Lemma 3.6. □

Remark 3.7. — We proved actually that

$$S^+_{\mathrm{naa}}(g_\alpha) = \bigcup_{i \in I} H_i \cap \overline{Q}_\alpha,$$

where I is a subset (possibly empty) of $\{1, \dots, n\}$.

We study now the case analogous to the case ii) of Lemma 3.3. Let $g_\alpha = f \circ \sigma_\alpha : V \to \mathbb{R}$, where V is an open subset of $\mathbb{R} \times H_1^+ \subset V_\alpha$. Assume that g_α has the following expansions (on the both sides of H_1)

$$g^+_\alpha = g_\alpha(x_1, x') = \sum_{\nu=0}^{\infty} a_\nu(x') x_1^{\nu/r_1}, \quad \text{for } x_1 > 0, \tag{3.5}$$

$$g^-_\alpha = g_\alpha(x_1, x') = \sum_{\nu=0}^{\infty} b_\nu(x')(-x_1)^{\nu/r_1}, \quad \text{for } x_1 < 0. \tag{3.6}$$

As before a_ν and b_ν are analytic functions in H_1^+ such that for any $\nu \in \mathbb{N}$, $a_\nu(x_2^{r_2}, \dots, x_n^{r_n})$ and $b_\nu(x_2^{r_2}, \dots, x_n^{r_n})$ extend to analytic functions on H_1.

Denote by $S^\pm_{\mathrm{naa}}(g_\alpha)$ the set of points $x \in H_1$ such that there exists an analytic arc

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow V, \quad \gamma(0) = x, \ \gamma(t) \in V, \text{ for } t \neq 0,$$

and such that the analytic extension of $g^+_\alpha \circ \gamma(t)$, $t > 0$, does not coincide with $g^-_\alpha \circ \gamma(t), t < 0$.

Now we have the following lemma.

Lemma 3.8. — *$S^\pm_{\mathrm{naa}}(g_\alpha)$ is a closed subanalytic, nowhere dense, subset of V_α. More precisely, if $S^\pm_{\mathrm{naa}}(g_\alpha)$ is nonempty, then $S^\pm_{\mathrm{naa}}(g_\alpha) = \overline{V \cap H_1}$.*

Proof. — Note that, if $S^\pm_{\mathrm{naa}}(g_\alpha)$ is nonempty, then there exists $\nu \in \mathbb{N}$ such that $a_\nu \not\equiv b_\nu$. Let ν_0 be the smallest such a ν. Put $c_{\nu_0}(x') = a_{\nu_0}(x') - b_{\nu_0}(x')$. Take $(0, x') \in H_1$, such that $c_{\nu_0}(x') = 0$. Choose, as in the proof of Lemma 3.6, an analytic arc $\eta(t)$, $t \in (-\varepsilon, \varepsilon)$ in H_1 such that $\eta(0) = x'$ and $\eta(t) \in H_1 \cap V$, $c_{\nu_0}(\eta(t)) \neq 0$ for $t \neq 0$ with a property that all $a_\nu(\eta(t^r))$ and $b_\nu(\eta(t^r))$ are analytic at $0 \in \mathbb{R}$, for any $\nu \in \mathbb{N}$. As in the proof of Lemma 3.6 we may assume that $\nu \geqslant \nu_0$ in the expansions (3.5) and (3.6).

Take an odd integer N greater than the order of $c_{\nu_0}(\eta(t))$. Then on the arc

$$\gamma(t) = (t^N, \eta(t)), \quad t \in (-\varepsilon, \varepsilon),$$

the analytic extension of $g_\alpha^+ \circ \gamma(t)$, $t > 0$ does not coincide with $g_\alpha^- \circ \gamma(t)$, $t < 0$. Hence $S^{\pm}_{\mathrm{naa}}(g_\alpha) = \overline{V \cap H_1}$, which proves Lemma 3.8. □

In Lemma 3.8 we considered only arcs which go from a quadrant Q_1 to a quadrant Q_2, and where the boundaries of those quadrants have a common part of dimension $n-1$ (*i.e.*, their boundaries have a common face). Clearly the same arguments are valid for any two arbitrary quadrants.

Hence Lemmas 3.6 and 3.8 imply:

Lemma 3.9. — *The set $S_{\mathrm{naa}}(g_\alpha) = \{x \in \sigma_\alpha^{-1}(U) \mid g_\alpha$ is not arc-analytic at $x\}$ is a closed subanalytic (even compact), nowhere dense, subset of V_α.*

We are now in a position to conclude the proof of Theorem 3.1. By the arc lifting property (*cf.* Lemma 2.1) of each σ_α it is clear that

$$S_{\mathrm{naa}}(f) = U \cap \bigcup_\alpha \sigma_\alpha(K_\alpha \cap S_{\mathrm{naa}}(g_\alpha)),$$

where $K_\alpha \subset V_\alpha$ are compact subanalytic sets such that $\bigcup \sigma_\alpha(K_\alpha) = \overline{U}$, *cf.* Theorem 3.4 (i). So, $S_{\mathrm{naa}}(f)$ is closed in U and subanalytic as a locally finite union of images of compact subanalytic sets by analytic mappings. □

To justify Remark 3.2 that for a semialgebraic continuous function $f : U \to \mathbb{R}$ the set $S_{\mathrm{naa}}(f)$ is semialgebraic, it is enough to recall that the Rectilinearization of functions holds in the real algebraic category (see [**6**],[**2**], [**16**]). In fact in this case we have global centres for blowings up.

Remark 3.10. — Note that Theorem 3.1 is no longer true if we do not assume continuity. Indeed let $f(x, y) = y$, $x \neq 0$, and $f(0, y) = 0$ otherwise. Clearly f is not continuous in any neighbourhood of the origin (but continuous at 0), it is subanalytic and arc-analytic at the origin, but not arc-analytic in any neighbourhood of the origin.

As an immediate consequence of Theorem 3.1, we have the following property, namely that arc-analyticity is an open property.

Corollary 3.11. — *Let U be an open neighbourhood of the origin in $\mathbb{R}^n$, and $f : (U, 0) \to (\mathbb{R}, 0)$ be a germ of a continuous, subanalytic function. Then f is arc-analytic in a neighbourhood of the origin, if and only if, for any germ of analytic arc $\alpha : ((-\varepsilon, \varepsilon), 0) \to (U, 0)$ $f \circ \alpha$ is analytic at the origin.*

Remark 3.12. — In general the set $S_{\mathrm{naa}}(f)$ is not analytic (neither arc-symmetric *cf.* [**9**]) it is only subanalytic and closed. Consider a continuous semialgebraic function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows: $z = f(x, y)$ is the smallest real root of the polynomial $z^3 + x^2yz - x^4$. Then

$$S_{\mathrm{naa}}(f) = S_{\mathrm{na}}(f) = \{x = 0,\, y \geqslant 0\},$$

so it is a closed half line.

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