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ASYMPTOTIC SOLUTIONS OF NON LINEAR WAVE EQUATIONS AND POLARIZED NULL CONDITIONS

by

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Dédié à la mémoire de Jean Leray,
un mathématicien exceptionnel et un grand homme.

Abstract. — The jump in generality made by Leray for the WKB type construction of high frequency asymptotic solutions of linear partial differential equations has allowed the treatment of arbitrary linear systems of partial differential equations. It also permitted the extension to quasilinear systems, and the appearance of new properties linked to the non linearities, in particular a distorsion of signals. The non linearity of a differential system is also an obstruction to the existence of global solutions of evolution problems. In the case of non linear wave equations on the Minkowski spacetime of dimension 4 it has been discovered by Christodoulou and Klainerman that a "null condition" satisfied by the non linearities leads to global existence results. The equations of the fundamental field equations (standard model, Einstein equations) are quasi linear second order partial differential equations, but not well posed due to gauge invariance. We introduce a "polarized null condition". We show it is satisfied by the standard model, but not quite by the Einstein equations. We construct for both systems asymptotic high frequency solutions with linear transport law along the rays. In the case of Einstein equations the wave inflicts a "back reaction" on the background metric.

***Résumé* (Conditions nulles polarisées).** — La généralisation faite par Leray de la méthode WKB pour la construction de solutions asymptotiques à haute fréquence de systèmes arbitraires d'équations aux dérivées partielles linéaires a permis le traitement de systémes quasilinéaires et l'apparition de propriétés nouvelles comme la distorsion des signaux. La non linéarité est aussi une obstruction à l'existence de solutions globales des systèmes d'évolution. On introduit une condition nulle polarisée, généralisation de la condition nulle de Christodoulou-Klainerman à des systèmes mal posés par suite de l'invariance de jauge. On montre qu'elle conduit à une équation de transport linéaire le long des rayons d'une solution asymptotique. Elle est satisfaite par le modèle standard, mais un terme résiduel dans le cas des équations d'Einstein conduit à une « réaction en retour » sur la métrique de base.

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1. Introduction

Leray [11], and Gårding Kotake Leray [7] have brought a fundamental improvement to the WKB construction of high frequency asymptotic solutions of linear partial differential equations as functions of the form $u = ve^{i\omega\varphi}$, with v a slowly varying amplitude, ω a large parameter and φ a scalar function called the phase. The method had be extended by Lax [10] to the construction of asymptotic solutions of first order linear systems as formal series

$$u = e^{i\omega\varphi}(v_0 + \frac{1}{\omega}v_1 + \cdots).$$

The jump in generality made by Leray allowed the treatment of arbitrary linear systems of partial differential equations. It also permitted the extension to quasilinear systems [2], and the appearance of new properties linked to the non linearities in some sense similar to shocks(1), in particular a distorsion of signals. The non linearity of a differential system is also an obstruction to the existence of global solutions of evolution problems. In the case of non linear wave equations on the Minkowski spacetime of dimension 4 it has been discovered by Christodoulou [6] and Klainerman [8] that a null condition satisfied by the non linearities leads to global existence results. The equations of the fundamental field equations (standard model, Einstein equations) are quasi linear second order partial differential equations, but not well posed due to gauge invariance. We introduce a polarized "null condition". We show it is satisfied by the standard model, but not quite by the Einstein equations. We construct for both these systems asymptotic high frequency solutions with linear transport law along the rays. In the case of Einstein equations the wave inflicts a "back reaction" [4] on the background metric, as was already noticed in[3].

2. The GKL linear theory

2.1. Linear systems. — We change slightly the notations of GKL to give it the geometrical aspects that it does possess. We write a linear differential system on a smooth pseudo riemannian manifold V under the form

$$L(x, D)u = b(x)$$

with x a point of V of local coordinates x^α, D the covariant derivative and u a field on V. The system reads in local coordinates and index notation

$$L^A_B(x, D)u^B \equiv \sum_{1\leqslant|a|\leqslant m_B-n_A} L^A_{B,a}(x)D^a u^B = b^A(x) \tag{2.1}$$

(1)See for instance [1].

where L^A_B a linear operator of order[2] $m_B - n_A$, summation over B and a is made and we denote as usual:

$$a = \alpha_1, \dots, \alpha_n, \qquad D^a = D^{|a|}_{\alpha_1 \cdots \alpha_n}, \qquad |a| = \alpha_1 + \cdots + \alpha_n.$$

We denote by H the principal part of L, represented in coordinates by the matrix of the terms of order $m_B - n_A$ in L^A_B (such a term may be absent):

$$H^A_B(x, D)u^B \equiv \sum_{|a|=m_B-n_A} H^A_{B,a}(x) D^a u^B.$$

GKL call wave any solution of the homogeneous system ($b \equiv 0$) associated with L.

2.2. Asymptotic waves. — Let $u^{(r)}(x,\xi)$, $r = 0, 1, \dots$ be a family of smooth fields defined on $V \times \mathbb{R}$. Let ω be a real parameter (called frequency by analogy with the WKB expansions). Let φ be a real function on V called phase. GKL consider a formal series on $V \times \mathbb{R}$ of the form

$$u^B(x,\xi) = \sum_{r=0}^{\infty} \omega^{-m_B - r} u^{B,r}(x,\xi). \tag{2.2}$$

For any field v on $V \times \mathbb{R}$ it holds that:

$$D_\alpha \{v(x,\xi)\}_{\xi=\omega\varphi(x)} = \{D_\alpha v(x,\xi) + \omega \varphi_\alpha v'(x,\xi)\}_{\xi=\omega\varphi(x)}$$

with

$$v' \equiv \frac{\partial v}{\partial \xi}, \qquad \varphi_\alpha \equiv \frac{\partial \varphi}{\partial x^\alpha}.$$

Inserting this identity in the formal computation of the action of the linear operator L on the formal series $u^B(x,\xi)_{\xi=\omega\varphi(x)}$ gives a formal series in powers of ω. The first term reads (summation in a and B, but not in A which labels the equation):

$$\sum_{|a|=m_B-n_A} \omega^{-n_A} H^A_{B,a}(x) \varphi^a \Big[\Big(\frac{\partial}{\partial \xi}\Big)^{m_B - n_A} u^{B,0}(x,\xi) \Big]_{\xi=\omega\varphi(x)}. \tag{2.3}$$

Definition 1. — A GKL asymptotic wave is a formal series of the type (2.2) such that the formal series obtained by its insertion in (2.1) is identically zero.

Neglecting terms irrelevant in the treatment obtained by n_A integrations with respect to ξ of each equation, the annulation of the term (2.3) is deduced from the equation

$$\sum_{|a|=m_B-n_A} H^A_{B,a}(x) \varphi^a \widetilde{u}^{B,0}(x,\xi) = 0, \qquad \widetilde{u}^B \equiv \Big(\frac{\partial}{\partial \xi}\Big)^{m_B} u^{B,0}.$$

[2] It can be shown that any linear system can be written under this form without modifying its characteristic polynomial. The numbers m and n are called Leray - Volevic indices.

A necessary and sufficient condition for these equations to have a solution $\widetilde{u}^{(0)}(x,\xi) \not\equiv 0$ is the vanishing of the following determinant:

$$\Delta(\varphi) \equiv \operatorname{Det}\Big(\sum_{|a|=m_B-n_A} H^A_{B,a}(x)\varphi^a\Big) = 0, \tag{2.4}$$

i.e. that $D\varphi$ be a solution of the characteristic (eikonal) equation of the operator L.

The phase φ being so chosen the first term $u^{(0)}$ of the asymptotic wave must be such that $\widetilde{u}^{(0)}$ belongs to the kernel of the linear homogeneous system:

$$\sum_{|a|=m_B-n_A} H^A_{B,a}(x)\varphi^a \widetilde{u}^{B,0}(x,\xi) = 0. \tag{2.5}$$

hence, supposing that the dimension of this kernel is 1 (simple characteristic), $u^{B,0}$ must be of the form

$$\widetilde{u}^{B,0} = U(x,\xi)h^B(x)$$

with h a particular solution of the system (2.5), depending only on x, and U a scalar function on $V \times \mathbb{R}$.

GKL show then that U must satisfy a linear propagation equation along the rays of the phase φ by writing the next term in the expansion, coefficient of ω^{-n_A-1}. Indeed the vanishing of this term reads (after n_A integrations with respect to ξ, $\widehat{a}_i$ means that α_i has been suppressed from the sequence a)

$$\begin{aligned}\sum_{|a|=m_B-n_A} \Big\{H^A_{B,a}(x)\varphi^a\widetilde{u}^{B,1}(x,\xi) + H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}\Big(\frac{\partial}{\partial\xi}\Big)^{m_B-1} D_{\alpha_i}u^{B,0}(x,\xi)\Big\}&\\ + \sum_{|a|=m_B-n_A-1} L^{1,A}_{B,a}\varphi^a\Big(\frac{\partial}{\partial\xi}\Big)^{m_B-1} u^{B,0}(x,\xi) = 0.&\end{aligned} \tag{2.6}$$

Since the determinant (2.4) is zero this equation can have a solution $\widetilde{u}^{(1)}$ only if the right hand side is orthogonal to the kernel $h^T(x)$ of the transposed linear system. Replacing $(\partial/\partial\xi)^{m_B-1}u^{B,0}$ by $\widehat{U}(x,\xi)h(x)$, with $\widehat{U}$ a primitive of U with respect to ξ leads to an ordinary first order differential system for $\widehat{U}$:

$$h^T_A(x)\{H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}D_{\alpha_i}[(\widehat{U}(x,\xi)h^B(x)] + \sum_{|a|=m_B-n_A-1} L^{1,A}_{B,a}\varphi^a\widehat{U}(x,\xi)h^B(x)\} = 0. \tag{2.7}$$

The identity

$$h^T_A(x)H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}D_{\alpha_i}h^B(x) \equiv D_{\alpha_i}\Delta(\varphi)$$

shows that the system is a propagation system for $\widehat{U}$ along the rays of the phase φ, bicharacteristics of the operator L.

When U is determined, solution of (2.7), the second term $u^{(1)}$ is determined, up to a solution $U^{(1)}(x,\xi)h(x)$, by solving the linear equation (2.6), and integration with respect to ξ.

GKL show that an analogous procedure can be applied to annul the following terms in the expansion, and a formal asymptotic series can be constructed, through always linear systems and integration. Such an asymptotic series give approximate solutions to any order in ω, under smoothness assumptions of the coefficients.

2.3. Quasilinear systems. — The GKL construction has been extended to quasilinear first order systems in [**2**] by using a Taylor expansion of the coefficients in a neighbourhood of a solution (background). The equation for U contains then derivatives along the rays of the background and derivatives with respect to ξ. It leads to "dispersions of signals" if the system does not satisfy the Boillat - Lax exceptionnality condition. Due to the non linearity it is in general possible to obtain asymptotic approximate solutions of the given system only by truncating the series at first order in ω.

In the next sections we will consider quasilinear second order systems, with characteristic determinant possibly identically zero, and apply the results to some physical fields.

3. Quasilinear second order systems

3.1. Definitions. — We consider quasilinear second order systems with unknown a set of tensor fields u on a C^∞ manifold V. We do not write an explicit dependence in x, though it may exist. The system reads:

$$F(u, Du, D^2u) \equiv G(u, Du) \cdot D^2u + f(u, Du) = 0. \tag{3.1}$$

where D is the covariant derivative in some given pseudo riemannian smooth metric on V.

In index notations, with $u \equiv (u^A), A = 1, \dots, N$, and x^α local coordinates on V the system reads:

$$F^A(u, Du, D^2u) \equiv G_B^{A,\alpha\beta}(u, Du) D^2_{\alpha\beta} u^B + f^A(u, Du)) = 0.$$

The system is said to be quasi diagonal if

$$G_B^{A,\alpha\beta}(u, Du) \equiv g^{\alpha\beta}(u, Du)\delta_B^A$$

with δ_B^A the Kronecker delta. The fundamental field equations (Yang Mills, Einstein) are not quasidiagonal if a particular gauge is not chosen.

3.2. Asymptotic solutions

3.2.1. Definitions. — A high frequency wave on V is a tensor field of the type

$$u(x) = \underline{u}(x) + \omega^{-1}\{v(x, \xi)\}_{\xi=\omega\varphi(x)} \tag{3.2}$$

with $\underline{u}$ a tensor field on V, called background, v a tensor field of the same type as $\underline{u}$, but depending on a real parameter $\xi \in \mathbb{R}$, ω a real parameter ("frequency"), and φ a real function (phase).

Definition 2. — The high frequency wave u is called an asymptotic solution of the equation $F(u, Du, D^2u) = 0$ on $W \subset V$ if it satisfies on W an equation of the form

$$F(u, Du, D^2u) = \omega^{-1}\{(\mathcal{R}(x, \xi)\}_{\xi=\omega\varphi}$$

with $|\mathcal{R}|$ bounded on $W \times \mathbb{R}$.

An asymptotic solution satisfies approximately the equation $F = 0$, arbitrarily nearly for ω large enough.

3.2.2. Fundamental ansatz. — In addition to the derivation law (2.3) the following elementary ansatz is of fundamental importance in the non linear case: if $f'(x, \xi) \equiv (\partial f/\partial \xi)(x, \xi)$ is continuous and bounded in ξ on $\mathbb{R}$ there exists $f(x, \xi)$ bounded in ξ on $\mathbb{R}$ only if

$$\lim_{\Xi=\infty} \frac{1}{\Xi} \int_0^{\Xi} f'(x, \xi) d\xi = 0. \tag{3.3}$$

The condition must be a fortiori satisfied if we want f to be periodic in ξ. If T is the period we have

$$\int_0^T f'(x, \xi) d\xi = 0.$$

4. Construction of asymptotic solutions

4.1. Taylor expansion. — We suppose that G and f are smooth in u and Du in a neighbourhood of some given smooth[(3)] $\underline{u}$ called background. We underline the value taken at $\underline{u}$ by a quantity depending on the field u, in particular

$$\underline{G} \equiv G(\underline{u}, D\underline{u}), \qquad \underline{f} = \underline{f}(\underline{u}, D\underline{u}),$$

The field G admits a Taylor expansion of the form:

$$G(u, Du) = \underline{G} + \underline{\delta G} + \frac{1}{2}\underline{\delta}^2\underline{G} + \mathcal{S}$$

where $\underline{\delta G}$, and $\underline{\delta}^2\underline{G}$ are respectively a linear and a quadratic form in:

$$\delta u \equiv u - \underline{u}, \quad \delta Du \equiv D(u - \underline{u}),$$

namely

$$\underline{\delta G} \equiv \underline{G}'_u \delta u + \underline{G}'_{Du} \delta Du$$

and

$$\underline{\delta}^2\underline{G} \equiv \underline{G}''_{uu}(\delta u, \delta u) + 2\underline{G}''_{uDu}(\delta u, \delta Du) + \underline{G}''_{DuDu}(\delta Du, \delta Du).$$

The remainder $\mathcal{S}$ can be written:

$$\mathcal{S} \equiv \int_0^1 G'''_t(\delta u, \delta Du) dt$$

(3) Less restrictive conditions can be imposed on $\underline{u}$ if appropriate to the problem at hand.

where G_t''' denotes the third derivative of G at the point $(\underline{u}+t\delta u, D\underline{u}+tD\delta u)$, i.e $G_t'''(\delta u, \delta Du)$ is an homogeneous cubic polynomial in δu and δDu.

The term $f(u, Du)$ admits a similar expansion.

4.2. Asymptotic expansion. — Using the expression (3.2) of a high frequency wave together with the derivation formula we see that δu, δDu and $\delta D^2 u$ are the following fields on V:

$$\delta u(x) \equiv \omega^{-1} v(x,\xi)_{\xi=\omega\varphi(x)},$$

and, using coordinates notation on V to make the computation more explicit

$$(\delta D_\alpha u)(x) \equiv \{v'(x,\xi)\varphi_\alpha(x) + \omega^{-1} D_\alpha v(x,\xi)\}_{\xi=\omega\varphi(x)}, \qquad \varphi_\alpha \equiv D_\alpha \varphi,$$

while, with $X_{(\alpha\beta)} \equiv X_{\alpha\beta} + X_{\beta\alpha}$,

$$(\delta D^2_{\alpha\beta} u)(x) \equiv \{\omega v''(x,\xi)\varphi_\alpha(x)\varphi_\beta(x) + D_{(\beta} v(x,\xi)\varphi_{\alpha)}(x) + \omega^{-1} D^2_{\alpha\beta} v(x,\xi)\}_{\xi=\omega\varphi(x)}.$$

Inserting these expressions in the Taylor expansion of G and f we obtain an equality of the form

$$F(u, Du, D^2u)(x) \equiv \{\omega F^{(-1)} + F^{(0)} + \omega^{-1}\mathcal{R}\}(x, \xi = \omega\varphi(x)).$$

The following lemma is an immediate consequence of the definitions.

Lemma 3. — *The high frequency wave u is an asymptotic solution of $F = 0$ if $F^{(-1)} = 0$, $F^{(0)} = 0$ and $|\mathcal{R}|$ is bounded on $V \times \mathbb{R}$.*

In the general quasilinear case, since δDu is of order zero in ω, the remainder of the Taylor expansions of G will contribute to the expression of $F^{(-1)}$ if $G'''_{DuDuDU} \neq 0$. The remainders will contribute to $F^{(0)}$ if $G'''_{uDuDu} \neq 0$ or if $f'''_{DuDuDu} \neq 0$. We suppose therefore that G is at most linear, and f at most quadratic, in Du, with coefficients functions of u.

A straightforward computation gives then that:

$$F^{(-1)}(x,\xi) \equiv \left\{(\underline{G}^{\alpha\beta} + \underline{G}^{\alpha\beta\prime}_{D_\gamma u}\varphi_\gamma v')v''\varphi_\alpha\varphi_\beta\right\}(x,\xi).$$

i.e.

$$F^{A(-1)}(x,\xi) \equiv \left\{(\underline{G}^{A\alpha\beta}_B + (\underline{G}^{A\alpha\beta}_B)'_{D_\gamma u^C}\varphi_\gamma v'^C)v''^B\varphi_\alpha\varphi_\beta\right\}(x,\xi).$$

In the case of a quasidiagonal system this equation splits into $v'' = 0$, which implies $v' = 0$ if we want v to be bounded on $V \times \mathbb{R}$, and

$$(\underline{g}^{\alpha\beta} + \underline{g}^{\alpha\beta\prime}_{D_\gamma u^C}\varphi_\gamma\varphi_\alpha\varphi_\beta v'^C = 0,$$

which implies by the ansatz (3.3), if we want v to be bounded on $V \times \mathbb{R}$,

$$\underline{g}^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0$$

and

$$\underline{g}'^{\alpha\beta}_{D_\gamma u^C}\varphi_\gamma\varphi_\alpha\varphi_\beta v^C = 0. \tag{4.1}$$

This equation is identically verified if g depends only on u, i.e if the system is hyperquasilinear.

For an hyperquasilinear system, terms in $F^{(0)}$ could come from the remainder $\mathcal{S}$ of the Taylor expansion of F if the third derivative f'''_{DuDuDu} was not zero. We will exclude this eventuality by restricting f to be a polynomial in Du of order at most two.

The hyperquasilinearity, as well as $f'''_{DuDuDu} = 0$ are satisfied by the fundamental field equations.

5. Quasidiagonal systems

We consider hyperquasilinear quasidiagonal systems with f quadratic in Du. They read, with q a quadratic form and a a linear one:

$$F(u, Du, D^2u) \equiv g^{\alpha\beta}(u)D^2_{\alpha\beta}u + q^{\alpha\beta}(u)(D_\alpha u, D_\beta u) + a^\alpha(u)D_\alpha u + b(u) = 0,$$

that is, in coordinates:

$$(5.1)\quad F(u, Du, D^2u)^A \equiv g^{\alpha\beta}(u)D^2_{\alpha\beta}u^A + q^{A,\alpha\beta}_{BC}(u)D_\alpha u^B D_\beta u^C + a^{A,\alpha}_B(u)D_\alpha u^B + b^A(u) = 0.$$

with g a Lorentzian metric in the neighborhood of a given field $\underline{u}$.

Theorem 4. — *The high frequency wave*

$$(5.2)\qquad u(x) = \underline{u}(x) + \omega^{-1}\{v(x,\xi)\}_{\xi=\omega\varphi(x)}$$

is an asymptotic solution of the system (5.1) *in a compact domain $W \subset V$ spanned by the trajectories of the vector field ℓ, with v' obtained by a linear propagation equation along these trajectories, if the following conditions are satisfied*

(1) *$\ell \equiv D\varphi$ is a null vector for $\underline{g} = g(\underline{u})$.*

(2) *The quantities $F, \underline{u}$ and φ satisfy the equations:*

$$\underline{g}'^{\alpha\beta}_u \varphi_\alpha\varphi_\beta = 0 \qquad \text{and} \qquad \underline{q}^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0.$$

(3) *$\underline{u}$ is a solution of the system.*

Proof. — Inserting the expression (5.2) in the left hand side of (5.1) we find an expansion in powers of ω of the type (4.1) with $F^{(-1)}$ given by:

$$F^{(-1)}(x,\xi) \equiv \underline{g}^{\alpha\beta}\varphi_\alpha\varphi_\beta v''(x,\xi)$$

The condition (1) insures that $F^{(-1)} = 0$, for all ξ.

The annulation of the coefficient of ω^0, $F^{(0)} = 0$ reads

$$F^{(0)}(x,\xi) \equiv \underline{F}(x) + \underline{g}^{a\beta}\{2\varphi_\alpha D_\beta v' + v' D_\alpha\varphi_\beta + 2\underline{q}^{\alpha\beta}\varphi_\beta D_\alpha \underline{u} v' + \underline{a}^\alpha\varphi_\alpha v' + \underline{q}^{\alpha\beta}\varphi_\alpha\varphi_\beta v'^2 + \underline{g}'^{\alpha\beta}_u\varphi_\alpha\varphi_\beta v v''\}(x,\xi) = 0,$$

it reduces to a linear, non homogeneous, propagation equation for v' along the rays φ_α if the conditions (2) are satisfied.

Under the conditions (1) and (2) the derivative $v'(x,\xi)$ is determined through a linear equation $F^{(0)}=0$, a primitive $v(x,\xi)$ is solution of the ordinary differential system in x, along the rays,

$$\left\{\underline{g}^{\alpha\beta}(2\varphi_\alpha D_\beta v+vD_\alpha\varphi_\beta)+2\underline{q}^{\alpha\beta}\varphi_\beta D_\alpha\underline{u}v+\underline{a}^\alpha\varphi_\alpha v\right\}(x,\xi)+\xi\underline{F}(x)=0$$

A solution $v(x,\xi)$ exists for a given ξ on V, spanned by the rays, taking given initial values on an $n-$ submanifold S transversal to these rays, smooth if these initial values are smooth. It is bounded for $\xi\in\mathbb{R}$ on any compact subdomain of V if it is so of the initial values for any compact subdomain of S, and if and only if $\underline{F}(x)=0$. The remainder $|\mathcal{R}(x,\xi)|$ is then bounded for $\xi\in\mathbb{R}$ for any compact subdomain of V. □

5.1. Weak null condition. — The null condition has been introduced by Christodoulou and Klainerman independently to prove global existence of solutions of non linear wave equations on Minkowski spacetime. It has also been used by Klainerman and collaborators to lower the regularity demanded of solutions. It says, in the case of Minkowski space, that the set of zero tensors $u\equiv 0$ being a solution of the system *i.e.*, $\underline{F}\equiv F(0,0,0)=0$, the linearisation at $\underline{u}=0$ of the system is the wave equation in the Minkowski metric:

$$\underline{\delta F}=\eta^{\alpha\beta}D^2_{\alpha\beta}\delta u,$$

while its second variation at $u=0$ is such that:

$$\underline{\delta}^2\underline{F}\equiv\underline{F}''(\delta u,\delta Du,\delta D^2u)=0$$

whenever $\delta u,\delta Du,\delta D^2u$ are replaced by the following tensors:

$$\delta u=X,\quad \delta Du=Y\otimes\ell,\qquad \delta D^2u=Z\otimes\ell\otimes\ell$$

with X, Y and Z arbitrary tensors and ℓ a covector null for the Minkowski metric.

We will extend part of this definition to an arbitrary Lorentzian manifold $(V,\underline{g})$.

Definition 5. — The second order system (3.1) is said to satisfy the weak null condition for $\underline{u}$ and φ if

(1) φ satisfies the eikonal equation of $\underline{g}$:

$$\underline{g}^{\alpha\beta}\varphi_\alpha\varphi_\beta=0.$$

(2a) The following equations hold for all fields[(4)] X and Z on V:

$$\underline{F}''_{uD^2u}(X,ZD\varphi\otimes D\varphi)\equiv X^BZ^A\underline{g}'^{\alpha\beta}_{u^B}\varphi_\alpha\varphi_\beta=0,$$

i.e.

$$X^B\underline{g}'^{\alpha\beta}_{u^B}\varphi_\alpha\varphi_\beta=0.$$

(4) Of the same type as u.

(2b)
$$\underline{F}''_{DuDu}(YD\varphi, YD\varphi) \equiv \underline{q}^{A,\alpha\beta}_{BC} Y^C Y^B \varphi_\alpha \varphi_\beta = 0,$$

i.e.
$$\underline{q}^{A,\alpha\beta}_{BC} Y^C Y^B \varphi_\alpha \varphi_\beta = 0,$$

for all fields Y on V.

As an immediate consequence of this definition we have the following theorem.

Theorem 6. — *The conditions* (2) *of the theorem 4 are satisfied if it is so of the weak null condition for* $\underline{u}, \varphi$.

6. Non quasidiagonal systems

6.1. Polarizations. — We consider an hyperquasilinear system with f quadratic in Du

$$F(u, Du, D^2u) \equiv G(u) \cdot D^2 u + q^{\alpha\beta}(u)(D_\alpha u, D_\beta u) + a^\alpha(u) D_\alpha u + b(u) = 0. \tag{6.1}$$

and we decompose its principal part into a diagonal part and a non diagonal one which we will call the gauge part. We write:

$$G(u) \cdot D^2 u \equiv g^{\alpha\beta}(u) D^2_{\alpha\beta} u + P(u) \cdot D^2 u$$

i.e.
$$(G(u) \cdot D^2 u)^A \equiv g^{\alpha\beta}(u) D^2_{\alpha\beta} u^A + P^{A\alpha\beta}_B(u) D^2_{\alpha\beta} u^B$$

We suppose that $g(u)$ is a Lorentzian metric in a neighbourhood of some smooth tensor field $\underline{u}$. As before we underline values taken for $u \equiv \underline{u}$, $Du \equiv D\underline{u}$ of tensor fields depending on u and Du.

Definition 7. — The tensor X is said to be polarized with respect to $\underline{u}$ and the vector field ℓ if

$$\underline{P} \cdot X \otimes \ell \otimes \ell = 0, \qquad i.e. \qquad P^{A,\alpha\beta}_B(\underline{u}) X^B \ell_\alpha \ell_\beta = 0. \tag{6.2}$$

Definition 8. — The non quasi diagonal system (6.1) is said to satisfy the polarized null condition if it holds that:

$$\underline{F}''_{uD^2u}(X, ZD\varphi D\varphi) = 0, \qquad i.e. \qquad (G^{A\alpha\beta}_B)'_{u^C} X^C Z^B \varphi_\alpha \varphi_\beta = 0$$

for all polarized fields X and Z on V, and:

$$\underline{F}''_{DuDu}(YD\varphi, YD\varphi) = 0, \quad i.e. \quad \underline{q}^{A,\alpha\beta}_{BC} Y^C Y^B \varphi_\alpha \varphi_\beta = 0,$$

for all polarized fields Y on V.

6.2. Polarized asymptotic solution. — It is convenient in the case of non quasi diagonal systems to introduce second order perturbations in order to find propagation equations satisfied by the first order perturbation, and if possible eliminate gauge terms from them. A high frequency wave will be of the form

$$u(x) = \underline{u}(x) + \{\omega^{-1}v(x,\xi) + \omega^{-2}w(x,\xi)\}_{\xi=\omega\varphi(x)}.$$

It will be an asymptotic solution of the system (6.1) on $W \subset V$ if its insertion in it gives :

$$F(u, Du, D^2u)(x) = \omega^{-1}\{\mathcal{R}(x,\xi)\}_{\xi=\omega\varphi(x)}$$

with $|\mathcal{R}|$ bounded on $W \times \mathbb{R}$.

The vanishing of the coefficient $F^{(-1)}$ resulting form the insertion of u reads:

$$F^{(-1)}(x,\xi) \equiv \underline{g}^{\alpha\beta}\varphi_\alpha\varphi_\beta v''^A + P_B^{A,\alpha\beta}(\underline{u}, D\underline{u})\varphi_\alpha\varphi_\beta v''^B = 0.$$

We suppose, for physical applications, that $\ell \equiv D\varphi$ is a null vector for the background metric $\underline{g} = g(\underline{u})$. The condition above reduces to the polarization of v'' with respect to $\underline{g}$ and $D\varphi$, equivalently to the polarization of v. Then v reads

$$v^A(x,\xi) = \sum_{1\leqslant i\leqslant N} h_{(i)}^A(x)U_{(i)}(x,\xi)$$

where the $h'_{(i)}s$ are a basis of the kernel of the linear operator $P_B^{A,\alpha\beta}(\underline{u}, D\underline{u})\varphi_\alpha\varphi_\beta$ and the $U'_{(i)}s$ are functions on $V \times \mathbb{R}$, at this stage arbitrary.

The annulation of the coefficient $F^{(0)}(x,\xi)$ reads, since $\ell_\alpha \equiv \varphi_\alpha$ is a null vector of $\underline{g}$:

$$\begin{aligned}(6.3)\quad F^{(0)A}(x,\xi) \equiv{}& \underline{P}_B^{A\alpha\beta}\varphi_\alpha\varphi_\beta w''^B + \mathcal{L}^A(v') \\ &+ \underline{q}^{\alpha\beta,A}\varphi_\alpha\varphi_\beta(v',v') + \underline{G}_{B\,u^C}^{A\,\alpha\beta}\varphi_\alpha\varphi_\beta v^C v^{B\prime\prime} + \underline{F}^A(x) = 0\end{aligned}$$

with $\mathcal{L}$ a linear operator given by:

$$\mathcal{L}(v') \equiv \mathcal{D}(v') + \mathcal{P}(v')$$

$$\mathcal{D}^A(v') \equiv 2\underline{g}^{\alpha\beta}\Big\{\varphi_\alpha D_\beta v'^A + \frac{1}{2}v'^A D_\alpha\varphi_\beta\Big\} + 2\underline{q}_B^{\alpha\beta,A}\varphi_\beta v'^B D_\alpha\underline{u}) + \underline{a}_B^{\alpha,A}\varphi_\alpha v'^B$$

and

$$\mathcal{P}^A(v') \equiv 2\underline{P}_B^{A\alpha\beta}\varphi_\alpha D_\beta v'^B.$$

Since $\mathcal{L}$ is linear in v' and the first term of (6.3) linear in w'' the equation (3.3) says that the system (6.3) can have solutions v and w bounded for all ξ only if

$$\lim_{T=\infty}\frac{1}{T}\int_0^T \Big\{\Big\{\underline{q}^{\alpha\beta,A}\varphi_\alpha\varphi_\beta(v',v') + \underline{G}'^{\alpha\beta}_{u^C}\varphi_\alpha\varphi_\beta v^C v^{A\prime\prime}\Big\}(x,\xi) + \underline{F}(x)\Big\}\,d\xi = 0$$

Sufficient (not necessary, see the case of Einstein equations section 8) conditions for this equation to be satisfied for polarized v are:

(1) $\underline{u}$ is a solution of the system: $\underline{F} = 0$.

(2) F satisfies the polarized null condition relative to $\underline{u}$ and φ.

Supposing these conditions satisfied the necessary and sufficient condition for the existence of w'' is that $\mathcal{L}(v')$ be orthogonal to the kernel of the dual ${}^*(\underline{P}^{\alpha\beta}\varphi_\alpha\varphi_\beta)$ of the linear operator $\underline{P}^{\alpha\beta}\varphi_\alpha\varphi_\beta$. Denote by $\widetilde{h}_{(\widetilde{\imath})}$, $\widetilde{\imath} = 1, \dots, \widetilde{N}$ a basis of this kernel the condition reads

$$\widetilde{h}_{(\widetilde{\imath})A}\mathcal{L}^A\Big(\sum_{1\leqslant i\leqslant N} h_{(i)}U'_{(i)}\Big) = 0, \qquad \widetilde{\imath} = 1, \dots, N,$$

A system of $\widetilde{N}$ linear differential equations for the N unknown $U'_{(i)}$, scalar functions on V depending on the parameter ξ.

We will not discuss the general case, but look at applications to some field equations of physics. It turns out in these cases that the operator ${}^*(\underline{P}^{\alpha\beta}\varphi_\alpha\varphi_\beta)$ is injective, and that the polarization of the field is conserved by a well chosen propagation equation for v and associated choice of w.

7. The standard model

7.1. Fields and equations. — The fields of the standard model on a spacetime V with given Lorentzian metric g and covariant derivative D are:

– A connection 1 form A with values in a Lie algebra $\mathcal{G}$ of $N \times N$ matrices whose curvature F is given by:

$$F_{\lambda\mu} \equiv D_\lambda A_\mu - D_\mu A_\lambda + [A_\lambda, A_\mu]$$

– A scalar multiplet ϕ, mapping[(5)] $V \to \mathbb{C}^N$ and a spinor multiplet, mapping $V \to \mathbb{C}^N \times \mathbb{C}^4$.

We denote by $u \equiv (u_{(A)} \equiv A, u_\phi = \phi, u_{(\psi)} = \psi)$ the set (A, ϕ, ψ).

The equations are, indices raised with g:

$$\begin{aligned}
Y^\mu &\equiv \widehat{D}_\lambda F^{\lambda\mu} - J^\mu = 0,\\
J^\mu &\equiv \mathrm{Re}\{\phi^*\widehat{D}\phi + i\psi^*\gamma^\mu\psi\}\\
\Phi &\equiv \widehat{D}^\lambda\widehat{D}_\lambda\phi - K(\phi,\psi) = 0,\\
\Psi &\equiv \widehat{D}^\alpha\widehat{D}_\alpha\psi - \gamma^\alpha\widehat{D}_\alpha H(\phi,\psi) = 0
\end{aligned}$$

where the γ^α are gamma matrices, while $\widehat{D}$ denotes the g metric and A connection derivative, that is

$$\widehat{D}_\lambda F^{\lambda\mu} \equiv D_\lambda F^{\lambda\mu} + [A_\lambda, F^{\lambda\mu}], \widehat{D}_\lambda\phi \quad \equiv D_\lambda\phi + A_\lambda\phi, \qquad \widehat{D}_\alpha\psi = D_\alpha\psi + A_\alpha\psi,$$

hence

$$\widehat{D}^\lambda\widehat{D}_\lambda\phi \equiv D^\lambda D_\lambda\phi + 2A_\lambda D^\lambda\phi + (D^\lambda A_\lambda + A^\lambda A_\lambda)\phi$$

(5) More general representation spaces can be considered, but they will make notations heavier without changing the essential results.

and an analogous formula for $\widehat{D}^{\lambda}\widehat{D}_{\lambda}\psi$. The terms H and K are smooth in their arguments.

The equations form a system of the studied type, semilinear if the metric g is given, linear in the first derivatives of the fields, non quasidiagonal. The gauge part contains only A and reads

$$D_{\lambda}D^{\mu}A^{\lambda}.$$

7.2. Polarizations. — A high frequency wave with $v \equiv (v_{(A)}, v_{(\phi)}, v_{(\psi)})$ will be an asymptotic solution of order zero, with $D\varphi$ a null vector of g if

$$\varphi_{\lambda}v^{\lambda}_{(A)} = 0.$$

The other components of v are unrestricted at this stage.

7.3. Transport equations

7.3.1. Yang Mills. — According to the general results the annulation of $F^{(0)}_{(A)}$ reads

$$Y^{(0)\mu} \equiv -2\varphi_{\lambda}\varphi^{\mu}w''^{\lambda}_{(A)} + \mathcal{D}^{\mu}_{(Y)}(v') + \mathcal{P}^{\mu}_{(Y)}(v') + \underline{Y}^{\mu} = 0$$

with, using $D_{\lambda}\varphi_{\mu} = D_{\mu}\varphi_{\lambda}$, and $v'^{\lambda}_{(A)}\varphi_{\lambda} = 0$,

$$\mathcal{D}^{\mu}_{(Y)}(v') \equiv 2\varphi^{\lambda}D_{\lambda}v'^{\mu}_{(A)} + v'^{\mu}_{(A)}D_{\lambda}\varphi^{\lambda} + ?2?[v'^{\mu}_{(A)}, \varphi_{\lambda}\underline{A}^{\lambda}] - \varphi^{\mu}\operatorname{Re}\{\underline{\phi}^{*}v'_{\phi}\}$$
$$\mathcal{P}^{\mu}_{(Y)}(v') \equiv -D_{\lambda}v'^{\lambda}\varphi^{\mu} - \varphi^{\mu}[\underline{A}_{\lambda}, v'^{\lambda}_{A}]$$

7.3.2. Scalar multiplet. — The terms of order zero are, if $v_{(A)}$ is polarized, $\mathcal{D}_{(\Phi)} + +\underline{\Phi}$, with

$$\mathcal{D}_{(\Phi)} \equiv 2\varphi^{\lambda}D_{\lambda}v'_{(\phi)} + v'_{(\phi)}D_{\lambda}\varphi^{\lambda} + 2\underline{A}_{\lambda}\varphi^{\lambda}v'_{(\phi)}.$$

7.3.3. Spinor multiplet. — The terms of order zero are, if $v_{(A)}$ is polarized, $\mathcal{D}_{(\Psi)} + \underline{\Psi}$, with

$$\mathcal{D}_{(\Psi)} \equiv 2\varphi^{\lambda}D_{\lambda}v'_{(\psi)} + v'_{(\psi)}D_{\lambda}\varphi^{\lambda} + 2\underline{A}_{\lambda}\varphi^{\lambda}v'_{(\psi)} + \frac{1}{2}\gamma^{\alpha}\gamma^{\beta}\underline{\psi}(\varphi_{\alpha}v'_{(A)\beta} - \varphi_{\beta}v'_{(A)\alpha})$$
$$- \gamma^{\alpha}\varphi_{\alpha}\{\underline{H}'_{\phi}v'_{(\phi)} + \underline{H}'_{\psi}v'_{(\psi)}\}$$

7.3.4. Conclusions. — The transport equations $\mathcal{D}(v') = 0$ are a linear homogeneous system, it has a solution v' bounded on any compact subdomain spanned by the rays, and for all $\xi \in \mathbb{R}$ if it is so of initial data. The same is true of v and its primitive in ξ. The polarization condition on $v_{(A)}$ is preserved by transport since this system implies, using $\varphi_{\mu}\varphi^{\mu} = 0$ and $\varphi^{\lambda}D_{\lambda}\varphi_{\mu} = \varphi^{\lambda}D_{\mu}\varphi_{\lambda} = 0$,

$$\varphi_{\mu}\mathcal{D}^{\mu}_{(Y)}(v') \equiv 2\varphi^{\lambda}D_{\lambda}(\varphi_{\mu}v'^{\mu}_{(A)}) + \varphi_{\mu}v'^{\mu}_{(A)}D_{\lambda}\varphi^{\lambda} + 2[\varphi_{\mu}v'^{\mu}_{(A)}, \varphi_{\lambda}\underline{A}^{\lambda}] = 0.$$

The linear system satisfied by the 1 form w''^{λ} reduces to

$$2\varphi_{\lambda}w''^{\lambda}_{(A)} = -D_{\lambda}v'^{\lambda} - [\underline{A}_{\lambda}, v'^{\lambda}_{A}]$$

it has many solutions, with $w_{(A)}$ bounded for all ξ if it is so of a primitive of v'.

We read on the transport equations the following theorem.

Theorem 9. — *A high frequency perturbation of the scalar multiplet generates in general a high frequency perturbation of the Yang Mills field and of the spinor field if its background $\underline{\phi}$ is not zero.*

Corollary 10. — *If the background $\underline{\psi}$ is not zero a high frequency perturbation of the Yang Mills field generates in general a high frequency perturbation of the spinor field.*

8. Einstein equations

8.1. Polarizations. — The vacuum Einstein equations satisfied by a Lorentzian metric g on a smooth manifold V of dimension $n+1$ are of the type (6.1). They read, V being endowed with a smooth background metric $\underline{g}$, and D denoting the covariant derivative in this background (hence $D\underline{g}=0$):

$$\mathrm{Ricci}(g, Dg, D^2g) \equiv G(g)\cdot D^2g + q(g)(Dg, Dg) + g\cdot \mathrm{Riemann}(\underline{g})$$

with $G(g)$ the linear operator with a diagonal and a gauge part:

$$(G(g)\cdot D^2g)_{\alpha\beta} \equiv -\frac{1}{2}g^{\lambda\mu}D^2_{\lambda\mu}g_{\alpha\beta} + (P(g)\cdot D^2g)_{\alpha\beta},$$

with (indices raised with the inverse matrix of $\underline{g}$):

$$(P(g)\cdot D^2g)_{\alpha\beta} \equiv \frac{1}{2}\left\{D^\lambda D_\alpha g_{\beta\lambda} + D^\lambda D_\beta g_{\alpha\lambda} - \underline{g}^{\lambda\mu}D_\alpha D_\beta g_{\lambda\mu}\right\}$$

while $q(g)(Dg, Dg)$ is an homogeneous quadratic form in Dg with coefficients depending only on g.

Both $G(g)$ and $q(g)$ are independent of the choice of $\underline{g}$. Both are analytic in g as long as g is non degenerate. We set

$$h \equiv \delta g \equiv g - \underline{g}, \qquad \textit{i.e.} \qquad h_{\alpha\beta} \equiv g_{\alpha\beta} - \underline{g}_{\alpha\beta}.$$

It holds that ([**12**]):

$$\underline{\delta R}_{\alpha\beta} \equiv -\frac{1}{2}D^\lambda D_\lambda h_{\alpha\beta} + \frac{1}{2}\left\{D^\lambda D_\alpha h_{\beta\lambda} + D^\lambda D_\beta h_{\alpha\lambda} - \underline{g}^{\lambda\mu}D_\alpha D_\beta h_{\lambda\mu}\right\}$$

According to the previous definition a symmetric 2-tensor X is said to be *polarized* at $\underline{g}$ for the null vector ℓ if it satisfies the equations (6.2) which read here

$$\frac{1}{2}\{\ell_\alpha p_\beta(X) + \ell_\beta p_\alpha(X)\} = 0, \quad \text{with} \quad p_\alpha(X) \equiv \ell_\lambda X^\lambda_\alpha - \frac{1}{2}\ell_\alpha X^\lambda_\lambda.$$

The polarization conditions reduce therefore the $n+1$ equations:

$$p_\alpha(X) = 0.$$

Remark 11. — Elementary calculus gives $g'^{\alpha\beta}_{g_{\lambda\mu}} = -g^{\alpha\lambda}g^{\beta\mu}$, from which follows that

$$\underline{g}'^{\alpha\beta}_{g}\cdot X\ell_\alpha\ell_\beta = 0$$

whenever X is a symmetric 2-tensor polarized for the Einstein equations, with respect to the vector ℓ, null for $\underline{g}$.

8.2. Polarized null condition. — A straightforward computation gives ([**4**])

$$\underline{\delta}^2\underline{R}_{\alpha\beta} \equiv -h^{\lambda\mu}\{D_\lambda(D_\alpha h_{\beta\mu}+D_\beta h_{\alpha\mu}-D_\mu h_{\alpha\beta})-D_\alpha D_\beta h_{\lambda\mu}\}$$
$$-D_\lambda h^{\lambda\mu}(D_\alpha h_{\beta\mu}+D_\beta h_{\alpha\mu}-D_\mu h_{\alpha\beta})+\frac{1}{2}D_\beta h^{\lambda\mu}D_\alpha h_{\lambda\mu}$$
$$+\frac{1}{2}D^\lambda h^\rho_\rho(D_\alpha h_{\beta\lambda}+D_\beta h_{\alpha\lambda}-D_\lambda h_{\alpha\beta})+D_\lambda h^\mu_\alpha D^\lambda h_{\beta\mu}-D_\lambda h^\mu_\alpha D_\mu h^\lambda_\beta.$$

The various components of the second derivative of the Ricci tensor with respect to g, Dg and D^2g can be read on this formula.

The Ricci operator does not quite satisfy the polarized null condition, as was shown in [**4**].

Theorem 12. — *When $h_{\alpha\beta}, D_\lambda h_{\alpha\beta}$ and $D_\lambda D_\mu h_{\alpha\beta}$ are replaced respectively by $X_{\alpha\beta}$, $\ell_\lambda Y_{\alpha\beta}$ and $\ell_\lambda\ell_\mu Z_{\alpha\beta}$ with X, Y and Z symmetric 2-tensors polarized for ℓ, a null vector for $\underline{g}$, the second derivative of the Ricci operator gives the following equalities:*

$$\{\underline{R}_{\alpha\beta}\}''_{gD^2g}(X,\ell\otimes\ell\otimes Z)=\ell_\alpha\ell_\beta\Big\{X^{\lambda\mu}Z_{\lambda\mu}-\frac{1}{2}X^\lambda_\lambda Z^\mu_\mu\Big\}$$
$$\{\underline{R}_{\alpha\beta}\}''_{DgDg}(\ell\otimes Y,\ell\otimes Y)=\frac{1}{2}\ell_\alpha\ell_\beta\Big\{Y^{\lambda\mu}Y_{\lambda\mu}-\frac{1}{2}Y^\lambda_\lambda Y^\mu_\mu)\Big\}.$$

8.3. Transport equation and back reaction. — In agreement with the general theory the condition $F^{(0)}=0$ reads here, when ℓ is a null vector of $\underline{g}$:

$$R^{(0)}_{\alpha\beta}=\frac{1}{2}\{\varphi_\alpha p_\beta(w'')+\varphi_\beta p_\alpha(w'')+\mathcal{L}(v')_{\alpha\beta}+N_{\alpha\beta}(v,v',v'')+\underline{R}_{\alpha\beta}$$

with $\mathcal{L}$ the sum $\mathcal{D}+\mathcal{P}$ of a linear propagation operator for v' along the rays of $D\varphi$, which reduces here to a propagation operator along the rays of the phase φ:

$$(\mathcal{D}v')_{\alpha\beta}\equiv-\{\varphi^\lambda\underline{D}_\lambda v'_{\alpha\beta}+\frac{1}{2}v'_{\alpha\beta}\underline{D}_\lambda\varphi^\lambda\}$$

and a gauge associated linear operator which reduces here to:

$$\mathcal{P}_{\alpha\beta}\equiv\frac{1}{2}\{\varphi_\alpha y_\beta(v')+\varphi_\beta y_\alpha(v')\},\qquad y_\alpha(v')\equiv\underline{D}_\lambda v'^\lambda_\alpha-\frac{1}{2}\underline{D}_\alpha v'^\lambda_\lambda.$$

Since the Einstein equations do not quite satisfy the null condition the non linear term N does not vanish. For polarized v it reduces to:

$$N_{\alpha\beta}(v,v',v'')\equiv\frac{1}{2}\varphi_\alpha\varphi_\beta\{v^{\lambda\mu}v''_{\lambda\mu}-\frac{1}{2}v^\lambda_\lambda v''^\mu_\mu+\frac{1}{2}(v\prime^{\lambda\mu}v'_{\lambda\mu}-\frac{1}{2}v'^\lambda_\lambda v'^\mu_\mu)\}.$$

Using the fact that for an arbitrary function f it holds that

$$ff''=(ff')'-f'^2$$

we see that the condition

$$\lim_{\Xi=\infty}\frac{1}{\Xi}\int_0^\Xi\{\underline{R}_{\alpha\beta}+N_{\alpha\beta}\}d\xi=0$$

can be written

$$\underline{R}_{\alpha\beta} = \overline{E}\varphi_\alpha\varphi_\beta, \qquad \text{with} \qquad \overline{E} \equiv \lim_{\Xi=\infty} \frac{1}{\Xi}\int_0^{\Xi} E d\xi \tag{8.1}$$

where E is the positive function of x and ξ given by

$$E \equiv \frac{1}{4}\Big\{v'^{\lambda\mu}v'_{\lambda\mu} - \frac{1}{2}v'^{\lambda}_{\lambda}v'^{\mu}_{\mu}\Big\} \tag{8.2}$$

One says that the wave v' inflicts a back reaction on the background $\underline{g}$ which must be a solution of (8.1), *i.e.* of Einstein equations with source a null fluid, if the high frequency wave is to be a vacuum asymptotic solution.

We can now prove the following theorem.

Theorem 13. — *The vacuum Einstein equations admit on the manifold V the high frequency asymptotic solution with v and w of period T in ξ*

$$\underline{g}_{\alpha\beta}(x) + \{\omega^{-1}v_{\alpha\beta}(x,\xi) + \omega^{-2}w_{\alpha\beta}(x,\xi)\}_{\xi=\omega\varphi(x)}$$

if:

(1) *The tensor v, periodic in ξ as well as its integral $\int_0^\xi v d\xi$, satisfies the linear, homogeneous, propagation equation on V along the rays of the phase φ, isotropic for the background $\underline{g}$:*

$$\mathcal{D}(v) = 0$$

and v satisfies the polarization conditions on a hypersurface Σ transversal to rays of the phase φ which span V.

(2) *The tensor v and the background metric $\underline{g}$ satisfy the following Einstein equations with source a null fluid:*

$$\underline{R}_{\alpha\beta} = \overline{E}\varphi_\alpha\varphi_\beta. \tag{8.3}$$

The tensor w is a periodic solution of the linear system:

$$p_\alpha(w'') = y_\alpha(v') + \frac{1}{4}\varphi_\alpha\{(v^{\lambda\mu}v'_{\lambda\mu} - v^{\lambda}_{\lambda}v'^{\mu}_{\mu})\}' + \frac{1}{2}\varphi_\alpha(E - \overline{E}),$$

with E and $\overline{E}$ given by (8.2), (8.1).

Proof. — (1) If v' satisfies the propagation equations $\mathcal{D}(v') = 0$ on $V \times \mathbb{R}$ and the polarization conditions on Σ transversal to rays which span V, then it satisfies the polarization conditions on $V \times \mathbb{R}$ because the equations $\mathcal{D}(v') = 0$ imply the propagation both of $\varphi^\alpha v_{\alpha\beta} = 0$ and $v^\alpha_\alpha = 0$, as can easily be checked. The coefficient $\mathrm{Ricci}^{(-1)}$ of ω in the asymptotic expansion of $\mathrm{Ricci}(g)$ is therefore zero.

(2) If $\mathcal{D}(v') = 0$ and $\underline{R}_{\alpha\beta}$ satisfies (8.3) then the annulation of $R^{(0)}_{\alpha\beta}$ reduces to the equation (8.1) This equation has a solution w'' because the dual of the linear system of operators p_α is the linear system acting on a vector Z^α which is injective, indeed:

$$\ell^\lambda Z^\mu - \frac{1}{2}\underline{g}^{\lambda\mu}Z^\alpha\ell_\alpha = 0 \qquad \text{implies} \qquad Z^\alpha = 0.$$

The tensor w' is solution of the algebraic linear system

$$p_\alpha(w') = p_\alpha(w'_0) + y_\alpha(v) + \frac{1}{8}\varphi_\alpha \left\{v^{\lambda\mu}v_{\lambda\mu} - (v^\lambda_\lambda)^2\right\}' + \frac{1}{2}\varphi_\alpha \int_0^\xi (E - \overline{E})d\xi,$$

it has also period T in ξ since the right hand side has a zero integral on ξ on the interval $0 \leqslant \xi \leqslant T$. We can choose w'_0 such that the integral in ξ of w' is also 0 on $[0,T]$, hence w bounded as were w' and w'': the remainder $\mathcal{R}$ is bounded for $\xi \in \mathbb{R}$. □

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