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ON IRREGULAR HOLONOMIC $\mathscr{D}$-MODULES

by

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Abstract. — One proves the existence of a canonical lattice for the meromorphic connections; as a consequence, one obtains the two following results:
First, the fact that such a connection, defined outside a set of codimension 3, can be extended everywhere.
Then, the existence of a global good filtration for the holonomic $\mathscr{D}$-modules.

***Résumé* (Sur les modules holonomes irréguliers).** — On démontre l'existence d'un réseau canonique pour les connexions méromorphes ; on en déduit deux résultats :
D'une part, le fait qu'une telle connexion, définie hors d'un ensemble de codimension 3, se prolonge partout.
D'autre part, l'existence d'une bonne filtration globale pour les $\mathscr{D}$-modules holonomes.

I. Meromorphic connections

1. Introduction. — Let X be a complex analytic manifold of dimension n, and let Z be an analytic hypersurface of X (*i.e.* a closed analytic subset of codimension one at each of its points). We denote by $\mathscr{O}_X$ (resp. Ω^p_X) the sheaf of holomorphic functions on X (resp. the sheaf of holomorphic p-forms on X). We denote also by $\mathscr{O}_X[\star Z]$ the sheaf of meromorphic functions on X with poles on Z: if $f = 0$ is a local equation of Z, one has, with the usual notations $\mathscr{O}_X[\star Z] = \mathscr{O}_X[f^{-1}]$; we put also $\Omega^p_X[\star Z] = \mathscr{O}_X[\star Z] \otimes_{\mathscr{O}_X} \Omega^p_X$. Sometimes, we omit "$X$" and we write $\mathscr{O}$, $\mathscr{O}[\star Z]$, Ω^p, etc.

It is well know that $\mathscr{O}$ has noetherian fibers, and that it is coherent (*i.e.* the kernel of a map $\mathscr{O}^q \to \mathscr{O}^p$ is locally of finite type); from this follows at once that $\mathscr{O}[\star Z]$ has the same properties. Then, one defines a $\mathscr{O}[\star Z]$ coherent module, in the usual way, as being locally the cokernel of a morphism of $\mathscr{O}[\star Z]$ modules, say $\mathscr{O}[\star Z]^q \xrightarrow{u} \mathscr{O}[\star Z]^p$.

Let E be a coherent $\mathscr{O}[\star Z]$-module. By definition a lattice of E is a coherent sub-$\mathscr{O}$-module $L \subset E$ such that $E = \mathscr{O}[\star Z]L$. Locally, E admits always lattices (take a

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presentation $u : \mathscr{O}[\star Z]^q \to \mathscr{O}[\star Z]^p \to E \to 0$, and multiply by f^r, $r \gg 0$, to remove the poles...). But E does not admit always global lattices, even if X is compact : see counterexamples below, in subsection 6.

Remark. — Suppose that X is a projective manifold, *i.e.* a closed analytic submanifold of $\mathbf{P}^n(\mathbf{C})$. Then, by a classical theorem of Chow, X "is algebraic", *i.e.* there exists a projective algebraic manifold $\widetilde{X}$ such that $X = \widetilde{X}^{\mathrm{an}}$, the analytic manifold associated to $\widetilde{X}$.

Now, let Z be an hypersurface of X, which is also "algebraic" in the same sense, and let E be a coherent $\mathscr{O}[\star Z]$-module; then, the following assertions are equivalent:

i) E admits a lattice L

ii) E is "algebraic", *e.g.* there exists a $\mathscr{O}_{\widetilde{X}}[\star \widetilde{Z}]$-module $\widetilde{E}$ such that $E = \widetilde{E}^{\mathrm{an}}$ ($= \widetilde{E} \otimes_{\mathscr{O}_{\widetilde{X}}} \mathscr{O}_X$)

i) $\Rightarrow$ ii) follows from "GAGA", which asserts that the coherent $\mathscr{O}_X$-modules "are algebraic"

ii) $\Rightarrow$ i) follows from a standard result of algebraic geometry with asserts that quasi-coherent sheaves on algebraic varieties which some mild finiteness assumptions (in particular, projective algebraic varieties) are inductive limits of coherent sheaves.

Now, we come back to the general case.

Definition 1.1. — Let E be a coherent $\mathscr{O}[\star Z]$-module. A connection on E is defined, in the usual way, as an operator $\nabla : E \to E \otimes_{\mathscr{O}} \Omega^1$ verifying the following properties

i) ∇ is $\mathbf{C}$-linear

ii) For $\phi \in \mathscr{O}$, $e \in E$ one has $\nabla(\phi e) = e \otimes d\phi + \phi \nabla e$

For such a ∇, one defines as usual its extension (denoted also ∇): $E \otimes_{\mathscr{O}} \Omega^p \to E \otimes_{\mathscr{O}} \Omega^{p+1}$. One says that ∇ is flat if $\nabla^2 : E \to E \otimes_{\mathscr{O}} \Omega^2$ vanishes; in that case $\nabla^2 : E \otimes_{\mathscr{O}} \Omega^p \to E \otimes_{\mathscr{O}} \Omega^{p+2}$ vanishes also for all p (proofs as usual in differential geometry).

Proposition 1.2. — *Let E be a coherent $\mathscr{O}[\star Z]$-module, with a connection ∇ (non necessary flat). Then, E is locally stably free, i.e. , for every point $x \in X$, there exists p such that $E_x \oplus \mathscr{O}_x[\star Z]^p$ is free*

The proof is similar to the proof of proposition 2.2 below (and in fact simpler!).

The main result of this chapter is the following : a coherent $\mathscr{O}[\star Z]$-module provided with a flat connection admits a global lattice and actually admits a canonical lattice; see the precise statement in section 4. To prove this result, we need the use of formal completions, which we will study now.

2. Formal completions. — Let X, Z, $\mathscr{O}_X, \dots$, be as before. We define, as usual, the formal completion $\widehat{\mathscr{O}}_{X|Z}$ (or $\widehat{\mathscr{O}}$, if there is no ambiguity) as the sheaf on Z associated to the presheaf $U \to \varprojlim \Gamma(U, \mathscr{O}/f^k\mathscr{O})$ with f a local equation of Z. It is obvious that $\widehat{\mathscr{O}}_{X|Z,a}$ is contained in the formal completion of $\mathscr{O}_a$ with respect to the powers of the maximal ideal $\mathscr{M}_a$; in particular, $\widehat{\mathscr{O}}_{X|Z,a}$ is an integral domain. Furthermore, it is noetherian and faithfully flat over $\mathscr{O}_a$; also $\widehat{\mathscr{O}}_{X|Z}$ is coherent.

I shall not prove these properties here, although I have no explicit reference. To prove that $\widehat{\mathscr{O}}_{X|Z,a}$ is noetherian and that $\widehat{\mathscr{O}}_{X|Z}$ is coherent one can for instance argue as in [**L-M**], where a more delicate case of formal completions is treated; the main ingredients are the "theorem of privileged neighborhoods" and the theorem of Frisch asserting that the ring of holomorphic functions on a closed polycylinder is noetherian. Then, the faithful flatness of $\widehat{\mathscr{O}}_a$ onto $\mathscr{O}_a$ follows from the fact that they have same completion (for the topology defined by the powers of the maximal ideal).

One defines $\widehat{\mathscr{O}}[*Z]$, $\widehat{\Omega}^p$, etc. as before. If F is a coherent $\widehat{\mathscr{O}}[*Z]$-module, one defines also a lattice of F as a coherent sub-$\widehat{\mathscr{O}}$-module L such that $F = \widehat{\mathscr{O}}[*Z]L$.

Let E be a coherent $\mathscr{O}[*Z]$-module, and put $\widehat{E} = E \otimes_{\mathscr{O}} \widehat{\mathscr{O}}$; if L is a lattice of E, $\widehat{L} = L \otimes_{\mathscr{O}} \widehat{\mathscr{O}}$ is a lattice of $\widehat{E}$. Note also that the natural map $E \to \widehat{E}$ is injective (use the exact sequence $0 \to \mathscr{O}_a \to \widehat{\mathscr{O}}_a \to \widehat{\mathscr{O}}_a/\mathscr{O}_a \to 0$ and the fact that $\mathrm{Tor}_1^{\mathscr{O}_a}(E, \widehat{\mathscr{O}}_a/\mathscr{O}_a) = 0$).

Proposition 2.1. — *The mapping $L \to \widehat{L}$ is a bijection "lattices of E" $\simeq$ "lattices of $\widehat{E}$". The inverse is the mapping $M \to M \cap E$ (extended by E outside Z).*

One proves this result as follows: as the result is local, one can suppose that one has already a lattice L' of E, and one equation $f = 0$ of Z. Then, locally, if L is a lattice of E, one has, for some q : $f^qL' \subset L$. Similarly, if M is a lattice of $\widehat{E}$, one has locally $f^q\widehat{L}' \subset M$. But the lattices of E (resp. $\widehat{E}$) which contain f^qL' (resp $f^q\widehat{L}'$) are in one to one correspondence with the coherent $\mathscr{O}$-sub modules of E/f^qL' (resp. with the coherent $\widehat{\mathscr{O}}$-sub modules of $\widehat{E}/f^q\widehat{L}'$); then, the result follows from the equality $E/f^qL' \simeq \widehat{E}/f^q\widehat{L}'$.

Let now F be a coherent $\widehat{\mathscr{O}}[*Z]$-module. One defines a connection ∇ on F as before, in the case of $\mathscr{O}[*Z]$-modules.

Proposition 2.2. — *Let F be a coherent $\widehat{\mathscr{O}}[*Z]$-module provided with a connection (not necessarily flat). Then, F is locally stably free.*

To prove this proposition, we need a lemma.

Lemma 2.3. — *For $i \geqslant 1$, one has $\mathrm{Ext}^i_{\widehat{\mathscr{O}}[*Z]_a}(F_a, \widehat{\mathscr{O}}[*Z]_a) = 0$ (a, a point of Z)*

Denote by P one of these Ext^i; it is finite over $\widehat{\mathscr{O}}[*Z]_a$ and is a torsion module (since it is annihilated by extension of $\widehat{\mathscr{O}}[*Z]_a$ to its fraction field). On the other hand, it is naturally provided with a connection: to prove this, we take an injective resolution $I^{\cdot}$ of $\widehat{\mathscr{O}}[*Z]_a$ over $\widehat{\mathscr{D}}[*Z]_a$ = the ring of differential operators with coefficients in $\widehat{\mathscr{O}}[*Z]_a$;

this resolution is also injective over $\widehat{\mathscr{O}}[*Z]_a$ (exercise: use the fact that $\widehat{\mathscr{D}}[*Z]_a$ in flat over $\widehat{\mathscr{O}}[*Z]_a$); then, one consider the obvious connection on $\mathrm{Hom}_{\widehat{\mathscr{O}}[*Z]_a}(F_a, I^k)$ and the cohomology groups of the corresponding complex.

Take now $g \in \mathrm{Ann}\, P$, and take $p \in P$; in local coordinates, one has $(\partial_i g)p + g(\nabla_{\partial_i} p) = 0$ ($\partial_i = \partial/\partial x_i$); therefore one has $(\partial_i g)p = 0$. Therefore $\mathrm{Ann}\, P$ is stable by the derivations ∂_i. As $\mathrm{Ann}\, P \neq 0$, it implies that one has $\mathrm{Ann}\, P = \widehat{\mathscr{O}}[*Z]_a$. (Exercise: choose a $g \in \mathrm{Ann}\, P$; multiplying it by f^p, $p \gg 0$, we can suppose that g has no pole; then, develop it in power series at a, and find a differential operator $b(x,\partial)$ such that $b(x,\partial)g$ is invertible in $\widehat{\mathscr{O}}_a$). Therefore one has $P = 0$ and the lemma is proved.

Now, the proof of the proposition follows a standard line. First, note the following facts.

i) The theorem of syzygies is true for $\widehat{\mathscr{O}}_a$, *i.e.* a finite module E over $\widehat{\mathscr{O}}_a$ has a free resolution of finite length (actually of length $\leqslant n = \dim X$). As $\widehat{\mathscr{O}}_a$ is local and noetherian, a standard argument shows that it suffices to prove the result for $E = \mathbf{C} = \widehat{\mathscr{O}}_a/\widehat{\mathscr{M}_a}$ ($\widehat{\mathscr{M}_a}$, the maximal ideal of $\widehat{\mathscr{O}}_a$); but this follows at once from the same result for $\mathscr{O}_a$, and the fact that $\widehat{\mathscr{O}}_a$ is flat over $\mathscr{O}_a$.

ii) From this, follows that the theorem of syzygies is also true for $\widehat{\mathscr{O}}[*Z]_a$; in fact, take E finite over $\widehat{\mathscr{O}}[*Z]_a$, and choose a lattice $L \subset E$, *i.e.* a finite $\widehat{\mathscr{O}}_a$ submodule such that $\widehat{\mathscr{O}}[*Z]_a L = E$; the natural mapping $L \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a \to E$ is bijective [the surjectivity is obvious; to prove the injectivity, note *e.g.* that the map $L \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a \to E \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a$ is injective since $\widehat{\mathscr{O}}[*Z]_a$ is flat over $\widehat{\mathscr{O}}_a$; on the other hand, the second term is equal to E: we leave the verification as an exercise]. Now, take a free resolution $\Phi^{\cdot}$ of L over $\widehat{\mathscr{O}}_a$; then $\Phi^{\cdot} \otimes_{\widehat{\mathscr{O}}_a} \widehat{\mathscr{O}}[*Z]_a$ is a free resolution of E over $\widehat{\mathscr{O}}[*Z]_a$; and if $\Phi^{\cdot}$ has finite length, the last one has also finite length, which proves ii).

Now, the proof of the proposition is done in two steps; take F as in the proposition 2.2, and take $a \in Z$.

i) F_a is projective; it is sufficient to prove the following result: if E is finite over $\widehat{\mathscr{O}}[*Z]_a$, then one has, for $i \geqslant 1$: $\mathrm{Ext}^i_{\widehat{\mathscr{O}}[*Z]_a}(F_a, E) = 0$. One proves this result by induction on the length of a free resolution of E. If *e.g.* E admits a resolution of length ℓ, one has an exact sequence $0 \to E' \to \widehat{\mathscr{O}}[*Z]_a^p \to E \to 0$, where E' has a free resolution of length $\ell - 1$; then the exact sequence of "Ext" imply that $\mathrm{Ext}^i(F_a, E) = \mathrm{Ext}^{i+1}(F_a, E')$ ($i \geqslant 1$), and the result follows.

ii) Any projective module G of finite type over $\widehat{\mathscr{O}}[*Z]_a$ is stably free. This is proved by induction on the length of a free resolution of G: if G has a free resolution of length ℓ, one has an exact sequence $O \to G' \to \widehat{\mathscr{O}}[*Z]_a^p \to G \to 0$, where G' has a free resolution of length $\ell - 1$; G being projective, the exact sequence splits, and G' is also projective. Then, the result follows from the induction hypothesis.

Remark. — I do not know if a stably free module of finite type over $\mathscr{O}[*Z]_a$ or $\widehat{\mathscr{O}}_{X|Z}[*Z]_a$ is actually free (of course, on $\mathscr{O}_a$ or $\mathscr{O}_{X|Z,a}$, this is true since these are local rings). If instead of $\mathscr{O}[*Z]_a$, we have a ring of polynomials $\mathbf{C}[x_1,\dots,x_n]$, then the similar statement is true according to a celebrated theorem of Quillen-Suslin. But I do not know if their methods can be extended to the cases considered here.

3. Extension of coherent sheaves. — Let X be an analytic manifold of dimension n, and let S be a closed analytic subset of X, of codimension $\geqslant 2$; we denote by i the injection $X-S\to X$. As before, $\mathscr{O}_X$ denotes the sheaf of holomorphic functions on X. Recall first the following result.

***Proposition 3.1* (“Hartogs property”)**. — *The natural morphism $\mathscr{O}_X\to i_*\mathscr{O}_{X-S}$ is an isomorphism.*

In other words, if a is a point of S and U an open neighborhood of a in X, then a holomorphic function on $U-S$ extends in a unique way to a holomorphic function on U. When S is smooth, this follows from a classical argument of Hartogs; in general, the result follows by using a stratification of S by smooth subvarieties, and by an argument of decreasing induction on the dimension of the strata.

Given an $\mathscr{O}_X$-coherent sheaf F, we call $F^\vee=\mathit{Hom}_{\mathscr{O}_X}(F,\mathscr{O}_X)$ the dual of F; we say that F is reflexive if the natural mapping $F\to F^{\vee\vee}$ is bijective. The following proposition is a simple particular case of a result of Serre [**Se**].

Proposition 3.2. — *Let F be a coherent $\mathscr{O}_{X-S}$-module, which is reflexive. Then, the following properties are equivalent:*

i) F admits a coherent extension to X (in that case, we say that “F is extendable”)

ii) i_*F is coherent.

The assertion “ii) implies i)” is obvious. Conversely, suppose that F is extendable; then $G=\mathit{Hom}_{\mathscr{O}_X}(F,\mathscr{O}_X)$ admits also an extension, say $\overline{G}$; I claim that $\overline{G}^\vee=i_*F$; actually, one has

$$\begin{aligned}\overline{G}^\vee &= \mathit{Hom}_{\mathscr{O}_X}(\overline{G},\mathscr{O}_X)\\ &= \mathit{Hom}_{\mathscr{O}_X}(\overline{G},i_*\mathscr{O}_{X-S}) && \text{(“Hartogs property”)}\\ &= i_*\,\mathit{Hom}_{\mathscr{O}_{X-S}}(i^*\overline{G},\mathscr{O}_{X-S}) && \text{(adjunction)}\\ &= i_*G^\vee\\ &= i_*F\end{aligned}$$

[For the adjunction formula which is purely sheaf-theoretic, we refer to the standard literature on sheaves]

In section 5, we will see deeper results on extension of sheaves; note that one interest of the property is the following fact : under the condition of prop. 3.2, the fact for a sheaf to be extendable is a local property.

We will now consider similar results for formal completions; as in section 1, let $Z\subset X$ be a closed hypersurface, let $S\subset Z$ be a closed analytic subset of codimension $\geqslant 2$ (with respect to X). We denote j the injection $Z-S\to Z$. The analogue of proposition 3.1 is here the following statement.

Proposition 3.3. — *Let $g \in \Gamma(Z, \widehat{\mathscr{O}}[*Z])$; suppose that, on $Z-S$, one has $g \in \Gamma(Z-S, \widehat{\mathscr{O}})$; then g has no pole i.e. $g \in \Gamma(Z, \widehat{\mathscr{O}})$.*

The statement is local, so it is enough to prove it in the neighborhood of a point $a \in S$. Using the fact that the map $\mathscr{O}[*Z]/\mathscr{O} \to \widehat{\mathscr{O}}[*Z]/\widehat{\mathscr{O}}$ is bijective, we can suppose that, in the neighborhood of a, we have $g \in \mathscr{O}[*Z]$; then g is holomorphic on $X-Z$ and on $Z-S$, therefore on $X-S$, and we can apply Hartogs.

Now, the analogue of prop. 3.2 is the following property; we give F a coherent $\widehat{\mathscr{O}}[*Z]$-module, which is reflexive (definition as before); on $Z-S$, we give a lattice L of F; denote by $j_{[*]}L$ the sheaf of sections ℓ of F verifying $j^*\ell \in L$. Then, we have the following result.

Proposition 3.4. — *With the preceding hypotheses, suppose that L is reflexive. The following properties are equivalent.*

i) L can be extended on Z as a lattice of F (in that case, we say that "L is extendable")

ii) $j_{[*]}L$ is a lattice of F

The proof of "i) implies ii)" is done by biduality as in prop. 3.2. We choose $\overline{L}$ an extension of L as a lattice of F; then $\overline{M} = \overline{L}^\vee$ is an extension of $M = L^\vee$, and this is a lattice of $F^\vee$ (I leave this point to the reader). Now, $\overline{M}^\vee$ is an extension of L, and a lattice of F, at it suffices to prove that $\overline{M}^\vee = j_{[*]}L$.

The proof can be done in the following way: on $Z \times \mathbf{R}_+$, one considers the sheaf $\widetilde{\mathscr{O}}$ which is $\widehat{\mathscr{O}}$ on $Z \times 0$, and (the inverse image of) $\widehat{\mathscr{O}}[*Z]$ elsewhere; on the other hand, one considers the sheaf $\widetilde{F}$ which is (the inverse image of) F on $Z \times \mathbf{R}_+^*$, and L on $(Z-S) \times \{0\}$. Then, one argues as in prop. 3.2 (note also, that, instead of $\mathbf{R}_+$, one could use a space of two-points, one closed, and one dense but it does not matter).

We will use the preceding properties in the following situation. Let X,Z and S be as above. Let E be a coherent $\mathscr{O}[*Z]$-module, locally stably free (therefore locally free on $X-Z$), and let L be a lattice of E on $X-S$, which we suppose to be reflexive; finally denote by $\widehat{E}$ and $\widehat{L}$ the formal completions of E and L along Z. As before, we denote by i and j the injections $X-S \to X$ and $Z-S \to Z$. One has the following result.

Proposition 3.5. — *Suppose that L is extendable; then i_*L is a lattice of E; furthermore $\widehat{L}$ is extendable and one has $\widehat{i_*L} = j_{[*]}\widehat{L}$. Conversely, if $\widehat{L}$ is extendable, then L is extendable.*

Suppose L extendable; then one has a natural map $i_*L \to i_*i^*E$; the next lemma shows that i_*L is a submodule of E.

Lemma 3.6. — *With the preceding hypotheses on E, X, Z, S, one has an isomorphism $E \to i_*i^*E$.*

As E is locally stably free, and therefore reflexive, the same argument as in prop. 3.2 will prove the result, provided we prove that one has $\mathscr{O}_X[*Z]\simeq i_*\mathscr{O}_{X-S}[*Z]$ ("Hartogs property for meromorphic functions"). To prove this result, note that it is local, and that we can enlarge S; so we can suppose that S contains the singular part of Z. Now, let a be a point of S, U an open neighborhood of a, and take $g \in \Gamma(U - S, \mathscr{O}_X[*Z])$; let f be a local equation of Z near a; on a connected component of $Z - S$, $f^r g$ has no pole for $r \gg 0$; as Z has only finitely many connected components adherent to a, we can choose r such that $f^r g$ has no pole on $Z - S$ near a; therefore (Hartogs) $f^r g$ is holomorphic near a, and the result follows.

Now, to prove that i_*L is a lattice of E, we observe that $E/i_*L[*Z]$ is $\mathscr{O}[*Z]$-coherent, and has its support in S, and a fortiori in Z; but a coherent $\mathscr{O}_X[*Z]$-module F with support in Z is necessarily equal to zero (take locally a lattice M of F; then there exist r such that $f^r M = 0$ and the result follows). Therefore on has $i_*L[*Z] = E$.

Finally, $\widehat{i_*L}$ is a lattice of $\widehat{E}$ which extends $\widehat{L}$; therefore $\widehat{L}$ is extendable, and we have only to prove that one has $\widehat{i_*L} = j_{[*]}\widehat{L}$. Let $\overline{M}$ be an extension of $L^\vee$; one has $\overline{M}^\vee = i_*L$ (*cf.* the proof of 3.2), and also $\widehat{\overline{M}^\vee} = (\widehat{\overline{M}})^\vee = j_{[*]}\widehat{L}$ (*cf.* proof of prop. 3.4).

Conversely, let $\widehat{L}$ be extendable, and let N be a lattice of $\widehat{E}$ which extends it; then $N \cap E$ (extended by E outside of Z) is an extension of L, (*cf.* prop 2.1). This ends the proof.

Remark. — Actually, we will have to use the preceding results in a slightly more general case: let $S \subset X$ be a closed set (for the usual "transcendental" topology); we say that "S has codimension $\geqslant 2$ in X if, in a neighborhood of each of its points, S is contained in an analytic set of codimension $\geqslant 2$. Then, the proposition 3.2 is still true for S closed of codimension $\geqslant 2$ in the preceding sense; the proof is the same; furthermore the extension i_*F is "independent of S" in the following sense: take $S \subset S'$, verifying the same hypothesis, and let k and i' be the injections $X - S' \to X - S$ and $X - S' \to X$; then, one has $i'_*k^*F = i_*F$: to prove this, it is sufficient to prove the isomorphism $F \simeq k_*k^*F$, which is proved by the same argument as prop. 3.2

The propositions 3.4 and 3.5 are also valid with the same hypotheses on S; I leave the details to the reader.

4. The canonical lattice of a meromorphic connection. — As in section 1, let X be a complex analytic manifold, and $Z \subset X$ a closed analytic hypersurface. Let (E, ∇) be a coherent $\widehat{\mathscr{O}}_{X|Z}[*Z]$-module, provided with a connection ∇; here, and in the next section, we suppose ∇ flat; for short, we call such a (E, ∇) a "formal meromorphic connection". We will say that (E, ∇) is regular if, in the neighborhood of any point $a \in Z$, (E, ∇) is equal to the formal completion $(\widehat{F}, \nabla)$ of a regular meromorphic connection in the sense of Deligne [**De**] (F, ∇) [Actually, one can prove that, in that case, (E, ∇) is globally the formal completion of a regular meromorphic connection, but we do not need it here].

In order to simplify the notations, we will often write E for (E,∇). Let Z' be the regular part of Z, and take a point $a\in Z'$; in the neighborhood of a, the description of regular connections is well-known: we choose local coordinates $(x_1,\dots,x_n)$ at a, with Z' defined by $\{x_1=0\}$; then in the neighborhood of a, E is free over $\widehat{\mathscr{O}}[*Z]$ and admits a basis $e_1,\dots,e_m$ in which one has

$$\nabla e_j=\frac{dx_1}{x_1}\sum_{i,j}c_{ij}e_i,\quad C=(c_{ij})\in\operatorname{End}(\mathbf{C}^m)$$

The matrix C is not entirely determined; only the "monodromy" is fixed, *i.e.* the matrix $\exp(2\pi iC)$, up to conjugacy; in particular, choose a section τ of the canonical map $\mathbf{C}\xrightarrow{p}\mathbf{C}/\mathbf{Z}$, *i.e.* a map $\tau:\mathbf{C}/\mathbf{Z}\to\mathbf{C}$ such that $p\circ\tau=id$; then we can choose C so that its eigenvalues belong to the image of τ; in that case, C is determined up to conjugacy (exercise). Furthermore, the lattice generated on $\widehat{\mathscr{O}}$ by $e_1,\dots,e_m$ depends only on τ and is called the "canonical lattice associated to τ" [This is proved as follows: if we have two bases $\{e_i\}$ and $\{e'_i\}$ having these properties, we consider the matrix S which transforms the first base into the second one; then an elementary calculation shows that S has no pole, and the same for S^{-1}; therefore the two bases define the same lattice]. Note also the following fact; the canonical lattice associated to τ is characterized by the following property: if $f_1,\dots,f_m$ is any basis of this lattice, then, in local coordinates as above,

(i) The matrix of ∇_{∂_i} ($i\geqslant 2$, $\partial_i=\partial/\partial x_i$) in this basis has no pole

(ii) The matrix of ∇_{∂_1} has a simple pole, and the eigenvalues of the polar part belong to $\operatorname{Im}(\tau)$.

These properties are well-known in one variable; the extension to the general case is easy, and we leave it as an exercise to the reader.

Now, we do not suppose (E,∇) regular anymore. If g is a meromorphic function with poles on Z, *i.e.* $g\in\Gamma(X,\mathscr{O}[*Z])$, we define the connection $e^g\otimes E$ as the $\widehat{\mathscr{O}}[*Z]$-module E provided with the connection $\phi\mapsto\nabla\phi+dg\otimes\phi$; in this context, it will often be convenient to write $e^g\otimes\phi$ instead of ϕ. Take $a\in Z'$, Z' the regular (=smooth) part of Z. We call "admissible decomposition" of E_a a decomposition $E=\oplus e^{g_i}\otimes F_i$, $g_i\in\mathscr{O}[*Z]_a$, F_i regular. In such a decomposition, the g_i's are only determined modulo $\mathscr{O}_a$; therefore we can put together the terms corresponding to equivalent g_i's and we can suppose that g_i-g_j has a pole for $i\neq j$; in that case, the factors $e^{g_i}\otimes F_i$ are well determined [These results are well-known in dimension 1; see *e.g.* [**Le**] or [**Ma2**]. The proof extends easily to the case considered here; the main point is the following; if U is a small open neighborhood of a, for $i\neq j$ one has $\operatorname{Hom}(U;e^{g_i}\otimes F_i,e^{g_j}\otimes F_j)=0$. Indeed this set is the set of sections on U of the connection $e^{g_j-g_i}\otimes \mathit{Hom}_{\mathscr{O}[*Z]}(F_i,F_j)$ which are horizontal, *e.g.* killed by ∇; and, from the fact that g_j-g_i does have a pole, and that the second factor is regular, one deduces easily the absence of horizontal sections $\neq 0$.]

If we have an admissible decomposition, we say that it is "good" if, for $i \neq j$, the most polar part of $g_i - g_j$ does not vanish at the point a; this improvement of the notion of admissible decomposition is not necessary for the definition of the canonical lattice but is useful for other purposes, more precisely for the theory of "Stokes structures" in several variables. Now the first main result is the following.

Theorem 4.1. — *There exists an open set $Z'' \subset Z'$, with $Z - Z''$ of codimension $\geqslant 2$ in the sense of section 3, having the following property: at any point $a \in Z''$, E admits, after a ramification along Z, a good decomposition*

"After ramification" means the following: in the neighborhood of a, one chooses local coordinates such that $Z' = \{x_1 = 0\}$. For p, integer > 1, one denotes ϕ_p the map $(y, x_2, \ldots, x_n) \mapsto (x_1 = y^p, \ldots, x_n)$ and $(\phi_p^* E, \phi_p^* \nabla)$ the obvious inverse image; then there exists p such that $(\phi_p^* E, \phi_p^* \nabla)$ admits a good decomposition at $\phi_p^{-1}(a)$. (It is probably true that one can choose Z'' Zariski-open in Z; but I have no proof of this result, which would not however improve the applications).

Now, we build a canonical lattice of E in the following way. One chooses the section τ of $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ whose image is $O < \operatorname{Re} \lambda \leqslant 1$. If E is decomposed in $\oplus e^{g_i} \otimes F_i$, one takes in each F_i the canonical lattice associated to τ, and one takes in E their direct sum; one verifies at once that this is independent of the choice of the g_i's (we recall that they are only determined mod $\mathscr{O}$). Now, if E is decomposed after the ramification ϕ_p, one does the same in $\phi_p^* E$, and one takes the restriction of this lattice to E. One verifies that the result is independent of the ramification that we have made, because of the special choice of τ. We have therefore defined a lattice L of E on Z'', with $S = Z - Z''$ closed and of codimension $\geqslant 2$ in X. The main result is now the following.

Theorem 4.2. — *The lattice L is extendable, in the sense of proposition 3.4*

Note that L is locally free, and therefore reflexive (we leave the proof as an exercise); on the other hand, E is stably locally free everywhere, after prop. 2.2; therefore the hypotheses of prop. 3.4 are satisfied.

Now, if F is a meromorphic (flat) connection with poles on Z, one obtains also a canonical lattice M on Z" by applying the preceding construction to $E = \widehat{F}$, and taking $M = E \cap L$, *cf.* 2.1. It follows from the results of section 3 that $i_* M$ is a lattice of E, with i the injection $X - S \to X$, $S = Z - Z''$; we call it again "the canonical lattice" of E; note that this lattice is also independent of the choice of S (*cf.* remark 3).

For the proof of 4.1 and 4.2, we refer to [**Ma4**], section 4 (there, these theorems are denoted respectively 3.2.1 and 3.3.1).

5. Applications. — We will give some applications to the meromorphic connections; I have no idea if some similar results exist or not for formal meromorphic connections. As in section 4, they are always supposed flat.

The first one concerns the algebraic character of projective meromorphic connections.

Theorem 5.1. — *Let X be a smooth projective algebraic variety over* $\mathbf{C}$*, and let E be a (flat) meromorphic connection on X^{an}; then E is algebraic, i.e. there exists a meromorphic connection F on X such that $E = F^{\mathrm{an}}$.*

Let L be a lattice of E, *e.g.* the canonical one; as L is $\mathscr{O}_{X^{\mathrm{an}}}$-coherent, by GAGA, there exists a coherent $\mathscr{O}_X$-module M such that $L = M^{\mathrm{an}}$; if we denote by Z the hypersurface of poles of E, Z is also algebraic, and we have $E = F^{\mathrm{an}}$, with $F = M[*Z]$. Finally, to prove that ∇ is also algebraic, we note that ∇ can be defined as a splitting of the exact sequence $0 \to E \to j^1E \to \Omega^1 \otimes_{\mathscr{O}} E \to 0$, where j^1E denote the sheaf of principal parts of order one (this can be defined in this way: take locally a lattice L in E, and take $j^1L \otimes_{\mathscr{O}} \mathscr{O}[*Z]$; one verifies that this is independent of the choice of L, and of the fact that we have taken the "left" or "right" structure of $\mathscr{O}$-module over j^1L). Then, we apply once more GAGA to show that this splitting is algebraic.

The next application is related with the extension problem. Let X be a complex analytic manifold, and let S be a closed analytic subset of X; first, note the following elementary result: if $\operatorname{codim} S \geqslant 2$, then any flat connection (without pole) on $X - S$ extends to X, since such connections are determined by their monodromy, and there is no monodromy around S. The same result is true for regular meromorphic connections: let E be such a connection on $X - S$, and let Y be the hypersurface of its poles: as $\operatorname{codim} S > \operatorname{codim} Y$ a classical theorem of Remmert-Stein says that the adherence $\overline{Y}$ of Yin X is analytic; now, we can extend E at the points of $S - \overline{Y}$ as above, and we are reduced to the case where $Z = \overline{Y}$ is a closed hypersurface of X, and S is contained in Z; but, then, the result follows from the solution of the "problem of Riemann-Hilbert" (any flat connection on $X - Z$ extends in a unique way to a regular connection on X, with poles on Z); *cf.* [**De**]

This result is not true for irregular meromorphic connections; see a counterexample below, in section 6. But one has the following result.

Theorem 5.2. — *Let S be a closed analytic subset of X; if* $\operatorname{codim} S \geqslant 3$*, any meromorphic connection on $X - S$ extends to a meromorphic connection on X.*

The uniqueness of this extension is obvious, since according to previous results, this extension is given by the direct image (*cf.* lemma 3.6).

The existence follows from difficult results on extension of $\mathscr{O}_X$-coherent sheaves, which we will now recall. If we have a coherent $\mathscr{O}_X$-module L, we call *projective dimension* of L at $a \in X$ the minimal length of a free resolution of L_a; and we put $\operatorname{depth}_a L = n - \operatorname{projdim} L_a$, $n = \dim X$ (the interest of the depth is that it is not changed by closed embedding). Given an L, we denote by $S_k(L)$ the set of points $a \in X$ where $\operatorname{depth}_a L \leqslant k$; one proves that $S_k(L)$ is an analytic subset of dimension $\leqslant k$; one says that L verifies (s_k) if one has

$$(s_k) \qquad \dim S_k(F) \leqslant k-2$$

One has the following theorem, due to Frisch-Guenot and Siu, [**F-G, Si**]; see also [**Do**].

Theorem 5.3. — *Let X be an analytic manifold, S a closed analytic subset of X, and denote by i the injection $X-S \to X$. Let L be a coherent $\mathscr{O}$-module on $X-S$, and suppose that L verifies (s_k) for $k \leqslant \dim S + 2$; then*

i) L is extendable to X

ii) i_*L is coherent; it satisfies (s_k) for $k \leqslant \dim S+2$, and it is the only one coherent extension to have this property.

Now we can prove the theorem 5.2 (the argument is close to an argument used by Mebkhout [**Me**] in the regular case, to reprove the Riemann-Hilbert correspondence). Suppose $\operatorname{codim} S \geqslant 3$, and let E be a meromorphic connection on $X' = X - S$, with poles on $Z' \subset X'$. As already noted, we can suppose that there exists a closed hypersurface $Z \subset X$ with $S \subset Z$, $Z' = Z - S$.

Let L be the canonical lattice of E on X'; to prove 5.2, it is sufficient to prove that i_*L is coherent; then $i_*L[*Z]$ is the required extension of E; to extend ∇, note that, in the neighborhood of a point $a \in S$, as $Z-S$ has only finitely many components, one has $f^r\nabla L \subset L \otimes_{\mathscr{O}} \Omega^1$, for some r; the extension of ∇ is then an obvious consequence of the definition of i_*.

According to theorem 5.3, to prove that i_*L is coherent, it suffices to prove that L verifies (s_k) for $k \leqslant n-1$, ($n = \dim X$). Let T be the set of points of Z' where L is not free; we have the following lemma.

Lemma 5.4. — $\operatorname{codim}_{X'} T \geqslant 3$.

If we admit this lemma, we are done; if we denote by j the injection $X' - T \to X'$ we have, by the definition of L, $L \simeq j_*j^*L$; on the other hand, j^*L is locally free on $X' - T$, and therefore satisfies s_k for $k \leqslant n-1$; then by theorem 5.3, L satisfies the same conditions on X'.

Now we prove the lemma; according to the proof of prop. 3.2, and the definition of L, L is reflexive; and it suffices to prove that the set of points where a reflexive sheaf F is not free has codimension $\geqslant 3$.

In dimension 2, this means that F is locally free; take $a \in X$; write $F = \mathcal{H}om(G, \mathscr{O})$ and take locally near a a free presentation $\mathscr{O}^q \to \mathscr{O}^p \to G \to 0$; one has an exact sequence

$$0 \longrightarrow F \longrightarrow \mathscr{O}^p \xrightarrow{\;{}^t u\;} \mathscr{O}^q;$$

and the theorem of syzygies shows that F_a is free. In the general case, the argument is similar: let T be the set of points where F is not free; suppose that there is a point $a \in T$ where $\dim T = n-2$ (or $n-1$, the argument is the same), and take p_a the ideal of a component T' of T at a of dimension $n-2$. The same argument shows that $F_a \otimes_{\mathscr{O}_a} \mathscr{O}_{p_a}$ is free: but this implies that, generically on T' near a, F is free, which contradicts the hypothesis. This prove the lemma and the theorem.

To end this section, we give another way to express the "Hartogs property", and theorem 5.2, which is interesting by itself. Let Z be a hypersurface of X, and denote by $\mathrm{conn}(X, Z)$ the category of (flat) meromorphic connections on X, with poles on Z; the "Hom" are defined in the obvious way, as morphism of $\mathscr{O}_X[*Z]$ modules which commute with ∇. Let S be a closed analytic subset of Z; one has the following result.

Theorem 5.5

i) *If* $\mathrm{codim}_X S \geqslant 2$, *the functor "forget"* $\mathrm{conn}(X, Z) \to \mathrm{conn}(X - S, Z - S)$ *is fully faithful.*

ii) *If* $\mathrm{codim}_X S \geqslant 3$, *this functor is an equivalence.*

The assertion ii) follows from i) and theorem 5.2; therefore it is sufficient to prove i). But, if we have two connections E and F with poles on Z, $\mathrm{Hom}(E, F)$ is just the set $\Gamma(X, \mathit{Hom}_{\mathscr{O}[*Z]}(E, F)^{\nabla})$, where $(.,.)^{\nabla}$ means "horizontal sections" *i.e.* sections killed by ∇. Therefore, to obtain the result, it suffices to use the Hartogs property of $\mathit{Hom}_{\mathscr{O}[*Z]}(E, F)$ (*cf.* lemma 3.6).

Note also that, by extension "with parameters" of the theory of irregular singularities in one variable, one has a geometric description ("Stokes structure") of a meromorphic connection at the generic points of Z. According to i), this gives a geometric description of $\mathrm{Hom}(E, F)$ for $E, F \in \mathrm{conn}(X, Z)$. However, a complete geometric description of the category $\mathrm{conn}(X, Z)$ is still missing. Roughly speaking, the statement ii) means that we have essentially to understand what happens generically in codimension 2.

6. Counterexamples. — They will be based on the following well-known fact, noted in the proof of lemma 5.4: in dimension 2, a coherent reflexive sheaf is locally free.

First, an example of a meromorphic connection E on $X = \mathbf{C}^2 - \{0\}$, which cannot be extended to $\mathbf{C}^2$; we cover $\mathbf{C}^2 - \{0\}$ by $U = \{x \neq 0\}$ and $V = \{y \neq 0\}$; we take L, a $\mathscr{O}_X$-module locally free of rank 1, by gluing $f \in \mathscr{O}_{U,a}$ and $g \in \mathscr{O}_{V,a}$ by $g = \exp(1/xy)f$; we take $Z = \{x = 0\}$, $E = L[*Z]$, and ∇ is defined in U by $\nabla f = df$; therefore, on V, on has $\nabla g = dg - gd(1/xy)$; this connection cannot be extended; for L is its canonical lattice and it is known that L cannot be extended. I reproduce here the argument of Douady [**Do**]. Suppose that L could be extended; it would admit an extension reflexive, and therefore locally free; and L itself would be free in $D - \{0\}$, D a bidisc. But this is not the case; the class defined by $1/xy$ in $H^1(D - \{0\}; \{U, V\}, \mathscr{O})$ is not 0; and its image by exp is not 0, for, in the exact sequence

$$H^1(D - \{0\}, \mathbf{Z}) \xrightarrow{2i\pi} H^1(D - \{0\}, \mathscr{O}) \xrightarrow{\exp} H^1(D - \{0\}, \mathscr{O}^*),$$

one has $H^1(D - \{0\}, \mathbf{Z}) = 0$.

Next, an example of a coherent $\mathscr{O}[*Z]$ sheaf, which has no global lattice (this example is due to Deligne); we take $X = \mathbf{P}^2(\mathbf{C})$, $Z \subset X$ a smooth curve of genus $\geqslant 1$, and E locally free of rank one; these E are classified by $H^1(X, \mathscr{O}[*Z]^*)$; one has an exact sequence $0 \to \mathscr{O}^* \to \mathscr{O}[*Z]^* \to \mathbf{Z}_Z \to 0$, therefore an exact sequence

$$H^1(X, \mathscr{O}^*) \longrightarrow H^1(X, \mathscr{O}[*Z]^*) \xrightarrow{\alpha} H^1(Z, \mathbf{Z}) \longrightarrow H^2(X, \mathscr{O}^*) \longrightarrow$$

One has $H^2(X, \mathscr{O}^*) = 0$ [use the exponential exact sequence $0 \to \mathbf{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0$ and the fact that $H^2(X, \mathscr{O}) = 0$, $H^3(X, \mathbf{Z}) = 0$]. Let now E be a $\mathscr{O}[*Z]$-module free of rank one, such that its image by α is not 0; E has no lattice. In fact, if there was such a lattice L, we could replace it by its bidual, which is locally free; but this is impossible, since it would be defined by a class in $H^1(X, \mathscr{O}^*)$.

Actually on has even more: $E_{|X-Z}$ cannot be the image by restriction to $X - Z$ of a class in $K_0^{\mathrm{top}}(X)$; otherwise the Chern class $c \in H^2(X - Z, \mathbf{Z})$ of $E_{|X-Z}$ would come from a class $\widetilde{c} \in H^2(X, \mathbf{Z})$; Therefore the image of c in $H^3_Z(X, \mathbf{Z})$ would be zero; but, by Thom's isomorphism, on has $H^3_Z(X, \mathbf{Z}) \simeq H^1(Z, \mathbf{Z})$; and one verifies easily that the class obtained in $H^1(Z, \mathbf{Z})$ in this way is egal, up to sign, to $\alpha[E]$, $[E]$ the class of E in $H^1(X, \mathscr{O}[*Z]^*)$.

Here is another example, which is related to the subject of this course; we take first $\widetilde{X} = \mathbf{C} \times \mathbf{P}^1(\mathbf{C})$ (coordinates x and t respectively); we put $\widetilde{Z} = \widetilde{Z}_0 \cup \widetilde{Z}_\infty$, $\widetilde{Z}_a = \{t = a\}$, and we put $\widetilde{E} = \mathscr{O}_{\widetilde{X}}[*\widetilde{Z}]$; we define now E on $X = \mathbf{C}^* \times \mathbf{P}^1(\mathbf{C})$ by the following identifications: we identify (x, t) and $(x + 1, t)$, and we identify $f \in \widetilde{E}_{x,t}$ and $g \in \widetilde{E}_{x+1,t}$ if $g = tf$. The same argument as before shows that E has no lattice near Z_0 or Z_∞ ($Z_a = \{t = a\}$, is the image of $\widetilde{Z}_a$ in X) because the class defined by E in $H^1(Z_0, \mathbf{Z})$ or in $H^1(Z_\infty, \mathbf{Z})$ is $\neq 0$.

Now, we provide $\widetilde{E}$ with the relative connection (for the projection ($\widetilde{X} \to \mathbf{C}$) defined by $\nabla f = \partial f/\partial t - (x/t)f$; this defines on E a relative connection which is, actually, the space of moduli for the connections of rank one on $\mathbf{P}^1(\mathbf{C})$ with regular singularities at 0 an ∞, and no other singularity. This example, which will be used also in the next chapter, shows that the theorem of existence of global lattices is no longer true for relative connections (note also that here, the eigenvalues of the polar part of the connection are no longer constant, and that this fact was basic in the definition of the canonical lattice).

II. Filtration of holonomic modules

1. Introduction. — Let X be a complex analytic manifold of dimension n; $\mathscr{O}_X$ and Ω^n_X have the same meaning as in chapter I. In addition, $\mathscr{D}_X$ denote the sheaf of linear differential operators with analytic coefficients.

Denote by T^*X the cotangent space of X, and let $T^*X \xrightarrow{\pi} X$ be the projection; we identify X with the zero-section of T^*X (often denoted $T^*_X X$). We denote as usual, by $\mathscr{E}_X$ (resp. $\widehat{\mathscr{E}_X}$) the sheaf of analytic microdifferential operators (resp. formal microdifferential operators on T^*X). For the properties of $\mathscr{E}_X$ (resp. $\widehat{\mathscr{E}_X}$) that we have to use, see the appendix of [**G-M**] and [**SKK**]; we will use $\mathscr{E}_X$ very little. If there is no confusion, we omit X and write $\mathscr{D}$, $\widehat{\mathscr{E}}$,...

For $p \in \mathbf{N}$ (resp $p \in \mathbf{Z}$), we denote by $\mathscr{D}_p$ (resp $\widehat{\mathscr{E}_p}$) the subsheaf of operators of order $\leqslant p$ of $\mathscr{D}$ (resp. $\widehat{\mathscr{E}}$).

Let M be a coherent $\mathscr{D}$-module; by definition, a good filtration of M is an increasing sequence $\{M_p\}$ of coherent sub-$\mathscr{O}$-modules of $\mathscr{D}$ verifying

i) $\cup M_p = M$

ii) $\mathscr{D}_p M_q \subset M_{p+q}$; furthermore, for every compact $K \subset X$, there exists q_0 such that $\mathscr{D}_p M_q = M_{p+q}$ on K for $p \geqslant 0$, $q \geqslant q_0$

These properties can be expressed in a slightly different way: call "standard" a filtration of $\mathscr{D}^k$ of the form $\deg(a_1, \ldots, a_k) = \sup(\deg a_i + \ell_i)$ where $\deg a_i$ is the usual order and $\ell_i \in \mathbf{N}$. Then, a filtration $\{M_p\}$ of M is good iff, locally, it is the quotient of a standard filtration for some surjection $\mathscr{D}^k \to M \to 0$.

Similarly, let $U \subset T^*X$ be an open set which we can suppose homogeneous, *i.e.* stable by the action of $\mathbf{C}^*$ on the fibers of $T^*X \xrightarrow{\pi} X$. If M is a coherent $\widehat{\mathscr{E}}_{|U}$-module, an increasing collection $\{M_p\}$ ($p \in \mathbf{Z}$) of $\widehat{\mathscr{E}}_0$-submodules is a good filtration if, locally, it is the quotient of a standard filtration of some surjection $\widehat{\mathscr{E}}^k \to M \to O$ (the definition of "standard" is the same as above). On $U - X$, this implies that one has $\widehat{\mathscr{E}}_q M_p = M_{p+q}$, $(p, q \in \mathbf{Z})$ and that M_0 is a lattice of M, *i.e.* a coherent $\widehat{\mathscr{E}}_0$-submodule such that one has $\widehat{\mathscr{E}} M_0 = M$; conversely, if one has a lattice L, then $\widehat{\mathscr{E}}_p L = M_p$ is a good filtration.

N.B. In [**Ma3**], it is incorrectly stated that the M_p are coherent on $\widehat{\mathscr{E}}_0$, if M_p is a good filtration of M; this is not true at the points of X (*e.g.* $\widehat{\mathscr{E}}_{-1}$ is not coherent on $\widehat{\mathscr{E}}_0$ at the points of X).

The aim of this chapter is to prove the following result: a holonomic $\mathscr{D}$-module admits globally a good filtration, which is canonical in the same sense as in chapter I, *i.e.* it depends on a section $\mathbf{C}/\mathbf{Z} \xrightarrow{\tau} \mathbf{C}$ of the canonical projection $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$. As in chapter I, this implies the algebraicity of holonomic $\mathscr{D}$-modules over projective smooth algebraic varieties. We leave to the reader the precise statement and the proof, which are similar to theorem I.5.1.

The strategy is to prove first the similar result for holonomic $\widehat{\mathscr{E}}$-modules outside of the zero-section; this is done by reducing it to the existence of lattices for formal-meromorphic connections (theorem I.4.2). The result for $\mathscr{D}$ is then deduced by the trick of "addition of a dummy variable" of [**K-K**]. Note also that this implies the existence of a good filtration for a holonomic $\mathscr{E}$-module (actually, if M is a coherent $\mathscr{E}_{|U}$-module, with $U \cap X = \varnothing$ there is a one-to-one correspondence between lattices of M an lattices of $\widehat{M} = \widehat{\mathscr{E}} \otimes_{\mathscr{E}} M$; in one sense, take $\widehat{L} = \widehat{\mathscr{E}}_0 \otimes_{\mathscr{E}_0} L$; in the other one, take $L = \widehat{L} \cap M$. The proof is similar to prop I.2.1; one has to use here the faithful flatness of $\widehat{\mathscr{E}}_0$ on $\mathscr{E}_0$).

In the case of regular singularities, the existence of canonical good filtrations for $\mathscr{E}$ and $\mathscr{D}$ holonomic modules was proved by Kashiwara-Kawai [**K-K**].

To end this section, we give an example of coherent $\mathscr{D}_X$-module without global good filtration; it does not even admit a good filtration on large compact sets. We start with the last example of section I.6, where $X = \mathbf{C}^* \times \mathbf{P}^1\mathbf{C}$ and E is the relative meromorphic connection defined there; we denote by $\mathscr{D}_{X/\mathbf{C}^*}$ the sheaf on X of relative differential operators w.r. to the projection $X \to \mathbf{C}^*$ (one differentiate only in the direction of the fibers); then, E is a $\mathscr{D}_{X/\mathbf{C}^*}$-module and one proves easily that it is coherent (add a parameter in the standard argument proving that a meromorphic connection in one variable is $\mathscr{D}$-coherent). But it has no good filtration; for, if $\{M_p\}$ would be one, on every compact $K \subset X$ we would have $M_p[*Z] = E$ for p large; therefore M_p would be a lattice on K; but the arguments of section I.6 show that such a lattice does not exist, *e.g.* for $K = S \times \mathbf{P}^1\mathbf{C}$, S the unit circle of $\mathbf{C}^*$.

Now, put $M = \mathscr{D}_X \otimes_{\mathscr{D}_{X/\mathbf{C}^*}} E$; E is embedded in M by $e \mapsto 1 \otimes e$ (this map is injective since $\mathscr{D}_X$ is free over $\mathscr{D}_{X/\mathbf{C}^*}$); if $\{M_p\}$ were a good filtration of M on K, the same argument would prove that $M_p \cap E$ is a lattice of E on K for p large; but this is impossible.

2. Holonomic $\mathscr{E}$-modules. — Let M be a holonomic $\widehat{\mathscr{E}}$-module on a homogeneous open subset of $T^*X - X$, and let Z be its support (or "characteristic variety"). Since M is holonomic, Z is homogeneous and lagrangian.

Let a be a point of Z, where Z is smooth; by a homogeneous canonical transformation we can suppose that a is transformed into the point $a' : x = 0, \xi = (1, 0, \dots, 0)$ and that, in the neighborhood of a, Z is transformed into Z' defined by the equations $x_1 = \xi_2 = \cdots = \xi_n = 0$ ($n = \dim X$). Using a Maslov transformation ("Fourier integral operator" or "quantified canonical transformation", *cf.* [**H**], [**SKK**]) which extends the preceding canonical transformation to microdifferential operators, one transforms M into a holonomic system M' with support on Z'. I will always suppose that the Maslov transformation preserves the "order" (*i.e.* it is of order 0 in the sense of Hörmander [**H**] or of order $n/2$ in the sense of [**SKK**]). Using the preparation theorem for $\widehat{\mathscr{E}}$, one sees that, in the neighborhood of a', M' is coherent over $\widehat{\mathscr{E}}(x_2, \dots, x_n, \partial_1)$, the subsheaf of $\widehat{\mathscr{E}}$ of elements having a symbol depending only on $(x_2, \dots, x_n, \partial_1)$. Denote by V a neighborhood in $\mathbf{C}^{n-1} \times \mathbf{P}^1(\mathbf{C})$ of the point $x_2 = \cdots = x_n = 0$, $\xi_1 = \infty$ and let T be the hypersurface $\xi_1 = \infty$; by identification $\partial_1 \to \xi_1$, $x_1 \to -(\partial/\partial\xi_1)$ ("formal Fourier transform w.r. to x_1"), we can identify $\widehat{\mathscr{E}}(x_2, \dots, x_n, \partial_1)$ near a' with $\widehat{\mathscr{O}}_{V|T}[*T]$; with this interpretation, M' becomes a formal meromorphic connection on V with poles on T; the derivations are given by $\nabla_{\partial/\partial\xi_1} = -x_1$, $\nabla_{\partial/\partial x_i} = \partial_i$, $i \geqslant 2$. We denote by E this connection (and we identify it with $M'_{|(\xi_1=1)}$).

Now, we are in position to apply the results of section 4. We will see in a few lines what means the theorem I.4.1 in this context. Now, let L be the canonical lattice of E, and let $L' \subset M'$ correspond to L; one has the following result :

Theorem 2.1

i) L' *is in a lattice of* M' *as* $\widehat{\mathscr{E}}$*-module.*

ii) L' *is independent of the Maslov transformation chosen (of course, with the restriction on the order indicated above).*

Let $T'' \subset T$ the set of points where E admits a good formal decomposition after ramification, and put Z'' the corresponding subset of Z'; denoting by j the injection $T'' \to T$, one has, with the notations of section I.3: $L = j_{[*]}j^*L$; the same is true for L' in an obvious sense. From this follows at once that it is sufficient to prove the theorem at the points of Z''. Take $c \in T''$; near c denote by ϕ_p the map $(x_2, \dots, x_n, \eta) \to (x_2, \dots, x_n, \eta^p)$ for a suitable p; one has a decomposition $\phi_p^* E = \oplus e^{g_i} \otimes F_i$, F_i regulars, $g_i \in \mathscr{O}_{\widetilde{V}}[*\widetilde{T}]$, with $\widetilde{V} = \phi_p^{-1}V$, $\widetilde{T} = \phi_p^{-1}T$; we can suppose that $g_i - g_j$ has no pole, and we can choose $g_i = \sum_1^q a_k \eta^k$, a_k holomorphic in $(x_2, \dots, x_n)$.

Furthermore, here, we have $q < p$ (*i.e.* the g_i are of degree < 1 w.r. to ξ_1); this follows from the fact that the action of x_1 on M (or, equivalent $\partial/\partial\xi_1$ on E) is "topologically nilpotent"; more precisely for any good filtration of M, one has $x_1^k = 0$ on $\operatorname{gr} M$ for $k \gg 0$; therefore the "irregularity in the sense of Katz" of the connection $\partial/\partial\xi_1$ on $E_{|x_2=\cdots=x_n=\mathrm{cte}}$ is < 2, and this implies the required result (this is a standard argument in the theory of formal meromorphic connections in one variable; I do not give the details here. See *e.g.* such an argument in [**De**] or [**Ma1**]).

The corresponding description of M' at any b, with $q(b) = c$, q the projection $(x_2, \dots, x_n, \xi_1) \mapsto (x_2, \dots, x_n, \infty)$ is as follows; we introduce the sheaf $\widehat{\mathscr{E}}^{(p)}$ of formal microdifferential operators of order multiple of $1/p$ (in an obvious sense; see *e.g.* [**Ma1**] for the case of one variable); then, one has a decomposition $\widehat{\mathscr{E}}^p \otimes_{\widehat{\mathscr{E}}} M' = \oplus N_i^{g_i}$ where N_i is regular, and $N_i^{g_i}$ is equal to N_i on which $\widehat{\mathscr{E}}^p$ acts through the automorphism $\sigma_{g_i} : \widehat{\mathscr{E}}^p \to v$ defined by $\sigma_{g_i} a = e^{-g_i} a e^{g_i} = \sum (-1)^k ((\operatorname{ad} g_i)^k / k!) a$ with, of course, $\operatorname{ad} g_i(a) = g_i a - a g_i$ (note that, since order $g_i < 1$, this series converges). This description has been obtained by Rodrigues [**Ro**], with a different argument.

Now the proof of the theorem is easy: since σ_{g_i} preserves the order (and even the principal symbol) of microdifferential operators, the proof of i) is reduced to the regular case, where it is obvious. The proof of ii) is also reduced to the regular case, and follows from the fact that, in the regular case, the "order" of the sections of a microdifferential module is preserved by Maslov transformation of the required order (we admit this point). This proves the theorem.

Now, we come back to our original holonomic module M, with support $Z \subset U$. On Z^{reg}, the smooth part of Z, we have a well defined lattice by the preceding construction; call it L^{reg} and call L its extension by "formal direct image" as in chapter I, *i.e.* near $a \in Z - Z^{\mathrm{reg}}$, we take for L_a the sections m of M near a such that $j^{-1}m \in L^{\mathrm{reg}}$ (j the injection $L^{\mathrm{reg}} \to L$); then the final result is the following:

Theorem 2.2. — L *is a lattice of* M.

It is obvious that L is an $\widehat{\mathscr{E}}_0$-module, and that $\widehat{\mathscr{E}}L = M$. The only one point to prove is the coherence of L over $\widehat{\mathscr{E}}_0$. To prove this, choose a point $a \in Z - Z^{\mathrm{reg}}$; according to a lemma of Kashiwara-Kawai [**K-K**], there exists a homogeneous canonical transformation ϕ defined near a such that $\phi(Z)$ is in "generic position" near $\phi(a)$, *i.e.* one has $\pi^{-1}\pi\phi(a) \cap \phi(Z) = \mathbf{C}^*\phi(a)$; using a Maslov transformation of the required order which extends ϕ to microdifferential operators, we can suppose that Z itself is in generic position at a.

We choose local coordinates at $\pi(a)$ such that a is the point $x = 0, \xi = (1, 0, \dots, 0)$. Put, in the neighborhood of a: $\pi(Z) = Y$, $X' = \{x \in X; x_1 = 0\}$ and denote by p (resp. q) the projection $X \to X'$ (resp $Z \to X' \times \mathbf{C}^*$) defined by $p(x_1, \dots, x_n) = (x_2, \dots, x_n)$, $q(x, \xi) = (p(x), \xi_1)$.

It is well-known (and easy to prove) that Y is an hypersurface, and Z its strict conormal bundle, *i.e.* the closure of the conormal bundle to Y^{reg}; furthermore, the projection $p_{|Y}$ and q are finite. Using the preparation theorem for $\widehat{\mathscr{E}}$ one sees that, in the neighborhood of a, q_*M is coherent over $\widehat{\mathscr{E}}(x_2, \dots, x_n, \partial_1)$.

Now, we argue as before by partial Fourier transform; V and T having the same meaning as above, we identify in the obvious way X' and T, and denote r the projection $X' \times \mathbf{C}^* \to T$; let E be the $\widehat{\mathscr{O}}_{V|T}[*T]$-connection defined by M, and let Λ be its canonical lattice; consider $r^{-1}\Lambda \subset r^{-1}E = M$. The end of the proof follows from the lemma below.

Lemma 2.3. — *One has* $q_*L = r^{-1}\Lambda$

If we admit the lemma, we have that q_*L is coherent over $\widehat{\mathscr{E}}_0(x_2, \dots, x_n, \partial_1)$; as q is finite, $q^{-1}q_*L \to L$ is surjective; therefore, L is locally of finite type over $q^{-1}\widehat{\mathscr{E}}_0(x_2, \dots, x_n, \partial_1)$, and a fortiori locally of finite type over $\widehat{\mathscr{E}}_0$; but as it is contained in M, which is locally a direct limit of coherent $\widehat{\mathscr{E}}_0$-modules, L is coherent.

To prove the lemma, one look first at the points where E has a good decomposition, and one compare this decomposition with the decomposition of M on the sheets of the projection q (or p, which is equivalent); this gives the equality of q_*L and $r^{-1}\Lambda$ at the generic points of $X' \times \mathbf{C}^*$; then one proves easily that this implies also the result at the other points.

Remark. — If one develops the last argument, one finds the following result: there is a set T of codimension one in Z (in the sense of remark I.3) such that M admits a good decomposition after ramification on $Z - T$. This improves a little bit the result of Rodrigues, who has such a decomposition on an open dense set. As in the case of connections, it seems to me likely that one can take for T an analytic subset, but I have no proof of that.

3. The case of $\mathscr{D}$-modules

N.B. This should replace the section (3.B) of [**Ma3**]

Theorem 3.1. — *Let X be a complex analytic manifold, and let M be a holonomic $\mathscr{D}_X$-module. Then M admits a good filtration (and actually, a canonical one).*

We will reduce to problem to the microlocal case, by an argument of Kashiwara-Kawai ([**K-K**], appendix A); put $Y = X \times \mathbf{C}$, with coordinates (x,t) and denote by j the injection $x \mapsto (x,0)$ into Y. Denote also by $\overline{j}$ and J the maps of the obvious diagram $T^*Y \xleftarrow{\overline{j}} T^*Y \times_Y X \xrightarrow{J} T^*X$ (J is the cotangent map).

Let M be a coherent $\mathscr{D}_X$-module, and let $\widehat{M} = \widehat{\mathscr{E}}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}M$ be its formal microlocalization. One takes the microlocal direct image $\widehat{M}'$ of $\widehat{M}$, defined by $\widehat{M}' = \overline{j}_*(\widehat{\mathscr{E}}_{Y\leftarrow X} \otimes_{J^{-1}\mathscr{E}_X} J^{-1}\widehat{M})$; this is equivalent to take the microlocalization of the direct image j_+M; explicitly, at a point $(x,t,\xi,\tau) \in T^*Y$, this consists of taking the equation of $\widehat{M}$ and adding the equation $tm = 0$. We restrict $\widehat{M}'$ to the set $\widetilde{T^*X} \subset T^*Y$ defined by $t = 0$, $\tau = 1$; by J, $\widetilde{T^*X}$ is isomorphic to T^*X; it is known (loc. cit.) that one reconstructs $\widehat{M}$ from $\widehat{M}'_{|\widetilde{T^*X}}$ by $M = J_* \ker(t, \widehat{M}'_{|\widetilde{T^*X}})$. In particular, one reconstructs M by $M = J_* \ker(t, \widehat{M}'_{|\widetilde{X}})$, $\widetilde{X} \subset \widetilde{T^*X}$ being defined by $\xi = 0$.

Now, take L, the canonical lattice of $\widehat{M}'$ on $T^*Y - Y$, which is defined in particular on $\widetilde{T^*X}$, and put $L_k = \widehat{\mathscr{E}}_k L$ (here, $\widehat{\mathscr{E}}$ means $\widehat{\mathscr{E}}_Y$). According to [**K-K**], prop. A.8, $M_k = J_* \ker(t, L_{k|\widetilde{X}})$ ($k \geqslant 0$) gives a good filtration of M if one proves that one has $tL \subset L_{-1}$ (this point was overlooked in [**Ma3**]). Therefore, to prove the theorem, the only remaining point is to prove the last assertion. Let Z be the characteristic variety of M, *i.e.* the support of $\widehat{M}$ and let Z' be support of $\widehat{M}'$ (in the subset $\tau \neq 0$ of T^*Y); one has $Z' = \{(x,0,\xi,\tau) \mid (x,\xi) \in Z,\ \tau \neq 0\}$. It is sufficient to prove the result at the generic points of Z', *i.e.* outside of a set of codimension one. There are two cases to consider

i) The points $(x,0,\xi,\tau)$ where $(x,0) \in Z$, $(x,0) \notin \overline{Z - X}$; near such a point, one has $M \simeq \mathscr{O}_X^p$, and the result is easy.

ii) The points $(x,0,\xi,\tau)$, $\xi \neq 0$, $\tau \neq 0$ such that (x,ξ) is a smooth point of Z where $\widehat{M}$ can be described by a "good decomposition after ramification" as explained in section 2. Using again the same Maslov transformation, extended trivially to the variables (t,τ), we are reduced to the case where Z is defined by $x_1 = \xi_2 = \xi_n = 0$, and $Z' = Z \times \mathbf{C}^*_\tau$; at such a point, we have $\widehat{M}' = \widehat{M} \widehat{\otimes} \widehat{\mathscr{E}}_T \delta(t)$ (*i.e.* $\widehat{M}'$ is obtained from $\widehat{M}$ by adding the equation $tm = 0$); using ramification and the automorphisms σ_{g_i} as in section 2, and using the fact that t commutes with σ_{g_i}, we are reduced to prove the result in the regular case; but in that case it is well known that a microdifferential operator of degree $\leqslant 0$ whose principal symbol vanishes on Z' reduces the order of sections by one. This ends the proof of the statement and of the theorem.

Remark. — In fact, as $\delta(t)$ has order $1/2$, the canonical filtration taken on $\widehat{M}'$ does not induce the canonical filtration (or lattice) of $\widehat{M}$; with the convention taken here, the canonical lattice of $\widehat{M}$ is the one whose elements, at the generic points, have order $< 1/2$ in the sense of [**SKK**] (modulo the automorphism σ_{g_i}). To recover it we should take on $\widehat{M}'$ the lattice L whose elements have over < 1. Then, taking $M_k = J_* \ker(t, \widehat{\mathscr{E}}_k L_{|\widetilde{X}})$, for $k \geqslant 0$, one can prove the following results :

i) The canonical lattice of $\widehat{M}$ on $T^*X - X$ is given by $\sum \widehat{\mathscr{E}}_{-k} M_k$

ii) At the points $(x,0) \in Z$, $(x,0) \notin \overline{Z - X}$, on has $M_k = M$, $\forall\, k \geqslant 0$

iii) At the other points $(x,0) \in Z$, a section m of M belongs to M_k if $\pi^{-1}m$ has order $< k + 1/2$ on $T^*X - X$ at the generic points close to $\pi^{-1}(x)$.

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