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CREPANT RESOLUTIONS OF GORENSTEIN TORIC SINGULARITIES AND UPPER BOUND THEOREM

by

Dimitrios I. Dais

***Abstract.* —** A necessary condition for the existence of torus-equivariant crepant resolutions of Gorenstein toric singularities can be derived by making use of a variant of the classical Upper Bound Theorem which is valid for simplicial balls.

1. Introduction

Let d be a positive integer, $\sigma \subset \mathbb{R}^{d+1}$ a rational, $(d+1)$-dimensional strongly convex polyhedral cone (w.r.t. the lattice $\mathbb{Z}^{d+1}$), and

$$U_\sigma = \operatorname{Spec}\left(\mathbb{C}\left[\sigma^\vee \cap (\mathbb{Z}^{d+1})^\vee\right]\right)$$

the associated affine toric variety, where $\sigma^\vee$ denotes the dual of σ. (For the usual notions of toric geometry, see [**7**]). As it is known (see e.g. [**10**, § 6]):

Theorem 1.1. — *U_σ is Gorenstein if and only if the set* $\operatorname{Gen}(\sigma)$ *of the minimal generating integral vectors of σ lies on a "primitive" affine hyperplane, i.e., iff*

$$\operatorname{Gen}(\sigma) \subset \mathbf{H}_\sigma = \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} \;\mid\; \langle \boldsymbol{m}_\sigma, \boldsymbol{x}\rangle = 1\right\},$$

where $\boldsymbol{m}_\sigma \in (\mathbb{Z}^{d+1})^\vee$ is a primitive vector belonging to the dual lattice.

Remark 1.2
(i) In this case, σ supports the d-dimensional lattice polytope

$$P_\sigma = \{\boldsymbol{x} \in \sigma \;\mid\; \langle \boldsymbol{m}_\sigma, \boldsymbol{x}\rangle = 1\} \subset \mathbf{H}_\sigma \cong \mathbb{R}^d \tag{1.1}$$

(w.r.t. the lattice $\mathbf{H}_\sigma \cap \mathbb{Z}^{d+1} \cong \mathbb{Z}^d$).
(ii) In fact, *every* lattice d-polytope $P \subset \mathbb{R}^d$ can be considered as supported by a cone

$$\sigma_P = \left\{(r, r\boldsymbol{x}) \in \mathbb{R} \oplus \mathbb{R}^d \;\mid\; \boldsymbol{x} \in P,\ r \in \mathbb{R}_{\geqslant 0}\right\} \subset \mathbb{R}^{d+1}$$

so that U_{σ_P} is Gorenstein.

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The $(d+1)$-dimensional Gorenstein toric singularities[(1)] $(U_\sigma, \mathrm{orb}(\sigma))$ constructed by cones σ which support lattice d-polytopes $P = P_\sigma$ are to be subdivided into three distinct classes[(2)]:

(A) *Terminal singularities* (whenever P is an elementary polytope but not a basic simplex).

(B) *Canonical, non-terminal singularities which do not admit any crepant resolution* (i.e., for which P is a non-elementary polytope having no basic triangulations).

(C) *Canonical, non-terminal singularities admitting crepant resolutions* (i.e., for which P is a non-elementary polytope possessing *at least one* basic triangulation).

Comments. — A complete classification of the members of class **(A)** (up to analytic isomorphism) is obviously equivalent to the classification of elementary polytopes (up to lattice automorphism). For constructions of several families belonging to **(C)**, the reader is referred to [**1**], [**2**], [**3**], [**4**], [**5**]. In fact, for one- or two-parameter Gorenstein cyclic quotient singularities, it is possible to decide definitely if they belong to class **(A)**, **(B)** or **(C)**, by just checking some concrete number-theoretic (necessary and sufficient existence-) conditions (see [**3**] and [**2**], respectively). On the other hand, for *general* Gorenstein toric (not necessarily quotient-) singularities, a *necessary condition* for the existence of crepant resolutions can be created via an UBT for simplicial balls, as we shall see below in Thm. 3.1. Hence, its "violation" may be used to produce families of such singularities belonging to **(B)**.

2. Basic facts about UBT's

Notation

(i) The ***f****-vector* $\boldsymbol{f}(\mathcal{S}) = (\mathfrak{f}_0(\mathcal{S}), \mathfrak{f}_1(\mathcal{S}), \ldots, \mathfrak{f}_{d-1}(\mathcal{S}))$ of a polyhedral $(d-1)$-complex $\mathcal{S}$ is defined by setting for all i, $0 \leqslant i \leqslant d-1$,

$$\mathfrak{f}_i(\mathcal{S}) := \# \ \{i\text{-dimensional faces of } \mathcal{S}\}$$

(under the usual conventional extension: $\mathfrak{f}_{-1}(\mathcal{S}) := 1$). The coordinates of the ***h****-vector* $\boldsymbol{h}(\mathcal{S}) = (\mathfrak{h}_0(\mathcal{S}), \mathfrak{h}_1(\mathcal{S}), \ldots, \mathfrak{h}_{d-1}(\mathcal{S}), \mathfrak{h}_d(\mathcal{S}))$ of such an $\mathcal{S}$ are defined by the equations

$$\mathfrak{h}_j(\mathcal{S}) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} \mathfrak{f}_{i-1}(\mathcal{S}). \tag{2.1}$$

(1) Without loss of generality, we may henceforth assume that the cones $\sigma \subset \mathbb{R}^{d+1}$ are $(d+1)$-dimensional, and that the singularities under consideration have maximal splitting codimension. (The orbit $\mathrm{orb}(\sigma) \in U_\sigma$ is the unique fixed closed point under the usual torus-action on U_σ.)

(2) A lattice polytope P is called *elementary* if the lattice points belonging to it are exactly its vertices. A lattice simplex is said to be *basic* (or *unimodular*) if its vertices constitute a part of a $\mathbb{Z}$-basis of the reference lattice (or equivalently, if its relative, normalized volume equals 1). A lattice triangulation $\mathcal{T}$ of a lattice polytope P is defined to be *maximal* (resp. *basic*), if it consists only of elementary (resp. basic) simplices.

(ii) For a d-dimensional polytope P, the *boundary complex* $\mathcal{S}_{\partial P}$ of P is defined to be the $(d-1)$-dimensional polyhedral complex consisting of the proper faces of P together with $\varnothing$ and having ∂P as its support. $\mathcal{S}_{\partial P}$ is a polyhedral $(d-1)$-sphere. $\mathcal{S}_{\partial P}$ is a geometric pure simplicial complex (in fact, a simplicial $(d-1)$-sphere) if and only if P is a simplicial polytope. The $\boldsymbol{f}$-vector $\boldsymbol{f}(P)$ of a d-polytope P is by definition the $\boldsymbol{f}$-vector $\boldsymbol{f}(\mathcal{S}_{\partial P})$ of its boundary complex.

(iii) We denote by $\mathrm{CycP}_d(k)$ the *cyclic d-polytope* with k vertices. As it is known, the number of its facets equals

$$\mathfrak{f}_{d-1}(\mathrm{CycP}_d(k)) = \binom{k-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor} + \binom{k-1-\lceil\frac{d-1}{2}\rceil}{\lfloor\frac{d-1}{2}\rfloor} \tag{2.2}$$

This follows from Gale's evenness condition and the fact that $\mathrm{CycP}_d(k)$ is $\lfloor\frac{d}{2}\rfloor$-neighbourly (cf. [**13**, p. 24]).

(iv) Classical UB and LB-Theorems for simplicial spheres (see [**9**] and [**6**]):

Theorem 2.1 **(Stanley's Upper Bound Theorem for Simplicial Spheres)**
The $\boldsymbol{f}$-vector coordinates of a simplicial $(d-1)$-sphere $\mathcal{S}$ with $\mathfrak{f}_0(\mathcal{S}) = k$ vertices satisfy the following inequalities:

$$\mathfrak{f}_i(\mathcal{S}) \leqslant \mathfrak{f}_i(\mathrm{CycP}_d(k)) \ , \ \forall\ i\ ,\ 0 \leqslant i \leqslant d-1.$$

Theorem 2.2 **(Lower Bound Theorem for Simplicial Spheres)**. — *The $\boldsymbol{h}$-vector coordinates of a simplicial $(d-1)$-sphere $\mathcal{S}$ with $\mathfrak{f}_0(\mathcal{S}) = k$ vertices satisfy the following inequalities*:

$$\mathfrak{h}_1(P) = k-d \leqslant \mathfrak{h}_i(P) \ , \ \forall\ i\ ,\ 2 \leqslant i \leqslant d.$$

Besides them we need certain variants for simplicial *balls*.

Proposition 2.3 **("*h* of ∂" –Lemma)**. — *Let $\mathcal{S}$ be a d-dimensional Cohen-Macaulay closed pseudomanifold with non-empty boundary $\partial\mathcal{S}$. Then*

$$\mathfrak{h}_{i-1}(\partial\mathcal{S}) - \mathfrak{h}_i(\partial\mathcal{S}) = \mathfrak{h}_{(d+1)-i}(\mathcal{S}) - \mathfrak{h}_i(\mathcal{S}), \quad \forall i,\ 0 \leqslant i \leqslant d+1 \tag{2.3}$$

(*under the convention*: $\mathfrak{h}_{-1}(\partial\mathcal{S}) = 0$).

Proof. — See Stanley ([**12**, 2.3]). □

Working with *Buchsbaum complexes*, Schenzel [**8**] proved the following:

Theorem 2.4 **(Schenzel's Upper Bound Theorem)**. — *Let $\mathcal{S}$ be a d-dimensional Buchsbaum complex*[3] *having $\mathfrak{f}_0(\mathcal{S}) = \mathfrak{b}$ vertices. Then for all i, $0 \leqslant i \leqslant d+1$, the $\boldsymbol{h}$-vector*

[3] A simplicial complex $\mathcal{S}$ is a Buchsbaum complex over a field $\boldsymbol{k}$ if and only if it is pure and the localizations $\boldsymbol{k}[\mathcal{S}]_{\wp}$ of $\boldsymbol{k}[\mathcal{S}]$ w.r.t. prime ideals $\wp \neq \boldsymbol{k}[\mathcal{S}]_+ (= \bigoplus_{\nu>0} (\boldsymbol{k}[\mathcal{S}]_\nu)$ are Cohen-Macaulay. (For instance, homology d-manifolds without boundary, or homology d-manifolds whose boundary is a homology $(d-1)$-manifold without boundary, are Buchsbaum). Moreover, $\mathcal{S}$ is Cohen-Macaulay over $\boldsymbol{k}$ if an only if $\mathcal{S}$ is Buchsbaum over $\boldsymbol{k}$ and $\dim_{\boldsymbol{k}} \widetilde{H}_j(\mathcal{S};\boldsymbol{k}) = 0$, for all $i, 0 \leqslant i \leqslant d-1$, while $\dim_{\boldsymbol{k}} \widetilde{H}_d(\mathcal{S};\boldsymbol{k}) = (-1)^d\ \widetilde{\chi}(\mathcal{S})$, with $\widetilde{\chi}(\mathcal{S})$ the reduced Euler characteristic.

coordinates of $\mathcal{S}$ *satisfy the inequalities*

$$\mathfrak{h}_i(\mathcal{S}) \leqslant \binom{\mathfrak{b}-d+i-2}{i} - (-1)^i \binom{d+1}{i} \sum_{j=-1}^{i-2} (-1)^j \dim_{\boldsymbol{k}} \widetilde{H}_j(\mathcal{S};\boldsymbol{k}) \tag{2.4}$$

(*where* $\widetilde{H}_j(\mathcal{S};\boldsymbol{k})$ *are the reduced homology groups of* $\mathcal{S}$ *with coefficients in a field* $\boldsymbol{k}$.)

Corollary 2.5. — *Let* $\mathcal{S}$ *denote a simplicial* d*-dimensional* ball *with* $\mathfrak{f}_0(\mathcal{S}) = \mathfrak{b}$ *vertices. Then for all* i, $0 \leqslant i \leqslant d$, *the* $\boldsymbol{f}$*-vector of* $\mathcal{S}$ *satisfies the following inequalities*:

$$\mathfrak{f}_i(\mathcal{S}) \leqslant \mathfrak{f}_i\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \sum_{j=d-i}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-i} \left(\mathfrak{h}_j(\partial\mathcal{S}) - \mathfrak{h}_{j-1}(\partial\mathcal{S})\right), \tag{2.5}$$

Proof. — Introduce the auxiliary vector $\widetilde{\boldsymbol{h}}(\mathcal{S}) = \left(\widetilde{\mathfrak{h}}_0(\mathcal{S}), \ldots, \widetilde{\mathfrak{h}}_{d+1}(\mathcal{S})\right)$ with

$$\widetilde{\mathfrak{h}}_i(\mathcal{S}) := \begin{cases} \mathfrak{h}_i(\mathcal{S}), & \text{for } 0 \leqslant i \leqslant \left\lfloor \frac{d+1}{2} \right\rfloor \\ \mathfrak{h}_i(\mathcal{S}) - \left(\mathfrak{h}_{d-i}(\partial\mathcal{S}) - \mathfrak{h}_{d+1-i}(\partial\mathcal{S})\right), & \text{for } \left\lfloor \frac{d+1}{2} \right\rfloor + 1 \leqslant i \leqslant d+1 \end{cases}$$

Since $\mathcal{S}$ is Cohen-Macaulay, $\mathcal{S}$ is a Buchsbaum complex. Moreover, all reduced homology groups $\widetilde{H}_j(\mathcal{S};\boldsymbol{k})$ are trivial, which means that

$$\mathfrak{h}_i(\mathcal{S}) \leqslant \mathfrak{h}_i\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) = \binom{\mathfrak{b}-d+i-2}{i}, \ \forall i, \ 0 \leqslant i \leqslant \left\lfloor \tfrac{d+1}{2} \right\rfloor,$$

by (2.4). On the other hand, (2.3) implies for the coordinates of $\widetilde{\boldsymbol{h}}(\mathcal{S})$:

$$\widetilde{\mathfrak{h}}_i(\mathcal{S}) = \widetilde{\mathfrak{h}}_{(d+1)-i}(\mathcal{S}), \ \forall i, \ 0 \leqslant i \leqslant d+1,$$

and therefore

$$\widetilde{\mathfrak{h}}_i(\mathcal{S}) \leqslant \mathfrak{h}_i\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right), \ \forall i, \ 0 \leqslant i \leqslant d+1 . \tag{2.6}$$

Hence,

$$\begin{aligned}
\mathfrak{f}_i(\mathcal{S}) &= \sum_{j=0}^{i+1} \binom{d+1-j}{d-i} \mathfrak{h}_j(\mathcal{S}) \\
&= \sum_{j=0}^{i+1} \binom{d+1-j}{d-i} \widetilde{\mathfrak{h}}_j(\mathcal{S}) + \sum_{j=\lceil \frac{d}{2} \rceil+1}^{i+1} \binom{d+1-j}{d-i} \left(\mathfrak{h}_{d-j}(\partial\mathcal{S}) - \mathfrak{h}_{d+1-j}(\partial\mathcal{S})\right) \\
&= \sum_{j=0}^{i+1} \binom{d+1-j}{d-i} \widetilde{\mathfrak{h}}_j(\mathcal{S}) + \sum_{j=d-i}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-i} \left(\mathfrak{h}_{j-1}(\partial\mathcal{S}) - \mathfrak{h}_j(\partial\mathcal{S})\right) \text{[by interchanging} \\
&\qquad (d+1)-j \text{ and } j, \text{ and using the Dehn-Sommerville relations for } \boldsymbol{h}(\partial\mathcal{S})] \\
&\leqslant \sum_{j=0}^{i+1} \binom{d+1-j}{d-i} \mathfrak{h}_i\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) + \sum_{j=d-i}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-i} \left(\mathfrak{h}_{j-1}(\partial\mathcal{S}) - \mathfrak{h}_j(\partial\mathcal{S})\right) \text{ [by (2.6)]} \\
&= \mathfrak{f}_i\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \sum_{j=d-i}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-i} \left(\mathfrak{h}_j(\partial\mathcal{S}) - \mathfrak{h}_{j-1}(\partial\mathcal{S})\right)
\end{aligned}$$

for all $i, 0 \leqslant i \leqslant d$. □

Corollary 2.6. — *Let $\mathcal{S}$ be a simplicial d-*ball *with* $\mathfrak{f}_0(\mathcal{S}) = \mathfrak{b}$ *vertices. Suppose that* $\mathfrak{f}_0(\partial\mathcal{S}) = \mathfrak{b}'$. *Then*:

$$\mathfrak{f}_d(\mathcal{S}) \leqslant \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - (\mathfrak{b}' - d) \tag{2.7}$$

Proof. — For $i = d$, (2.5) gives

$$\begin{aligned} \mathfrak{f}_d(\mathcal{S}) &\leqslant \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \left(\mathfrak{h}_j(\partial\mathcal{S}) - \mathfrak{h}_{j-1}(\partial\mathcal{S})\right) \\ &= \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \mathfrak{h}_{\lfloor \frac{d}{2} \rfloor}(\partial\mathcal{S}) \\ &\leqslant \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\mathfrak{b})\right) - \mathfrak{h}_1(\partial\mathcal{S}), \end{aligned}$$

where the latter inequality comes from the LBT 2.2 for the simplicial sphere $\partial\mathcal{S}$. Now obviously, $\mathfrak{h}_1(\partial\mathcal{S}) = \mathfrak{b}' - d$. □

3. Crepant Resolutions and UBT

Let $(U_\sigma, \mathrm{orb}(\sigma))$ be a Gorenstein toric singularity as in § 1 (cf. Thm. 1.1).

***Theorem 3.1* (Necessary Existence Condition).** — *If U_σ admits a crepant desingularization, then the normalized volume of the lattice polytope P_σ (defined in (1.1)) has the following upper bound*[(4)]

$$\mathrm{Vol}_{\mathrm{norm}}(P_\sigma) \leqslant \mathfrak{f}_d\left(\mathrm{CycP}_{d+1}(\#(P_\sigma \cap \mathbb{Z}^d))\right) - \left(\#(\partial P_\sigma \cap \mathbb{Z}^d) - d\right) \tag{3.1}$$

Proof. — If U_σ admits a crepant desingularization, then there must be a basic triangulation, say $\mathcal{T}$ of P_σ. Since this $\mathcal{T}$ is, in particular, maximal, we have

$$\mathrm{vert}(\mathcal{T}) = P_\sigma \cap \mathbb{Z}^d, \quad \mathrm{vert}(\partial\mathcal{T}) = \partial P_\sigma \cap \mathbb{Z}^d. \tag{3.2}$$

On the other hand,

$$\mathrm{Vol}_{\mathrm{norm}}(P_\sigma) = \mathfrak{f}_d(\mathcal{T}). \tag{3.3}$$

Finally, since $\mathcal{T}$ is a simplicial d-ball, one deduces (3.1) from (2.7), (3.2), (3.3). □

Example 3.2. — Let

$$\sigma = \mathbb{R}_{\geqslant 0}\, e_1 + \mathbb{R}_{\geqslant 0}\, e_2 + \mathbb{R}_{\geqslant 0}\, e_3 + \mathbb{R}_{\geqslant 0}\, (-3, -7, -9, 20) \subset \mathbb{R}^4$$

be the four-dimensional cone supporting the lattice 3-simplex

$$\boldsymbol{s}_\sigma = \mathrm{conv}\left(\{e_1, e_2, e_3, (-3, -7, -9, 20)\}\right) = \{\boldsymbol{x} \in \sigma \mid \langle \boldsymbol{m}_\sigma, \boldsymbol{x} \rangle = 1\},$$

(4) By abuse of notation, we write $\mathbb{Z}^d$ instead of $\mathbf{H}_\sigma \cap \mathbb{Z}^{d+1}$ ($\cong \mathbb{Z}^d$)

where $\boldsymbol{m}_\sigma = (1,1,1,1)$. Obviously, $\mathrm{Vol}_{\mathrm{norm}}(\boldsymbol{s}_\sigma) = 20$. On the other hand, since $\boldsymbol{s}_\sigma$ has 8 lattice points (its 4 vertices and further 4 integer points lying in its relative interior), computing the right-hand side of (3.1), we obtain by (2.2):

$$\binom{8-\lceil \frac{4}{2}\rceil}{\lfloor \frac{4}{2}\rfloor} + \binom{8-1-\lceil \frac{3}{2}\rceil}{\lfloor \frac{3}{2}\rfloor} - (4-3) = \binom{6}{2} + \binom{5}{1} - 1 = 19.$$

This means that $(U_\sigma, \mathrm{orb}(\sigma))$ necessarily belongs to the class **(B)** described in § 1.

Remark 3.3. — The upper bound (3.1) will be improved considerably in [**5**], in the case in which P is a *simplex*, by using a different combinatorial-topological technique.

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D.I. Dais, Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, CY-1678 Nicosia, Cyprus • *E-mail :* `ddais@ucy.ac.cy` • *Url :* `http://www.ucy.ac.cy/~ddais/`