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GENERALIZED GRADIENTS AND POISSON TRANSFORMS

by

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Abstract. — For G a semisimple Lie group and P a parabolic subgroup we construct a large class of first-order differential operators which are G-equivariant between certain vector bundles over G/P. These are intertwining operators from one generalized principal series representation for G to another. We also study the relation with Poisson transforms to the Riemannian symmetric space G/K.

***Résumé* (Gradients généralisés et les transformations de Poisson).** — Pour G un groupe de Lie semi-simple et pour P un sous-groupe parabolique, nous construisons une grande famille d'opérateurs différentiels G-équivariants du premier ordre entre certains fibrés vectoriels sur G/P. Il s'agit d'opérateurs d'entrelacement des représentations de série principale généralisée. Nous étudions également la relation avec l'espace symétrique riemannien, G/K, en utilisant les transformations de Poisson.

1. Introduction

This paper is partly motivated by differential geometry, partly by representation theory for semi-simple Lie groups. We give a generalization of the results by Fegan [**4**], which dealt with the group $\mathrm{SO}(n,1)$, to the case of an arbitrary semisimple Lie group G and an arbitrary parabolic subgroup P. At the same time we give a new proof of Fegan's case, and place it in the framework of analysis on Lie groups. Our method of constructing intertwining first-order differential operators between generalized principal series representations for G has its origin in the generalized gradients of Stein and Weiss [**8**], suitably generalized to the setting of flag manifolds.

We expect our construction of these gradients to have applications in other parabolic geometries, and also in the construction of small unitary representations of semi-simple Lie groups. By duality our problem is related to finding embeddings between

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generalized Verma modules; this was studied for parabolic geometries in great generality by Cap, Slovák and Souček, see [**2**], and previously by Baston and Eastwood. Closest to our approach is the recent work by Korányi and Reimann [**6**] who with a different (and independent) method treat the case of a minimal parabolic subgroup. Note also [**10**] where a related family of operators is constructed and applied to the problem of finding composition series for real rank one groups.

Let us here briefly state in rough form our main result: Denote by $C^\infty(\mathbb{E})$ the smooth sections of a homogeneous vector bundle over the real flag manifold G/P, where G is a semi-simple Lie group, P a parabolic subgroup, and $\mathbb{E}$ induced by a representation E of P, i.e.

$$\mathbb{E} = G \times_P E.$$

We assume E irreducible, and denote by $\mathbb{T}^*$ the cotangent bundle over G/P with fiber T^* at the base point. The goal is to find a first-order differential operator on smooth sections

$$D\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{F})$$

which is G-equivariant between two such bundles. This is done by first finding an equivariant connection (actually in the first instance only equivariant w.r.t. the maximal compact subgroup K of G)

$$\nabla\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{E} \otimes \mathbb{T}^*)$$

and second to decompose the tensor product $E \otimes T^*$ and project on a suitable quotient F, invariant for the P-action:

$$\mathrm{proj}\colon E \otimes T^* \longrightarrow F.$$

Then our gradient is the composition $D = \mathrm{proj} \circ \nabla$ and we have

Theorem 1.1. — *In the setting above D is G-equivariant if and only if the Casimir operator of G has the same value in $C^\infty(\mathbb{E})$ and in $C^\infty(\mathbb{F})$.*

In the last section we show that these gradients can in some sense be extended to the Riemannian symmetric space G/K in a canonical way, which is consistent with natural vector-valued Poisson transforms from $C^\infty(\mathbb{E})$ to sections of bundles over G/K.

From a representation theory point of view the gradients D are useful in studying the lattice of invariant subspaces in $C^\infty(\mathbb{E})$, i.e. the composition series for generalized principal series. Though we shall not go into discussing higher order equivariant differential equations in this paper, it is clear that there will exist such by composing our first-order operators. The Poisson transforms relate to both representation theory and to geometric problems — we have at the end added one such example and also a case of a symplectic analogue on S^2 of the Dirac operator, equivariant for the double cover of the projective group.

2. Construction of gradients

Fix a semi-simple Lie group G with finite center, a maximal compact subgroup K, a corresponding Cartan decomposition of the Lie algebra of G:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

and a maximal Abelian subspace $\mathfrak{a}_0 \subseteq \mathfrak{s}$. We have a corresponding minimal parabolic subgroup $P_0 = M_0A_0N_0$ constructed in the usual way, and we fix a parabolic subgroup $P \supseteq P_0$ with Langlands decomposition

$$P = MAN$$

and for the Lie algebras $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ of M, A, N we get

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, θ the Cartan involution, and

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{n}_\alpha$$

the decomposition into the positive root spaces and $\alpha > 0$ means $\alpha \in \Delta^+ \subseteq \Delta$ for a choice of positive roots of $\mathfrak{a}$ in $\mathfrak{g}$. For this, see [**5**]. We shall also need the simple roots $S \subseteq \Delta^+$ (note that we can still talk about simple roots, even though we may not have a root system here). The flag manifold is the compact space

$$G/P = K/K \cap P = K/K \cap M$$

and this is where we shall construct equivariant first-order differential operators. Fix an irreducible finite-dimensional representation (σ, E_σ) of M in the Hilbert space E_σ (we shall later relax this condition); for $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, the complex dual space to $\mathfrak{a}$, consider the generalized principal series representation

$$\pi_{\sigma,\nu} = \mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes I)$$

induced from the P-representation

$$(\sigma \otimes e^\nu \otimes I)(man) = \sigma(m)a^\nu.$$

The smooth vectors are

$$C^\infty(\mathbb{E}_{\sigma,\nu}) = \\ \big\{f\colon G \to E_\sigma \mid f \in C^\infty,\ f(gman) = \sigma(m)^{-1}a^{-\nu}f(g) \text{ for all } g \in G,\, man \in MAN\big\}$$

which we identify with the smooth sections of the homogeneous vector bundle

$$\mathbb{E}_{\sigma,\nu} = G \times_P E_{\sigma,\nu}$$

where $E_{\sigma,\nu} = E_\sigma$ with the P-action considered above. We call ν the weight of the representation/bundle.

As usual G acts by left translation:

$$(\pi_{\sigma,\nu}(g_0)f)(g) = f(g_0^{-1}g) \quad (g_0, g \in G)$$

and we may let this representation act in a Hilbert space by setting

$$\|f\|^2 = \int_K \|f(k)\|^2_{E_\sigma} dk$$

– but we shall not need to do so here. Recall that a first-order differential operator is a homomorphism from the first jet bundle to the image bundle, see **[7]**, so we are looking for σ' and ν' with a

$$D\colon J^1(\mathbb{E}_{\sigma,\nu}) \longrightarrow \mathbb{E}_{\sigma',\nu'}$$

where the fiber at a point $x \in G/P$ of the first jet bundle is

$$J^1(\mathbb{E}_{\sigma,\nu})_x = C^\infty(\mathbb{E}_{\sigma,\nu})/Z^1_x(\mathbb{E}_{\sigma,\nu})$$

where

$$Z^1_x(\mathbb{E}_{\sigma,\nu}) = \{f \mid f^{(\alpha)}(x) = 0, \quad |\alpha| \leq 1\}$$

is the space of sections vanishing to first order at the point.

Now the first jet bundle is also a homogeneous vector bundle, and at the base point the fiber is (suppressing the σ and ν)

$$J^1(E) \cong E \oplus \mathrm{Hom}(\bar{\mathfrak{n}}, E)$$

and for a section $f \in C^\infty(\mathbb{E})$ we have the natural map

$$j^1\colon f \longrightarrow (f, df)|_{eP} \in J^1(E)$$

specifying for a section its value and first derivative at the base point. Here we identify the tangent space at the base point $T^* \cong \bar{\mathfrak{n}}$ and the cotangent space $T^* \cong \mathfrak{n}$ via the duality induced by the Killing form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. It is convenient to consider the derivative of a section as the following covariant derivative

$$(\nabla_X f)(g) = \frac{d}{dt} f(g \exp tX)|_{t=0} \quad (f \in C^\infty(\mathbb{E}), X \in \bar{\mathfrak{n}}, g \in G)$$

which defines a connection

$$\nabla\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{E} \otimes \mathbb{T}^*).$$

Our goal is to compose this with a projection from $E \otimes \mathfrak{n}$ onto some subspace F invariant under the action of M — this is the generalized gradient construction of the desired

$$D\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{F})$$

with an appropriate choice of weights. So we are looking for a G-map

$$D\colon G \times_P J^1(E) \longrightarrow G \times_P F$$

which means looking for a P-map

$$D\colon J^1(E) \longrightarrow F.$$

The main problem is to construct D as an $\mathfrak{n}$-map, i.e. we have to study the action of $\mathfrak{n}$ on the module $J^1(E)$. This is done in the following

Lemma 2.1. — *Let $v \in E$ and $A \in \mathrm{Hom}(\bar{\mathfrak{n}}, E)$ correspond to the section $f \in C^\infty(\mathbb{E})$ via the map j^1; then for all $Y \in \mathfrak{n}$ the action is*

$$Y \cdot (v, A) = (0, [Y, \cdot]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot v + A([Y, \cdot]_{\bar{\mathfrak{n}}}))$$

where an element $Z \in \mathfrak{g}$ is decomposed

$$Z = Z_{\bar{\mathfrak{n}}} + Z_{\mathfrak{m}\oplus\mathfrak{a}} + Z_{\mathfrak{n}}$$

according to the direct sum

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Proof. — Let $X \in \bar{\mathfrak{n}}$, $Y \in \mathfrak{n}$, $n = \exp sY$ and $f \in C^\infty(\mathbb{E})$, then the N-action on the differential of f is

$$\frac{d}{dt} f(n^{-1} \exp tX)|_{t=0} = \frac{d}{dt} f(\exp(\mathrm{Ad}(n^{-1})tX))|_{t=0}$$

and we also have by differentiation of this the action of Y as

$$\frac{d}{dt}\frac{d}{ds} \exp(\mathrm{Ad}(n^{-1})tX)|_{s=0}|_{t=0} = -\mathrm{ad}(Y)X = [X, Y]_{\bar{\mathfrak{n}}} + [X, Y]_{\mathfrak{m}\oplus\mathfrak{a}} + [X, Y]_{\mathfrak{n}}.$$

Since f is a section, it transforms trivially from the right under $\mathfrak{n}$ and according to the action in E under $\mathfrak{m} \oplus \mathfrak{a}$. Hence we get the $\mathfrak{n}$-action

$$[Y, X]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot f(e) + A([Y, X]_{\bar{\mathfrak{n}}})$$

as stated, since

$$A([Y, X]_{\bar{\mathfrak{n}}}) = [Y, X]_{\bar{\mathfrak{n}}} \cdot df(e).$$

□

Following Fegan we first consider the "$\mathfrak{m} \oplus \mathfrak{a}$" part in this action, namely the term

$$[Y, X]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot v$$

which may be thought of as a map

$$\beta\colon \mathfrak{n} \longrightarrow \mathrm{Hom}(E, \mathfrak{n} \otimes E)$$

hence an element

$$\beta \in \mathrm{Hom}(\mathfrak{n}, E^* \otimes \mathfrak{n} \otimes E) \cong \mathrm{Hom}(\mathfrak{n} \otimes E, \mathfrak{n} \otimes E).$$

Now the image of β will be an $\mathfrak{n}$-submodule of $\mathfrak{n} \otimes E$ since β exactly encodes the action of $\mathfrak{n}$. The "$\bar{\mathfrak{n}}$" part is

$$A([X, Y]_{\bar{\mathfrak{n}}})$$

which can be made to vanish, namely by observing that if α is a simple root, then

$$\forall\, Y \in \mathfrak{n} \quad \forall\, X \in \bar{\mathfrak{n}}_\alpha\colon [X, Y]_{\bar{\mathfrak{n}}} = 0.$$

Hence for α a simple root, the image of

$$\beta\colon \mathfrak{n}_\alpha \otimes E \longrightarrow \mathfrak{n}_\alpha \otimes E$$

is $\mathfrak{n}$-invariant. Our next lemma gives a formula from which we can get a simple criterion for this image to be strictly smaller than $\mathfrak{n}_\alpha \otimes E$.

Lemma 2.2. — *Let $\beta\colon \mathfrak{n}\otimes E \longrightarrow \mathfrak{n}\otimes E \cong \mathrm{Hom}(\bar{\mathfrak{n}}, E)$ be the map above, i.e.*

$$\beta(Y\otimes v)(X) = [Y,X]_{\mathfrak{m}\oplus\mathfrak{a}}\cdot v \quad (Y\in\mathfrak{n}, X\in\bar{\mathfrak{n}}, v\in E),$$

then β can be expressed in terms of Casimir operators as follows for $Y\in\mathfrak{n}_\alpha$, $\alpha\in\Delta^+$:

$$\beta = \langle\alpha,\nu\rangle - \frac{1}{2}(C(\mathfrak{n}\otimes E) - C(\mathfrak{n}) - C(E))$$

where ν is the weight in $E = E_{\sigma,\nu}$ and $C(E)$ denotes the Casimir operator of M in the representation E etc. relative to the Killing form $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$.

Proof. — Choose a basis $X_1,\dots,X_r$ of $\mathfrak{a}$ with $\langle X_i, X_{i'}\rangle = \delta_{ii'}$ and a basis $X_{r+1},\dots,X_n$ of $\mathfrak{m}$ with $\langle X_j, X_{j'}\rangle = \varepsilon_j\delta_{jj'}$ where $\varepsilon_j = \pm 1$. Then, since $\mathfrak{m}$ centralizes $\mathfrak{a}$, and $\mathfrak{a}$ is Abelian, the $\mathfrak{m}\oplus\mathfrak{a}$ projection in question can be calculated as

$$\begin{aligned}[Y,X]_{\mathfrak{m}\oplus\mathfrak{a}} &= \sum_{i=1}^{r}\langle [Y,X], X_i\rangle X_i + \sum_{j=r+1}^{n}\varepsilon_j\langle [Y,X], X_j\rangle X_j \\ &= \sum_{i=1}^{r}\langle [X_i,Y], X\rangle X_i + \sum_{j=r+1}^{n}\varepsilon_j\langle [X_j,Y], X\rangle X_j\end{aligned}$$

where $Y\in\mathfrak{n}$, $X\in\bar{\mathfrak{n}}$. Hence with $v\in E$ we get

$$[Y,X]_{\mathfrak{m}\oplus\mathfrak{a}}\cdot v = \sum_{i=1}^{r}\langle [X_i,Y], X\rangle X_i\cdot v + \sum_{j=r+1}^{n}\varepsilon_j\langle [X_j,Y], X\rangle X_j\cdot v$$

so that the action on $Y\otimes v\in\mathfrak{n}\otimes E$ is

$$\beta = \sum_{i=1}^{r} X_i\otimes X_i + \sum_{j=r+1}^{n}\varepsilon_j X_j\otimes X_j.$$

The first term becomes, on $\mathfrak{n}_\alpha\otimes E$ for $\alpha\in\Delta^+$:

$$\sum_{i=1}^{r}\alpha(X_i)\nu(X_i) = \langle\alpha,\mu\rangle$$

(identifying $\mathfrak{a}\cong\mathfrak{a}^*$ via $\langle\cdot,\cdot\rangle$), and the second term

$$\frac{1}{2}\sum_{j=r+1}^{n}\varepsilon_j[(1\otimes X_j + X_j\otimes 1)^2 - X_j^2\otimes 1 - 1\otimes X_j^2] = \frac{1}{2}(-C(\mathfrak{n}\otimes E) + C(\mathfrak{n}) + C(E)).$$

Note our sign convention in the Casimirs here. $\square$

Now we remark that for α a simple positive root the image $\widetilde{W}$ of

$$\beta\colon \mathfrak{n}_\alpha\otimes E \longrightarrow \widetilde{W}\subseteq \mathfrak{n}_\alpha\otimes E$$

is $\mathfrak{n}$-invariant, and also $\mathfrak{m} \oplus \mathfrak{a}$-invariant. Furthermore, the weight in $\widetilde{W}$ is $\nu + \alpha$, so by choosing ν properly we obtain a non-trivial quotient $F = (\mathfrak{n}_\alpha \otimes E)/\widetilde{W}$ and hence a non-trivial map $D\colon J^1(E) \to F$. Namely, we have

Proposition 2.3. — *Let $E = E_{\sigma,\nu}$ and the weight $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ satisfy*

$$2\langle \alpha, \nu\rangle = C(F) - C(\mathfrak{n}_\alpha) - C(E)$$

for the simple root α, where

$$F \subseteq \mathfrak{n}_\alpha \otimes E$$

is an M-submodule consisting of the irreducible submodules with the same value $C(F)$ of the $\mathfrak{m}$-Casimir. Then the image $\widetilde{W} = \beta(\mathfrak{n}_\alpha \otimes E) \subseteq \mathfrak{n}_\alpha \otimes E$ is $\mathfrak{n}$-invariant. Furthermore, the weight of F being $\alpha + \nu$ we have the P-equivariant quotient mapping

$$\widetilde{D}\colon \mathfrak{n}_\alpha \otimes E \longrightarrow (\mathfrak{n}_\alpha \otimes E)/\widetilde{W} \cong F.$$

Proof. — F is simply the kernel of β, and $\widetilde{W} \cong (\mathfrak{n}_\alpha \otimes E)/F$. We already checked that $\widetilde{W}$ is an $\mathfrak{n}$-submodule, and it is also a submodule for M and $\mathfrak{a}$. □

In order to state the result in the simplest way, let us assume that decomposing the tensor product $\mathfrak{n}_\alpha \otimes E$ for α simple into irreducible M-modules $F = F_1, F_2, \ldots, F_N$ we have

$$\begin{cases} \mathfrak{n}_\alpha \otimes E = F \oplus F_2 \oplus \cdots \oplus F_n \\ C(F) \neq C(F_i) \quad \text{for } i = 2, \ldots, n. \end{cases} \tag{2.1}$$

Note that our considerations of differentials of sections amount to applying first

$$\nabla\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{E} \otimes \mathbb{T}^*)$$

and then a projection pointwise in the fiber

$$\mathrm{proj}_F\colon \mathfrak{n} \otimes E \longrightarrow \mathfrak{n}_\alpha \otimes E \longrightarrow F.$$

The conclusion is our main

Theorem 2.4. — *Fix an irreducible representation $E = (\sigma, E_\sigma)$ of M and let $F = (\sigma', E_{\sigma'})$ be an irreducible M-module occurring in $\mathfrak{n}_\alpha \otimes E$ satisfying (2.1) with α a simple root. Suppose the weight ν satisfies*

$$2\langle \alpha, \nu\rangle = C(F) - C(\mathfrak{n}_\alpha) - C(E).$$

Then $D = \mathrm{proj}_F \circ \nabla\colon C^\infty(\mathbb{E}) \to C^\infty(\mathbb{F})$ is a non-trivial first-order equivariant differential operator, i.e.

$$D\ \pi_{\sigma,\nu}(g) = \pi_{\sigma',\nu'}(g) D \qquad (g \in G)$$

where $\nu' = \nu + \alpha$, acting between the generalized principal series representations.

Proof. — At the base point in G/P the operator D coincides with $\widetilde{D}$ in Proposition 2.3, and this operator is a P-map on $J^1(E)$. Hence D defines a G-map

$$D\colon J^1(\mathbb{E}) \longrightarrow \mathbb{F}$$

which on sections $f \in C^\infty(\mathbb{E})$ is the same as D in the theorem. □

It is interesting to note that a generalized principal series $\pi_{\sigma,\nu}$ has an infinitesimal character given by

$$\Lambda = \lambda + \delta_M + \nu - \rho_{\mathfrak{a}}$$

where λ is the highest weight of E_σ, δ_M the half-sum of positive roots in $\mathfrak{m}$, letting $\mathfrak{t} \subseteq \mathfrak{m}$ be a θ-stable Cartan subalgebra, so that $(\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, $\widetilde{\Delta} = \widetilde{\Delta}((\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ its roots so that $\Delta =$ the set of roots of $(\mathfrak{g}, \mathfrak{a})$ is obtained by restriction from $(\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}$ to $\mathfrak{a}$. By extending an ordering from $\mathfrak{a}_{\mathbb{C}}$ to $(\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}$ we can ensure that Δ^+ arises by restriction from $\widetilde{\Delta}^+$. The members of $\widetilde{\Delta}$ vanishing on $\mathfrak{a}$ gives $\Delta_M = \Delta(\mathfrak{t}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}})$ and $\Delta_M^+ = \widetilde{\Delta}^+ \cap \Delta_M$ with δ_M the corresponding half-sum.

Finally, $\delta = \delta_M + \rho_{\mathfrak{a}}$ is the half-sum of all roots in $\widetilde{\Delta}^+$. See [**5**, p.225]. According to Proposition 8.20 [**5**] we have in the setting of our theorem that

$$\exists\, w \in W_G\colon w(\lambda + \delta_M + \nu - \rho_{\mathfrak{a}}) = \lambda' + \delta_M + \nu + \alpha - \rho_{\mathfrak{a}}$$

where W_G is Weyl group of G (i.e. of $\widetilde{\Delta}$) and $\Lambda' = \lambda' + \delta_M + \nu + \alpha - \rho_{\mathfrak{a}}$ is the corresponding infinitesimal character of $\pi_{\sigma',\nu'}$, $\nu' = \nu + \alpha$. Actually this last relation allows us to get a good deal of information on both the weight ν and also on the decomposition of the tensor product $\mathfrak{n}_\alpha \otimes E_\sigma$. We shall not pursue that here.

Note also that (still in the setting of the theorem) the value of the Casimir operator of G in $\pi_{\sigma,\nu}$ is given by [**5**, p. 463], letting $\rho = \rho_{\mathfrak{a}}$:

$$\begin{aligned}\Omega_G &= -\langle \nu - \rho, \nu - \rho\rangle + \langle \rho, \rho\rangle + C(E_\sigma)\\ &= -\langle \nu, \nu - 2\rho\rangle + C(E_\sigma).\end{aligned}$$

This value must be the same in $\pi_{\sigma,\nu}$ and in $\pi_{\sigma',\nu'}$, $\nu' = \nu + \alpha$, hence

$$-\langle \nu, \nu - 2\rho\rangle + C(E_\sigma) = -\langle \nu + \alpha, \nu + \alpha - 2\rho\rangle + C(E_{\sigma'})$$

so that we must have the relation

$$2\langle \nu, \alpha\rangle = C(E_{\sigma'}) - C(E_\sigma) - \langle \alpha, \alpha - 2\rho\rangle.$$

Compare this with our previous sufficient condition for the existence of the intertwining operator D, viz.

$$2\langle \nu, \alpha\rangle = C(E_{\sigma'}) - C(E_\sigma) - C(\mathfrak{n}_\alpha)$$

and we conclude

Corollary 2.5. — *Consider a simple root α of $\mathfrak{a}$ in $\mathfrak{g}$ and the corresponding root space $\mathfrak{n}_\alpha$ as an M-module. Then*

$$C(\mathfrak{n}_\alpha) = \langle \alpha, \alpha - 2\rho\rangle.$$

Remark. — An independent proof of this fact was kindly communicated to us by T. Kobayashi; it does not seem to be in the literature. Note that in particular $\langle\alpha, \alpha - 2\rho\rangle = 0$ in the split case, minimal parabolic, in agreement with a well-known fact for root systems.

3. Poisson transforms

In this section we study extensions of our gradients to the Riemannian symmetric space G/K where $K = G^\theta$ is the maximal compact subgroup. For simplicity we only consider the case of the minimal parabolic subgroup

$$P = MAN = M_0A_0N_0$$

so in particular M is compact. The general case presents no real complication, except the notation is more cumbersome. G/P is sometimes called the maximal boundary of G/K, and our aim is to establish a commuting diagram

$$\begin{array}{ccc} C^\infty(\mathbb{E}) & \xrightarrow{D} & C^\infty(\mathbb{F}) \\ \mathcal{P}\Big\downarrow & & \Big\downarrow\mathcal{P} \\ C^\infty(\widetilde{\mathbb{E}}) & \xrightarrow[\widetilde{D}]{} & C^\infty(\widetilde{\mathbb{F}}) \end{array}$$

where D is one of our gradients on G/P, $\widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{F}}$ homogeneous vector bundles over G/K, and $\widetilde{D}$ a first-order G-equivariant differential operator. The transform $\mathcal{P}$ is an integral transform generalizing the classical Poisson transform, and it provides a G-equivariant "extension" of sections over G/P to sections over G/K. Such extensions and also the commuting diagram was studied in the setting of quasi-conformal geometry on the sphere (and also the CR-analogue) by Korányi and Reimann, see the references in [**6**]. This geometric case means that the gradient D is the Ahlfors operator

$$SX = L_X h - \frac{2}{n}(\mathrm{div} X)h$$

on vector fields X on the n-sphere with standard metric h.

Over G/K we have homogeneous bundles $\mathbb{V} = G \times_K V$ with smooth sections

$$C^\infty(\mathbb{V}) = \big\{f \colon G \longrightarrow V \mid f \in C^\infty, f(gk) = \gamma(k)^{-1}f(g)(g \in G, k \in K)\big\}$$

where (γ, V) is a representation of K. Again we have a natural covariant derivative

$$\widetilde{\nabla} \colon C^\infty(\mathbb{V}) \longrightarrow C^\infty(\mathbb{V} \otimes \mathbb{T}^*)$$

where $\mathbb{T}^* \cong G \times_K \mathfrak{s}^*$ is the cotangent bundle, namely:

$$(\widetilde{\nabla}_X f)(g) = \frac{d}{dt} f(g \exp tX)|_{t=0} \quad (X \in \mathfrak{s}, g \in G, f \in C^\infty(\mathbb{V})).$$

It is worth to record (perhaps known, but not explicit in the literature)

Proposition 3.1. — *$\widetilde{\nabla}$ defines a G-equivariant covariant derivative with zero torsion. In particular the canonical metric on G/K is parallel, so that on tensors or spinors $\widetilde{\nabla}$ is the canonical Levi-Civita connection.*

Proof. — Note that $\widetilde{\nabla}$ is well-defined and maps between the indicated homogeneous vector bundles. To calculate the torsion consider $\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X$ where X and Y are vector fields on G/K. Since $[\mathfrak{s},\mathfrak{s}] \subseteq \mathfrak{k}$ we get zero torsion from the definition of $\widetilde{\nabla}$. The conclusion now follows from the characterization of the Levi-Civita connection by its invariance and zero torsion. □

It is interesting to see how this formula for $\widetilde{\nabla}$ fits well with the harmonic analysis over G/K, namely suppose Harish-Chandra's Plancherel formula is written, with $d\mu$ the Plancherel measure on $\widehat{G}$

$$L^2(G) \cong \int^{\oplus} \pi_\mu \otimes \pi_\mu^* \; d\mu$$

so that we for the L^2-sections of $\mathbb{V}$ have the decomposition

$$L^2(\mathbb{V}) \cong \int^{\oplus} \pi_\mu \otimes (\pi_\mu^* \otimes V)^K d\mu.$$

Then on each irreducible constituent $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}(\xi \otimes \xi^* \otimes v)(X) = \xi \otimes X \cdot \xi^* \otimes v \quad (\xi \in \pi_\mu, \xi^* \in \pi_\mu^*, v \in V, X \in \mathfrak{s})$$

and this indeed is a left G-, right K-map, and

$$\widetilde{\nabla} \colon \pi_\mu \otimes (\pi_\mu^* \otimes V)^K \longrightarrow \pi_\mu \otimes (\pi_\mu^* \otimes V \otimes \mathfrak{s}^*)^K.$$

As a corollary we obtain the formula for the μ-component of $\widetilde{\nabla}^* \widetilde{\nabla}$

$$(\widetilde{\nabla}^* \widetilde{\nabla})_\mu = C_G(\pi_\mu) - C_K(V)$$

where C_G is the Casimir operator of G and C_K that of K, both relative to the Killing form of G. This formula was first found by Branson, see e.g. [**1**]. Note finally, that for affine symmetric spaces we have similar results, primarily the formula for $\widetilde{\nabla}$ as a right derivative (the Lie derivative being the left derivative).

To prepare for the Poisson transform we assume that (σ, E_σ) is an irreducible representation of M and (γ, V_γ) an extension of this, i.e. we have an M-equivariant map

$$I \colon E_\sigma \longrightarrow V_\gamma$$

where (γ, V_γ) is an irreducible representation of K. Corresponding to this we define the vector-valued Poisson transform

$$\mathcal{P} \colon C^\infty(G \times_P E_{\sigma,\nu}) \longrightarrow C^\infty(G \times_K V_\gamma)$$

depending on a weight ν, as follows:

$$(\mathcal{P}f)(g) = \int_K \gamma(k) I(f(gk)) \, dk.$$

We shall also need the projection

$$I_\theta(X) = X - \theta(X) \quad (X \in \overline{\mathfrak{n}})$$

which is an M-equivariant map

$$I_\theta \colon \bar{\mathfrak{n}} \longrightarrow \mathfrak{s}$$

and also by duality we have (same notation and same formula, using $\mathfrak{s}^* \cong \mathfrak{s}$ via the Killing form)

$$I_\theta \colon \mathfrak{n} \longrightarrow \mathfrak{s}^*.$$

Suppose finally that $E_{\sigma'}$ is an irreducible constituent of $E_\sigma \otimes \mathfrak{n}_\alpha$, α a simple root, and $V_{\gamma'}$ an irreducible constituent of $V_\gamma \otimes \mathfrak{s}^*$, and we have a commutative diagram

$$\begin{array}{ccc} E_\sigma \otimes \mathfrak{n}_\alpha & \xleftarrow{\mathrm{proj}^*_{\sigma'}} & E_{\sigma'} \\ {\scriptstyle I\otimes I_\theta}\Big\downarrow & & \Big\downarrow{\scriptstyle I'} \\ V_\gamma \otimes \mathfrak{s}^* & \xleftarrow[\mathrm{proj}^*_{\gamma'}]{} & V_{\gamma'} \end{array} \tag{3.1}$$

where I' is an M-map,

$$\mathrm{proj}_{\sigma'} \colon E_\sigma \otimes \mathfrak{n}_\alpha \longrightarrow E_{\sigma'}$$

the M-equivariant projection, and

$$\mathrm{proj}_{\gamma'} \colon V_\gamma \otimes \mathfrak{s}^* \longrightarrow V_{\gamma'}$$

the K-equivariant projection. Corresponding to $I \otimes I_\theta$ and by restriction to I' we have a Poisson transform, again denoted $\mathcal{P}$, and we consider the following diagram, using the projections $\mathrm{proj}_{\sigma'}$ and $\mathrm{proj}_{\gamma'}$:

$$\begin{array}{ccccc} C^\infty(\mathbb{E}_{\sigma,\nu}) & \xrightarrow{\nabla} & C^\infty(\mathbb{E}_{\sigma,\nu} \otimes \mathbb{T}^*) & \longrightarrow & C^\infty(\mathbb{E}_{\sigma',\nu'}) \\ {\scriptstyle \mathcal{P}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{P}} & & \Big\downarrow{\scriptstyle \mathcal{P}} \\ C^\infty(\mathbb{V}_\gamma) & \xrightarrow[\widetilde{\nabla}]{} & C^\infty(\mathbb{V}_\gamma \otimes \mathfrak{s}^*) & \longrightarrow & C^\infty(\mathbb{V}_{\gamma'}). \end{array}$$

Theorem 3.2. — *In the setting above we define the gradients*

$$\begin{aligned} D &= \mathrm{proj}_{\sigma'} \circ \nabla \colon C^\infty(\mathbb{E}_{\sigma,\nu}) \longrightarrow C^\infty(\mathbb{E}_{\sigma',\nu'}) \\ \widetilde{D} &= \mathrm{proj}_{\gamma'} \circ \widetilde{\nabla} \colon C^\infty(\mathbb{V}_\gamma) \longrightarrow C^\infty(\mathbb{V}_{\gamma'}) \end{aligned}$$

where $\nu' = \nu + \alpha$, α simple, and (as in the previous section, same assumptions in force)

$$2\langle \alpha, \nu \rangle = C(E_{\sigma'}) - C(\mathfrak{n}_\alpha) - C(E_\sigma).$$

Then (up to a normalizing constant) we have the commuting diagram of G-equivariant maps, i.e.

$$\mathcal{P} \circ D = \widetilde{D} \circ \mathcal{P}.$$

Proof. — Let us first consider right N-invariant smooth functions $f\colon G\to E_\sigma$, i.e.

$$f(gn)=f(g)\quad (g\in G, n\in N).$$

On such a function

$$\begin{aligned}
\mathcal{P}\nabla_X f(g) &= \int_K (\gamma(k)\otimes \mathrm{Ad}^*(k))(I\otimes I_\theta)\frac{d}{dt}f(gk\exp tX)|_{t=0}\,dk\\
&= \frac{d}{dt}\int_K (\gamma(k)\otimes \mathrm{Ad}^*(k))If(g\exp t(\mathrm{Ad}(k)X-\theta\mathrm{Ad}(k)X)k)\,dk|_{t=0}\\
&= \frac{d}{dt}\int_K \gamma(k)If(g\exp t(X-\theta X)k)\,dk|_{t=0}\\
&= \widetilde{\nabla}_{X-\theta X}\mathcal{P}f(g)
\end{aligned}$$

where $X\in\bar{\mathfrak{n}}$ and $\mathrm{Ad}^*(k)=\mathrm{Ad}(k^{-1})^*$ denotes the coadjoint action of K on $\mathfrak{s}^*$ (and on $\mathfrak{g}^*$); note that for $\beta\in\mathfrak{g}^*$ and $Y\in\mathfrak{g}$ this means that $\langle \mathrm{Ad}^*(k)\beta, Y\rangle=\langle\beta,\mathrm{Ad}(k^{-1})Y\rangle$ for all $k\in K$. Hence with the M-equivariant $I_\theta\colon\mathfrak{n}\to\mathfrak{s}^*$ we have

$$\mathcal{P}\nabla=\widetilde{\nabla}\mathcal{P}$$

with I_θ built in the $\mathcal{P}$ on the left-hand side and the connections are on G/P respectively G/K; also the $\mathcal{P}$ on the right-hand side is the original corresponding to $I\colon E_\sigma\to V_\gamma$. Thus we have

$$\begin{array}{ccc}
C^\infty(G\times_{MN}E_\sigma) & \xrightarrow{\ \nabla\ } & C^\infty(G\times_{MN}(E_\sigma\otimes\mathfrak{n}))\\
{\scriptstyle\mathcal{P}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{P}}\\
C^\infty(G\times_K V_\gamma) & \xrightarrow[\ \widetilde{\nabla}\ ]{} & C^\infty(G\times_K(V_\gamma\otimes\mathfrak{s}^*))
\end{array}$$

as a commuting diagram where we have also built in the M-equivariance, e.g.

$$\begin{aligned}
&C^\infty(G\times_{MN}E_\sigma)\\
&\qquad=\big\{f\colon G\to E_\sigma \mid f\in C^\infty,\ f(gmn)=\sigma(m)^{-1}f(g)\ (g\in G, m\in M, n\in N)\big\}
\end{aligned}$$

still having the diagram, since all maps are M-equivariant. Now we restrict to sections of $\mathbb{E}_{\sigma,\nu}$ and compose with the projections as in (3.1); then we obtain the commuting diagram

$$\begin{array}{ccc}
C^\infty(\mathbb{E}_{\sigma,\nu}\otimes\mathbb{T}^*) & \xrightarrow{\ \mathrm{proj}_{\sigma'}\ } & C^\infty(\mathbb{E}_{\sigma',\nu'})\\
\big\downarrow{\scriptstyle\mathcal{P}} & & \big\downarrow{\scriptstyle\mathcal{P}}\\
C^\infty(\mathbb{V}_\gamma\otimes\mathbb{T}^*) & \xrightarrow[\ \mathrm{proj}_{\gamma'}\ ]{} & C^\infty(\mathbb{V}_{\gamma'})
\end{array}$$

with the pointwise projections

$$\begin{aligned}
\mathrm{proj}_{\sigma'} &:\quad E_\sigma\otimes\mathfrak{n}\longrightarrow E_\sigma\otimes\mathfrak{n}_\alpha\longrightarrow E_{\sigma'}\\
\mathrm{proj}_{\gamma'} &:\quad V_\gamma\otimes\mathfrak{s}^*\longrightarrow V_{\gamma'}.
\end{aligned}$$

Taking compositions of these two last diagrams we obtain the horizontal operators

$$D = \mathrm{proj}_{\sigma'} \circ \nabla, \qquad \widetilde{D} = \mathrm{proj}_{\gamma'} \circ \widetilde{\nabla}$$

over G/P resp. G/K, and they satisfy $\mathcal{P} \circ D = \widetilde{D} \circ \mathcal{P}$. □

4. Examples

For the group $G = \mathrm{SL}(3, \mathbb{R})^{\sim}$, the double (universal) covering of $\mathrm{SL}(3, \mathbb{R})$ we can illustrate the use of a generalized gradient D in constructing an exceptional unitary irreducible representation π of G. D will be a kind of "symplectic Dirac operator" on $S^2 = G/P$ where $P = MAN$ is a maximal parabolic with $M \cong \mathrm{SL}(2, \mathbb{R})^{\sim}$, the double covering of $\mathrm{SL}(2, \mathbb{R})$, i.e. the metaplectic group, $A \cong \mathbb{R}^+$, $N \cong \mathbb{R}^2$. Let $\pi_{1/2} \oplus \pi_{3/2}$ be the metaplectic representation of M with $\pi_{1/2}$ the even and $\pi_{3/2}$ the odd part (in this example we illustrate the fact that the inducing representation E_σ may be infinite-dimensional — the important property is that it has an infinitesimal character, the arguments are the same as before). Note that $\mathfrak{n} \cong \mathbb{R}^2$, and the extension of the metaplectic representation from M to the semi-direct product $H \times_s M$, where H is the Heisenberg group, means that we have an M-equivariant map

$$\pi_{1/2} \otimes \mathbb{R}^2 \longrightarrow \pi_{3/2}$$

given by

$$\varphi \otimes e \longrightarrow d\pi_{1/2}(e)\varphi.$$

The corresponding gradient

$$D\colon \mathrm{Ind}_{MAN}^{G}(\pi_{1/2} \otimes e^{\nu} \otimes I) \longrightarrow \mathrm{Ind}_{MAN}^{G}(\pi_{3/2} \otimes e^{\nu+\alpha} \otimes I)$$

will in the $\bar{N}$-picture of the induced representation be

$$D = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y}$$

where $\gamma_x = d\pi_{1/2}(e_1)$, $\gamma_y = d\pi_{1/2}(e_2)$ are the "Dirac gamma-matrices" and e_1, e_2 the canonical basis of $\mathbb{R}^2 \cong \overline{N}$. If we take the Schrödinger model we have

$$\gamma_x = \frac{1}{i}\frac{\partial}{\partial t}, \quad \gamma_y = t$$

acting in $L^2(\mathbb{R})$. A direct calculation gives that the kernel of D (at the value of ν as in the theorem) consists of the K-types

$$\ker D = \sum_{j=\frac{1}{2}, \frac{5}{2}, \dots}^{\oplus} V_j$$

where V_j is the $(2j+1)$-dimensional irreducible representation of $K = \mathrm{SU}(2)$. But then (we thank D. Vogan for this observation) $\ker D$ is a unitary irreducible representation equivalent with the exceptional representation π constructed by P. Torasso, associated with the minimal coadjoint orbit, see [**9**]. The point is that both the K-types and the infinitesimal characters agree, and that the representations are "small" in a certain sense. Hence our gradient provides an interesting part of the composition series for the induced representation, and it would be nice to see also the unitary structure in terms of D. We may conjecture that these gradients will fulfil a similar role in other interesting situations as well — note for example the case of unitary highest weight modules, see [**3**], where the relevant gradient corresponds to the so-called PRW-component.

Let us also illustrate the Poisson transform and the extension of gradients in the case of $G = \mathrm{SO}_0(n+1,1)$ where $G/P = S^n$ and $G/K = H^{n+1}$, the hyperbolic ball. Consider the defining representations $\mathbb{R}^n$ of $M = \mathrm{SO}(n)$ and $\mathbb{R}^{n+1}$ of $K = \mathrm{SO}(n+1)$, with the obvious embedding $I \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$. This also defines an embedding

$$\mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$$

which respects the decompositions

$$\mathbb{R}^n \otimes \mathbb{R}^n = \mathbb{R}^n \wedge \mathbb{R}^n \oplus [\mathbb{R}^n \otimes_s \mathbb{R}^n]_0 \oplus \mathbb{R}$$
$$\mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \wedge \mathbb{R}^{n+1} \oplus [\mathbb{R}^{n+1} \otimes_s \mathbb{R}^{n+1}]_0 \oplus \mathbb{R}$$

where $[\mathbb{R}^n \otimes_s \mathbb{R}^n]_0$ denotes the trace-free symmetric tensors. The projection on this part gives the gradient $X \to SX$ acting on vector fields X on S^n, this is exactly the Ahlfors operator, and similarly the gradient $Y \to \widetilde{S}Y$, the Ahlfors operator on H^{n+1}. Now our theorem amounts to the relation

$$\mathcal{P}SX = \widetilde{S}\mathcal{P}X$$

for all vector fields on S^n, where $\mathcal{P}$ on the right-hand side is the Poisson transform on vector fields, corresponding to

$$\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$$

and $\mathcal{P}$ on the left-hand side is the Poisson transform on trace-free symmetric two-tensors, corresponding to

$$[\mathbb{R}^n \otimes_s \mathbb{R}^n]_0 \longrightarrow [\mathbb{R}^{n+1} \otimes_s \mathbb{R}^{n+1}]_0.$$

In particular, if X is conformal, i.e. $SX = 0$, then so is $Y = \mathcal{P}X$ i.e. $\widetilde{S}Y = 0$. Furthermore (the point in [**6**]) if X is quasi-conformal, i.e. we have an estimate on the size of SX, then so is the extended vector field $Y = \mathcal{P}X$. Again, we may hope that our result can be applied in other similar geometric situations — see [**6**] for references to the case of CR geometry.

Finally our gradients could be useful in discussing the non-injectivity of Poisson transforms, and also provide composition series for vector bundles over G/K defined by invariant differential operators.

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