Quantum Hyberboloid and Braided Modules

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Abstract
We construct a representation theory of a "quantum hyperboloid" in terms of so-called braided modules. We treat these objects in the framework of twisted Quantum Mechanics.

Résumé
Nous construisons une théorie de représentations pour « l'hyperboloïde quantique » en termes de modules tressés. Nous traitons ces objets dans le cadre de la mécanique quantique tordue.

1 Introduction

In the present paper we study a quantum hyperboloid from the point of view of the generalized framework for quantum mechanics suggested in [GRZ]. The main idea of that paper is the following. Quantizing a degenerate Poisson bracket we have, in general, to modify the ordinary notions of quantum mechanics, namely, those of Lie algebra, trace and conjugation (involution) operators.

Meanwhile, all objects and operators disscused in [GRZ] were connected to an involutive $S^2 = id$ solution to the quantum Yang-Baxter equation (QYBE)

$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}.$$

In particular, such objects arise as a result of a quantization of some Poisson brackets (P.b.) generated by a skew-symmetric ($R \in \wedge^2(g)$) solution to the classical Yang-Baxter equation (CYBE),

$$[[R, R]] = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0,$$

where g is a Lie algebra. Another family of examples of such a type of objects is related to non-quasiclassical (or, in other words, non-deformational) solutions of the QYBE, cf. [G1], [GRZ].

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More precisely, given a representation $\rho : g \to Vect(M)$ of a Lie algebra g in the space of vector fields on a manifold or algebraic variety M, then the bracket

$$\{f,g\}_R = \mu < \rho^{\otimes 2}(R), df \otimes dg >, \ f,g \in Fun(M)$$

is Poisson. Hereafter μ denotes the product in the algebra under question and $< \, , \, >$ denotes the pairing between the vector fields and the differential forms extended on their tensor powers. Quantizing this Poisson bracket, we get an algebra belonging to a twisted, i.e., equipped with a Yang-Baxter twist S, tensor symmetric category ("symmetric" means that this twist is involutive). Moreover, this algebra is S-commutative, i.e., the product μ in it satisfies the relation $\mu = \mu S$.

Thus, by deforming the commutative algebra $Fun(M)$ "in the direction" of the above P.b. we get a S-commutative algebra. It is more interesting to deform in a similar way the non-commutative algebras, for example, those arising from a quantization of the Kirillov-Kostant-Souriau (KKS) bracket on a given coadjoint orbit in g^*.

Let us assume that $\rho = ad^*$. Then the bracket $\{ \, , \, \}_R$ is well defined on g^* as well as on any orbit in g^*. It is not difficult to see that the KKS bracket and this "R-matrix bracket" $\{ \, , \, \}_R$ are compatible and this problem can be thought of as one of simultaneous quantization of the whole of the Poisson pencil (P.p.) generated by these two brackets.

In this connection the following question arises: what is a quantization of the KKS bracket? There exist (at least) two ways to represent the quantum objects. On the one hand it is possible to think of these objects as the quotient algebras of the enveloping algebras $U(g)_h$ (h means here that this parameter is introduced as a factor in the Lie bracket in the definition of the enveloping algebra).

On the other hand the quantum object can be represented into $End(V)$ where V is a suitable Hilbert space. Such a representation can be constructed by means of a geometric quantization method or by means of an orbit method, but in numerous cases both approaches provide similar results.

We treat the algebra structure arising from the quantization of the KKS bracket in one or in other way, and we are interested in a further deformation of this algebra. In what follows the latter procedure will be called "twisting" to distinguish the two types of quantization. Roughly speaking, a twisting is a passage to a twisted category instead of the "classical" one. When a twisting arises from the above mentioned solutions of the CYBE, it can be performed by means of an operator $F = F_\nu$ (represented as a series in a parameter ν) such that $S = F^{-1}\sigma F$ where σ denotes the flip. Existence of such a series F has been established by V.Drinfeld in [D].

As a result, the principal objects and operators of the ordinary quantum mechanics can be twisted by means of F_ν. In particular, a usual trace becomes

S-commutative, i.e., such that $tr(A \circ B) = tr \circ S(A \otimes B)$ where $\circ$ denotes the operator product. A Lie bracket turns into an S-Lie bracket in the sense of [G1], [GRZ]. etc.

Our principal aim is to generalize this approach to the case when R is a solution of the modified CYBE. This means that the above element $[[R, R]]$ is g-invariant. In this case the R-matrix bracket is Poisson only on certain orbits in g^* which are called, according to the terminology of [GP], *the orbits of R-matrix type.* However, if $g = sl(2)$, all orbits in g^* are of the R-matrix type.

The result of the quantization of the above P.p. on a given orbit in g^* can be represented as a three parameter algebra $A^c_{h,q}$ where h is a parameter of quantization of the KKS bracket, q a parameter of twisting and c labels the orbits. $c = 0$ corresponds to the cone.

The algebras of such type have been considered in plenty of papers. We refer the reader to [P] where these algebras (equipped with a traditional involution) have appeared under the name of "quantum spheres" (see the discussion of involutions in Section 5).

It was shown in [DG1] that these algebras represent flat deformations of their classical counterparts. In this paper we realize the second step of the quantization procedure and develop a representation theory for the algebras $A^c_{h,q}$ in terms of *braided modules.*

Roughly speaking, a braided module is a $U_q(g)$-module equipped with a representation $\rho : A^c_{h,q} \to End(V)$ in such a way that the map ρ is a $U_q(g)$-morphism.

In this sense we treat the triple $(A^c_{h,q}, V, \rho)$ as an object of twisted quantum mechanics (more precisely, of the particular case, connected to the quantum group $U_q(g)$). In the present paper we consider the simplest example of such twisted quantum mechanics, namely, the one connected to the quantum hyperboloid and its modules.

Although an axiomatic approach to such a version of quantum mechanics has not yet been adequately developed, it is clear that the traditional involution approach is not reasonable for such a type of objects, since the maps of these algebras into $End(U_k)$, where U_k are the braided modules mentioned above, do not respect such an involution. In the present paper we suggest another way to coordinate the involution with a braided structure.

The paper is organized as follows. In the next section we recall the constructions of [DG1]. In Section 3 we develop a representation theory for this algebra in terms of braided modules. In Section 4 we consider the so-called braided Casimir, i.e., an invariant (with respect to the action of the quantum group) element and assign to it operators acting in braided modules. We prove that the latter operators are scalar, and we compute the eigenvalues of the braided Casimir. The last section is devoted to a discussion of the braided (twisted) traces and involutions as ingredients of twisted quantum mechanics.

Throughout the paper $U_q(g)$-Mod will denote the category of $U_q(g)$-modules. We include in it, besides the finite-dimensional modules, their inductive limits. The parameter q is assumed to be generic, and the basic field k is C or R (in the latter case we consider the normal form of the Lie algebra g).

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2 Quantum hyperboloid: basic notions

To construct a quantum hyperboloid it is sufficient to fix a representation of the quantum group $U_q(sl(2))$ into a three dimensional space V, decompose the space $V^{\otimes 2}$ into a direct sum of irreducible $U_q(sl(2))$-modules and impose a few natural equations on elements of $V^{\otimes 2} \oplus V \oplus k$ which are compatible with the action of the quantum group $U_q(sl(2))$ and are similar to their classical counterparts.

Thus, let us consider the algebra $U_q(sl(2))$ generated by the elements H, X, Y satisfying the well-known relations

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

Let us equip this algebra with a coproduct defined on the basic elements in the following way

$$\Delta(X) = X \otimes 1 + q^{-H} \otimes X,\ \Delta(Y) = 1 \otimes Y + Y \otimes q^H,\ \Delta(H) = H \otimes 1 + 1 \otimes H.$$

It is well-known that this algebra has a Hopf structure, being equipped with the antipode γ defined by

$$\gamma(X) = -q^H X,\ \gamma(H) = -H,\ \gamma(Y) = -Y q^{-H}.$$

Let us consider a linear space V with the base $\{u, v, w\}$, and turn V into a $U_q(sl(2))$-module by setting

$$Hu = 2u,\ Hv = 0,\ Hw = -2w,\ Xu = 0,\ Xv = -(q + q^{-1})u,\ Xw = v,$$

$$Yu = -v,\ Yv = (q + q^{-1})w,\ Yw = 0.$$

It is easy to show that the above relations for H, X, Y are satisfied.

We want to stress that throughout this paper we deal with a coordinate representation of module elements. We consider the endomorphisms as matrices and their action as left-multiplication by these matrices.

Using the coproduct we can equip $V^{\otimes 2}$ with a $U_q(sl(2))$-module structure as well. This module is reducible and can be decomposed into a direct sum of three irreducible $U_q(sl(2))$-modules

$$V_0 = \mathrm{span}((q^3+q)uw + v^2 + (q+q^{-1})wu),$$

$$V_1 = \mathrm{span}(q^2uv - vu,\ (q^3+q)(uw-wu) + (1-q^2)v^2,\ -q^2vw + wv),$$

$$V_2 = \mathrm{span}(uu,\ uv + q^2vu,\ uw - qvv + q^4wu,\ vw + q^2wv,\ ww)$$

of spins 0, 1 and 2 (hereafter the sign $\otimes$ is omitted).

Then only the following relations imposed on the elements of the space $V^{\otimes 2} \oplus V \oplus k$ are compatible with the $U_q(sl(2))$-action:

$$C_q = (q^3+q)uw + vv + (q+q^{-1})wu = c,\ q^2uv - vu = -2hu,$$

$$(q^3+q)(uw-wu) + (1-q^2)v^2 = 2hv,\ -q^2vw + wv = 2hw$$

with arbitrary h and c. The element C_q will be called a *braided Casimir*.

Therefore it is natural to introduce a *quantum hyperboloid* as the quotient algebra of the free tensor algebra $T(V)$ over the ideal generated by elements

$$(q^3+q)uw + v^2 + (q+q^{-1})wu - c,\ q^2uv - vu + 2hu,$$

$$(q^3+q)(uw-wu) + (1-q^2)v^2 - 2hv,\ -q^2vw + wv - 2hw.$$

This quotient algebra will be denoted by $A^c_{h,q}$.

The quotient algebra of $T(V)$ over the ideal generated by the latter three elements will be denoted by $A_{h,q}$. This algebra is another (compared with the quantum algebra $U_q(sl(2))$) q-analogue of the enveloping algebra $U(sl(2))$.

In [DG2] it has been shown that both algebras $A^c_{h,q}$ and $A_{h,q}$ represent the flat deformations of their classical counterparts. Let us make some comments on the proof.

Concerning the algebra $A^c_{h,q}$, the proof of flatness is based on the two following statements:

1. The algebra $A^0_{0,q}$ is Koszul (see [BG] for definition). This fact was proved in [DG1] "by hands". Now there exists (for the case $q = 1$ and hence for a generic q since the deformation $A^0_{0,1} \to A^0_{0,q}$ is flat) a more conceptual proof valid for any simple Lie algebra (see [Be], [Bo]).

2. It is possible to describe the algebra $A^c_{h,q}$ as the enveloping algebra of a generalized Lie algebra in the following sense which is slightly different from that of [DG2]. Let us consider the space $I = V_1 \oplus V_0$ and introduce two maps, $\alpha : I \to V$ and $\beta : I \to k$, as follows: $\alpha : V_0 \to 0$, $\beta : V_1 \to 0$,

$$\alpha(q^2uv - vu) = -2hu,\ \alpha((q^3+q)(uw-wu)+(1-q^2)v^2) = 2hv,$$

$$\alpha(-q^2vw + wv) = 2hw,\ \beta((q^3+q)uw + v^2 + (q+q^{-1})wu) = c.$$

It is easy to see that the data $(V,\, I \subset V^{\otimes 2},\, \alpha,\, \beta)$ satisfies the following relations
a. $Im(\alpha \otimes id - id \otimes \alpha)(I \otimes V \bigcap V \otimes I) \subset I$,
b. $(\alpha(\alpha \otimes id - id \otimes \alpha) + \beta \otimes id - id \otimes \beta)(I \otimes V \bigcap V \otimes I) = 0$,
c. $\beta(\alpha \otimes id - id \otimes \alpha)(I \otimes V \bigcap V \otimes I) = 0$.

Then, in virtue of the main result of [BG], we can deduce that its graded adjoint algebra $Gr\, A^c_{h,q}$ is isomorphic to $A^0_{0,q}$, in spirit of the PBW theorem.

Let us remark that the above conditions a., b., c. represent the most general analogue of the Jacobi identity related to deformation theory. However, they are useless from the representation theory point of view. On the contrary, the Jacobi identity presented in the next Section is related to representation theory of the algebra $A_{h,q}$.

As for the algebra $A_{h,q}$ itself, the proof of its flatness follows the same outline. The only difference is that the space V_1 plays the role of I, and only the map $\alpha : I \to V$ is considered. Then the result of [BG] mentioned above can be applied, assuming β to be equal zero, since the algebra $Gr\, A_{h,q}$ is also Koszul (cf. [DG1]).

It is not difficult to see that the quasiclassical terms of these flat deformations are just the Poisson pencils under consideration, where the R-matrix bracket is given by $R = {}^1\!/_2 X \wedge Y$.

Remark 2.1. For other simple Lie algebras g, it is possible to define algebras that are analogous to $A_{h,q}$ (cf. [DG2]). However, these algebras are no longer flat deformations of their classical counterparts. The only hope is to prove that some of their quotient algebras are flat deformations of their classical analogues which correspond to the orbits in g^* of R-matrix type according to the terminology of [GP].

Let us consider more closely the algebra $A^c_{0,q}$, arising from quantization of the R-matrix bracket. It possesses a particular property: it is a commutative algebra in the category $U_q(sl(2))$-Mod in the following sense.

The category $U_q(sl(2))$-Mod is balanced, cf. [CP], page 154. This yields the fact that for any two finite dimensional objects, U_1, U_2, of this category, there exists an involutive $U_q(sl(2))$-morphism $\widetilde{S} : U_1 \otimes U_2 \to U_2 \otimes U_1$ which is a deformation of the flip. For example, if $U_1 = U_2 = V$, then $\widetilde{S} = id \quad (resp. \widetilde{S} = -id)$ restricted to $V_0 \oplus V_2 \quad (resp. V_1)$. Decomposing the algebra $A^c_{h,q}$ into a direct sum of finite dimensional $U_q(sl(2))$-modules, we can define the operator $\widetilde{S} : (A^c_{h,q})^{\otimes 2} \to (A^c_{h,q})^{\otimes 2}$. Then the product operator μ in this algebra satisfies the relation $\mu\widetilde{S} = \mu$.

This property is established in fact in [DS] where a formal deformational quantization of the R-matrix bracket is represented. It only remains to show that our quantization is equivalent to that constructed in [DS] (the details are left to the reader).

Let us remark that although the algebra $A^c_{h,q}$ belongs to the category $U_q(sl(2))$-Mod for any h, the case $h = 0$ is an exceptional point from the representation theory point of view, as in the classical case. It is well known that the representation theory of the algebra $U(g)_h$ is different for the case $h = 0$ and $h \neq 0$. In the next sections, we shall disregard the case $h = 0$.

3 q-Lie bracket and braided modules

We will recall first of all the construction of braided (or q-)Lie bracket introduced in [DG1] for the $sl(2)$-case and in [DG2] for other simple Lie algebras.

Attempts to find a proper definition of Lie algebra-like objects connected to the quantum groups $U_q(g)$ or, more generally, to non-involutive solutions to the QYBE are known since the creation of quantum group theory. We shall disregard here all these approaches. We only want to remark that one usually looks for such a type of object as a subset in $U_q(g)$ itself. On the contrary, we do not need any quantum group. We only use it to define the objects and morphisms of the category, but it is possible, to define them in another way without using $U_q(g)$.

Roughly speaking, we define a q-Lie bracket as a $U_q(g)$-morphism $V^{\otimes 2} \to V,\ V = g$ deforming the usual Lie bracket. As a first step we equip the Lie agebra g with a structure of a $U_q(g)$-module, which is a deformation of g-module structure with respect to the adjoint action. Then the q-Lie bracket is defined in a unique way (up to a constant) if $g \neq sl(n),\ n > 2$. (For the $sl(n)$-case, see [DG2].)

In the $sl(2)$-case, the q-Lie bracket is defined in the following way. We set $[\ ,\ ] = \alpha$ on I and $[\ ,\ ] = 0$ on V_2. Thus, the bracket $[\ ,\ ]$ is well defined on the whole $V^{\otimes 2}$ and it is a $U_q(sl(2))$-morphism. It is evident that for $q = 1$ we get the ordinary $sl(2)$ bracket, up to a factor.

In [DG1] a multiplication table for this bracket has been calculated in the base $\{u,\ v,\ w\}$. Let us reproduce the result

$$[u,u] = 0,\ [u,v] = -q^2Mu,\ [u,w] = (q+q^{-1})^{-1}Mv,$$

$$[v,u] = Mu,\ [v,v] = (1-q^2)Mv,\ [v,w] = -q^2Mw,$$

$$[w,u]=-(q+q^{-1})^{-1}Mv,\ [w,v]=Mw,\ [w,w]=0,$$

where $M=2h(1+q^4)^{-1}$.

The space V equipped with this bracket will be called a *braided (or q-) Lie algebra* and is denoted by $\overline{sl(2)}_M$. The classical Lie algebra $sl(2)$ corresponds to $q=1$ and $M=2$.

A natural question arises: what is a reasonable definition of an enveloping algebra for it? Or, in other words, what is a suitable choice of a factor τ if we define the enveloping algebra of $\overline{sl(2)}_M$ as a quotient algebra,

$$T(V)/\{q^2uv-vu+\tau u,\ (q^3+q)(uw-wu)+(1-q^2)v^2-\tau v,\ -q^2vw+wv-\tau w\}.$$

Definition 3.1 — *We say that this quotient is enveloping algebra of the braided Lie algebra given by the above multiplication table if the left adjoint operator, $\rho(x)z=[x,z]$, defines a representation of this quotient algebra, i.e., the following relations are satisfied:*

$$q^2\rho(u)\rho(v)-\rho(v)\rho(u)=-\tau\rho(u),\ (q^3+q)(\rho(u)\rho(w)-\rho(w)\rho(u))+$$

$$(1-q^2)\rho(v)^2=\tau\rho(v),\ -q^2\rho(v)\rho(w)+\rho(w)\rho(v)=\tau\rho(w)$$

The envelopping algebra of the braided Lie algebra $\overline{sl(2)}_M$ will be denoted by $U(\overline{sl(2)}_M)$.

It is easy to find this value of the parameter: $\tau=M(1-q^2+q^4)$. Let us remark that, in the classical case, $q=1$, $\tau=M$.

Let us note that the algebra $U(\overline{sl(2)}_M)$ coincides in fact with $A_{h,q}$ when the parameter h is replaced by $\tau/2$.

Definition 3.2 — *We say that a map $\rho: V\to End(U)$ where U is a $U_q(sl(2))$-module is an almost representation of the q-Lie algebra $\overline{sl(2)}_M$ if it is a $U_q(sl(2))$-morphism and there exists a factor $\nu\neq 0$ such that*

$$q^2\rho(u)\rho(v)-\rho(v)\rho(u)=\nu(-\rho(u)),\ (q^3+q)(\rho(u)\rho(w)-\rho(w)\rho(u))+$$

$$(1-q^2)\rho(v)^2=\nu\rho(v),-q^2\rho(v)\rho(w)+\rho(w)\rho(v)=\nu\rho(w)$$

An almost representation will be called a representation if $\nu=\tau$.

Thus, by the above construction, we obtain at least one representation of the braided Lie algebra $\overline{sl(2)}_M$, namely, the adjoint one. Let us represent the above relations from Definition 3.1 in the following form

$$q^2[u,[v,z]]-[v,[u,z]]=-\tau[u,z],\ (q^3+q)([u,[w,z]]-[w,[u,z]]+$$

$$(1-q^2)[v,[v,z]]=\tau[v,z],\ -q^2[v,[w,z]]+[w,[v,z]]=\tau[w,z].$$

This is another q-analogue of the Jacobi identity which is valid for the braided Lie algebra $\overline{sl(2)}_M$. However, unlike the above form of Jacobi identity related to a deformation theory the latter one is connected to a representation theory of the braided Lie algebra $\overline{sl(2)}_M$. A similar form of Jacobi identity for "braided counterparts" of other simple Lie algebras is discussed in [G2].

Now we will describe a method to construct the other representations of $\overline{sl(2)}_M$. In the classical case, if we have a representation of a Lie algebra then by means of Leibniz rule we can construct a series of other ones (namely, whose spins are multiples of the spin of the initial module). In the q-case there exists a "truncated version" of the Leibniz rule, which is discussed in [G2]. Here we want to discuss another way of constructing all spin representations of the braided Lie algebra $\overline{sl(2)}_M$.

This way is based on the following observation: if we have an almost representation ρ of the braided Lie algebra $\overline{sl(2)}_M$ with the factor ν, then by rescaling, i.e., passing to the map $\tau\nu^{-1}\rho$, we get a representation of the q-Lie algebra under question. Thus, it suffices for us to construct almost representations of all spins.

Let us fix a spin k-irreducible $U_q(sl(2))$-module, $U = U_k$, and consider the space $End(U)$ of endomorphisms of U as an $U_q(sl(2))$-module. This means that if $\rho : U_q(sl(2)) \to End(U)$ is a representation of the quantum group $U_q(sl(2))$, then $\rho_{End} : U_q(sl(2)) \to End(End(U))$ is defined as follows:

$$\rho_{End}(a)M = \rho(a_1) \circ M \circ \rho(\gamma(a_2)),\ a \in U_q(sl(2)), M \in End(U),$$

where $\circ$ denotes the matrix product, γ is the antipode in $U_q(sl(2))$ and $a_1 \otimes a_2$ is the Sweedler's notation for $\Delta(a)$.

Let us remark that this structure of $U_q(sl(2))$-module on $End(V)$ is compatible with the matrix product, i.e.,

$$\rho_{End}(a)(M_1 \circ M_2) = \rho_{End}(a_1)M_1 \circ \rho_{End}(a_2)M_2.$$

Let us give the explicit form of the representation ρ_{End}:

$$\rho_{End}(X)M = \rho(X) \circ M - \rho(q^{-H}) \circ M \circ \rho(q^H) \circ \rho(X),$$

$$\rho_{End}(H)M = \rho(H) \circ M - M \circ \rho(H),\ \rho_{End}(Y)M = (\rho(Y) \circ M - M \circ \rho(Y)) \circ \rho(q^{-H}).$$

Let us decompose the $U_q(sl(2))$-module $End(U)$ into a direct sum of irreducible $U_q(sl(2))$-modules. It is evident that, for any spin k in this decomposition, there is a unique module isomorphic to V.

Let us define in a natural way a $U_q(sl(2))$-morphism $\overline{\rho} : V \to End(U)$ sending V to the mentioned component of $End(U)$. This morphism is defined up to a factor.

Proposition 3.3 — *The map $\overline{\rho}$ is an almost representation (for generic q).*

Proof. By construction, $\overline{\rho}$ is a $U_q(sl(2))$-morphism. It is evident that the elements

$$q^2\overline{\rho}(u)\overline{\rho}(v) - \overline{\rho}(v)\overline{\rho}(u),\ (q^3+q)(\overline{\rho}(u)\overline{\rho}(w) - \overline{\rho}(w)\overline{\rho}(u)) + (1-q^2)\overline{\rho}(v)^2,$$

$$-q^2\overline{\rho}(v)\overline{\rho}(w) + \overline{\rho}(w)\overline{\rho}(v) \in End(U)$$

generate a $U_q(sl(2))$-module isomorphic to V and therefore that they coincide respectively with $-\overline{\rho}(u)$, $\overline{\rho}(v)$, $\overline{\rho}(w)$, up to a factor, since the component of $End(V)$ isomorphic to V is unique. This factor is non-trivial for generic q since it is so for $q=1$. This completes the proof.

Let us consider two examples. The map

$$\rho(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \rho(v) = \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix},\ \rho(w) = \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix}$$

is a spin 1/2 almost representation of $\overline{sl(2)}_M$. It becomes a representation if we multiply it by the factor $M(1-q^2+q^4)(q^3+q^{-1})^{-1}$. A spin 1-representation is given by the following matrices

$$(q+q^{-1})^{-1}M \begin{pmatrix} 0 & q^2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix},\ M \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is interesting to compare the spin 1/2-representation of the quantum hyperboloid and that of the quantum group $U_q(sl(2))$. Both algebras are represented into the same space, but the former algebra is represented by the above matrices and the latter one by the same matrices as the Lie algebra $sl(2)$. □

4 Braided Casimir

Now, once we have constructed the representations of the algebra $U_q(sl(2))$, we can assign to any element $a \in U(\overline{sl(2)}_M)$ its image with respect to a given representation. We are interested in the distinguished element C_q. We call this element *braided Casimir*.

It differs from the so-called quantum Casimir which belongs to the quantum group $U_q(sl(2))$. In some sense the braided Casimir looks rather like a super-Casimir.

In this section we will generalize to the braided case the well-known property of the ordinary Casimir element showing that its image is a scalar operator in any irreducible $sl(2)$-module, and computing the corresponding values of the braided Casimir.

Proposition 4.1 — *Let $\overline{\rho} = \overline{\rho}_k : A_{h,q} \to End(U_k)$ be the spin k representation of the algebra $A_{h,q}$ in module U_k.*

Then the image $\overline{\rho}(C_q)$ of the braided Casimir is a scalar, nontrivial operator (let us recall that q is generic).

Proof. Since $\overline{\rho}$ is a $U_q(sl(2))$-morphism and C_q generates the trivial $U_q(sl(2))$-module,

$$\overline{\rho}(\rho(a)C_q)) = \rho_{End}(a)\overline{\rho}(C_q) = 0, \;\; a \in U_q(sl(2)),$$

where $\rho : U_q(sl(2)) \to End(U_k)$ is a representation of the quantum group $U_q(sl(2))$. For generic q, the elements $\rho_{End}(a)$, $a \in U_q(sl(2))$, generate the algebra $End(U_k)$. Using the above explicit form of the representation ρ_{End} it is easy to see that $\overline{\rho}(C_q)$ commutes with all elements of $End(U_k)$. This yields the conclusion of the proposition.

It is well-known that $dim\, U_k = l + 1$, where $l = 2k$. Let us introduce some notations.

We denote by $diag(a_1, a_2, ..., a_{l+1})$ the diagonal matrices, by $diag_+(a_1, a_2, ..., a_l)$ the matrices with main overdiagonal $(a_1, a_2, ..., a_l)$ and by $diag_-(a_1, a_2, ..., a_l)$ the matrices with main subdiagonal $(a_1, a_2, ..., a_l)$.

Let us fix the base in the $U_q(sl(2))$-module U_k such that the corresponding representation, $\rho = \rho_k : U_q(sl(2)) \to End(U_k)$, is of the form

$$\rho(X) = diag_+(1, 1, ..., 1), \; \rho(H) = diag(l, l-2, ..., -l),$$

$$\rho(Y) = diag_-(y_1, y_2, ..., y_l),$$

where y_i can be found by solving the following system,

$$y_1 = b_l, \; y_2 - y_1 = b_{l-2}, ..., y_l - y_{l-1} = b_{-l+2}, \; -y_l = b_{-l}, \; b_i = (q^i - q^{-i})(q - q^{-1})^{-1}.$$

It is easy to show that the matrix $U = diag_+(q^{2(l-1)}, q^{2(l-2)}, ..., 1)$ satisfies the following conditions: $\rho_{End}(X)U = 0$ and $\rho_{End}(H)U = 2U$. Let us consider two matrices V and W such that $-V = \rho_{End}(Y)U$, $(q + q^{-1})W = \rho_{End}(Y)U$. One can see that $V = diag(v_1, v_2, ..., v_{l+1})$ and $W = diag_-(w_1, ..., w_l)$.

Using the explicit form of the representation ρ_{End} given above, it is possible to find the values of all v_i and w_i, but we need only those of v_1 and v_2. We obtain $v_1 = y_1 q^{l-2}, \; v_2 = y_2 q^{l-2} - y_1 q^l$.

Let us consider the map $u \to U, v \to V, w \to W$. It defines an almost representation. Therefore the relations

$$q^2UV - VU = -\theta U, \; (q^3 + q)(UW - WU) + (1 - q^2)V^2 = \theta V,$$

$$-q^2WV - WV = \theta W$$

are satisfied for some θ. Let us compute this factor.

Substituting U and V in the first relation and computing the first non-trivial matrix element, we obtain

$$\theta = v_1 - q^2 v_2 = y_1 q^{l-2} - q^2(y_2 q^{l-2} - y_1 q^l) = y_1(q^{l+2} + q^{l-2}) - y_2 q^l = q^{2l+1} + q^{-1}.$$

By the same argument we obtain $(q^3+q)u_1 w_1 + (1-q^2)v_1^2 = \theta\, v_1$, using the second relation.

Therefore the first matrix element of the scalar operator $(q^3+q)UW + V^2 + (q + q^{-1})WU$ is equal to

$$(q^3+q)u_1 w_1 + v_1^2 = \theta v_1 + q^2 v_1^2 = y_1 q^{l-2}(\theta + y_1 q^l) = b_l b_{l+2} q^{2l-2}.$$

Thus, we see that the image of the braided Casimir under the above almost representation is equal to $b_l b_{l+2} q^{2l-2}\, Id$. We obtain a representation of the braided Lie algebra $\overline{sl(2)}_M$ if we put

$$\rho_{End}(u) = \tau\theta^{-1} U,\ \rho_{End}(v) = \tau\theta^{-1} V,\ \rho_{End}(w) = \tau\theta^{-1} W.$$

□

This leads to the following

Proposition 4.2 — *The value of the braided Casimir C_q corresponding to the braided $\overline{sl(2)}_M$-module U_k is equal to*

$$c_k = b_l b_{l+2} q^{2l-2}(\tau\theta^{-1})^2,\ \text{where } \theta = q^{2l+1} + q^{-1},\ l = 2k.$$

By this method we have constructed a series of representations of the algebra $A^{c_k}_{h,q}$ where c_k is given by the formula above where τ is replaced by $2h$.

5 Discussion: braided trace and braided involution

Let us remark that the non-braided algebra $A^c_{h,1}$ is multiplicity free, that is the multiplicity of any $sl(2)$-module in $A^c_{h,1}$ is at most 1. In fact only the integer-spin modules "live" in this algebra. The algebra $A^c_{h,q}$ has a similar property. So there exists a unique (up to a factor) way compatible with the $U_q(sl(2))$-action to introduce a *braided (twisted) trace* in this algebra as a non-trivial operator $A^c_{h,q} \to k$, killing all $U_q(sl(2))$-modules apart from the trivial one. We will denote this operator $tr^c_{h,q}$. It is defined up to a normalization.

Let us remark that if $h = 0$, this trace is an analogue of the integral over a sphere with respect to a symplectic measure. In this sense we prefer to work with the compact form of the Lie algebra $sl(2)$. However, in our setting (we are dealing only with polynomials on braided homogeneous space), a concrete real form of a homogeneous space under consideration has no importance.

By a method of [NM] it is easy to obtain the following

Proposition 5.1 — *In the algebra $A^c_{0,q}$ one has*

$$tr^c_{0,q} v^m = \frac{q^2-1}{2(q^{2m+2}-1}) (1+(-1)^m)(q\sqrt{c})^{-m}\, tr^c_{0,q} 1.$$

Using a modification of the method from [NM] it is also possible to obtain a similar formula for $tr^c_{h,q}$ for all h, but it is much more complicated and we do not reproduce it.

It is evident that the trace $tr^{c_k}_{h,q}$ regarded in the space $End(U_k)$ is just the famous quantum trace (cf. f.e. [CP], page 122). It would be interesting to calculate it directly using this fact. This calculation can be useful from a hypothetical braided (or quantum) orbit method point of view.

It is well known that in the framework of the orbit method one assigns to some orbits in g^* for a Lie algebra g certain g-modules. Meanwhile, the character formula compares the integrals of some special functions on these orbits with the traces of their images in the corresponding modules. We do not know what may be a reasonable analogue of the above correspondence in the braided case. As for a braided version of the character formula, it must be much more complicated.

We complete this section with a discussion of involution operators in the algebras under consideration. As we mentioned, the involution of the algebra $A^c_{h,q}$ constructed in [P] is not respected by the above respresentations.

What is a reasonable way to introduce an involution into the space $End(V)$ where V is a (finite-dimensional) object of a twisted category? Let us assume for the moment that the category is symmetric and there exists a pairing $V^{\otimes 2} \to k$ which is a morphism in the category. This means that it commutes with the twist S. Then the spaces $V^{\otimes 2}$ and $End(V)$ can be canonically identified, and the involution $*$ is the image of the twist $S : V^{\otimes 2} \to V^{\otimes 2}$ under this identification.

This yields the fact that such an involution satisfies the relations $*\mu = \mu(* \otimes *)S$ and $S(id \otimes *) = (* \otimes id)S$. Then, taking into account the fact that the "S-Lie bracket" $[\ ,\ ]$ is defined in the space $End(V)$ by $\mu(id - S)$, we obtain the relation

$$[\ ,\ ](* \otimes *) = - * [\ ,\ ]. \tag{1}$$

More precisely, we consider the space V over the field $k = R$ and assume the twist S to be real. Then we extend it to the space $V_C^{\otimes 2}$ where $V_C = V \otimes C$ by linearity. Under an involution (conjugation) we mean an involutive operator $* : V_C \to V_C$ such that $(\lambda z)^* = \overline{\lambda} z^*$, $\lambda \in C$, $z \in V_C$.

Definition 5.2 — *We say that the involution $*$ is compatible with the q-Lie bracket $[\,,\,]$ if relation (1) is satisfied.*

The following proposition is a straightforward calculation:

Proposition 5.3 — *The odd elements with respect to this involution (i.e. $z^* = -z$) form a subalgebra, that is the element $[a,\, b]$ is odd if a and b are.*

Remark. One often considers involutions which differ from ours by a sign. For such involutions we have to change the sign in relation (1) and consider the even elements instead of odd ones in Proposition 5.3.

Now we will classify all involutions $* : V_C \to V_C$ compatible with the q-Lie bracket.

Proposition 5.4 — *For a real $q \neq 1$ there exist only two involutions in the space V_C compatible with the q-Lie bracket, namely, $a^* = -\overline{a}$ for any $a \in V_C$, and $u^* = u$, $v^* = -v$, $w^* = w$.*

Proof. Choose a decomposition of u^*, v^*, w^* over the base

$$u^* = \alpha_1 u + \beta_1 v + \gamma_1 w$$

$$v^* = \alpha_2 u + \beta_2 v + \gamma_2 w$$

$$w^* = \alpha_3 u + \beta_3 v + \gamma_3 w,$$

where α_i, β_i, γ_i, $i = 1, 2, 3$ are complex coefficients. We want to find them in accordance with the compatibility condition (1).

It is easy to see that the relation

$$[u^*,\, u^*] = -[u,\, u]^* = 0$$

implies $\beta_1 = 0$. Similarly, from

$$[w^*,\, w^*] = -[w,\, w]^* = 0$$

we obtain $\beta_3 = 0$.

From the relation

$$[v^*,\, v^*] = -[v,\, v]^* = -(1 - q^2)Mv^*$$

we deduce that $\beta_2^2 + \beta_2 = 0$, i.e. $\beta_2 = 0$ or $\beta_2 = -1$.

The relation

$$[w^*,\, u^*] = -[w,\, u]^* = (q + q^{-1})^{-1} v^*$$

implies $\alpha_2 = \gamma_2 = 0$. If $\beta_2 = 0$, then $v^* = 0$, hence $\beta_2 = -1$.

Finally, from

$$[u^*, v^*] = -[u, v]^* = q^2 M u^* \text{ and } [w^*, v^*] = -[w, v]^* = -M w^*$$

we obtain $\gamma_1 = \alpha_3 = 0$.

Thus, we have $v^* = -v$, $u^* = \alpha_1 u$, $w^* = \gamma_3 w$. It is easy to see that only two cases are possible $\alpha_1 = \gamma_3 = -1$ and $\alpha_1 = \gamma_3 = 1$. This yields the conclusion. □

Although these conjugations are rather trivial, they are, together with the above traces, the ingredients of the twisted quantum mechanics in the sense of the following definition.

Definition 5.5 — *We say that an associative algebra is an object of twisted quantum mechanics if it belongs to a twisted category, is represented in the space End(V) equipped with a twisted Lie bracket, a trace and a conjugation as above and if the representation map is a morphism in this category.*

We cannot give a complete axiomatic system for twisted quantum mechanics. However, we want to stress that the quantum hyperboloid provides us with a completely new type of representation theory (and hence of quantum mechanics). It would be interesting to generalize this approach to infinite dimensional algebras and to use the above ingredients of twisted quantum mechanics in calculations of partition functions.

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