RAMIFICATIONS OF THE CLASSICAL SPHERE THEOREM

Karsten GROVE

Department of Mathematics
University of Maryland
College Park, MD 20742 (USA)

Abstract. The paper describes old and new developments, within as well as outside of Riemannian geometry, originating from the classical sphere theorem.

Résumé. Cet article décrit des développements anciens et récents, en géométrie riemannienne et ailleurs, provenant du classique théorème de caractérisation des sphères.

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TABLE OF CONTENTS

INTRODUCTION

Although Comparison Geometry can be traced back to the previous century, it did not really take root as a discipline until the 1930's through the work of Morse, [M1,2], Schoenberg [S], Myers [My] and Synge [Sy]. The real breakthrough came in the 1950's with the pioneering work of Rauch [R] and the foundational work of Alexandrov and Toponogov [T]. Since then, the simple idea of comparing the geometry of an arbitrary Riemannian manifold with the geometries of constant curvature spaces has witnessed a tremendous evolution.

Sphere Theorems have often played a pivotal role in this evolution. In fact many of the powerful ideas and techniques known today were first conceived in connection with investigations around potential sphere theorems (cf. also [Sh]). Their significance is also measured by their implications for the local structure of general Riemannian manifolds and other related, but more singular spaces.

Our aim here is to trace out paths of developments, still under construction, originating from the classical sphere theorem [R,K2] and the associated rigidity theorem by Berger [B1]. In doing so, it is our hope to reveal that there is an abundance of challenging open problems in this area whose solutions will yet again involve the conception of new ideas and tools.

1. DEVELOPMENTS FROM WITHIN

In this section we describe evolutions associated with constructions on a fixed Riemannian manifold.

It all began with Rauch's comparison theorem for the length of Jacobi fields [R] and subsequently with the global Alexandrov-Toponogov triangle comparison theorem [T]. And it culminated in the now classical theorem.

Theorem 1.1 (Rauch-Berger-Klingenberg). — *Let M be a closed simply connected Riemannian manifold whose sectional curvature satisfies $1 \leq \sec M \leq 4$. Then, either*

(i) M is a twisted sphere, or

(ii) M is isometric to a rank one symmetric space.

Under the assumptions stated in the theorem, one of the key ingredients is the injectivity radius estimate, $\operatorname{inj} M \geq \frac{\pi}{2}$. In the original approach, this was achieved via Morse theory of geodesics [K1,2], [CG] and [KS] (cf. also [E]). Before moving on to the natural generalization suggested by this estimate, let us point out that so far, no positively curved exotic spheres are known!

Quite recently, it was shown by M. Weiss that some exotic spheres do not admit 1/4-pinched metrics [W]. His method is based on the observation that a 1/4-pinched sphere M has maximal so called *Morse perfection*, i.e., there is a $\dim M$-dimensional ($\mathbb{Z}_2$ -equivariant) spherical family of Morse functions on M. On the other hand, sophisticated methods from algebraic K-theory reveal that some exotic spheres have smaller Morse perfection. It is interesting to note that this is also related to the so called *Gromoll-filtrations* of homotopy spheres, an idea which arose in the first proof that there are no exotic δ_n-pinched n-dimensional spheres when δ_n is sufficiently close to 1 [G]. Another completely different method to prove the same result was conceived independently by Shikata [S2]. He constructed a distance between differentiable structures [S1], an idea which has since been expanded tremendously (cf. [Gr]). The best estimate for $\delta_n = \delta$ is due to Suyama [Su]. His method combines the earlier methods for achieving a dimension independent constant, the first due to Shiohama [SS] and the second to Ruh [R1,2]. Here, Ruh's method of approximating an almost flat connection with a flat connection has evolved quite far and has had many subsequent applications (cf. [R3]).

Another natural question related to the classical sphere theorem is: what happens if M is not simply connected? So far, all known (strictly) 1/4-pinched manifolds are diffeomorphic to space forms. Moreover, at least these are the only manifolds which admit a δ-pinched riemannian metric, with δ sufficiently close to 1 [GKR1,2], [IR]. The general nonlinear "Riemannian center of mass" was developed in connection with the first proof of this result [GK].

Recall that the *radius* $\operatorname{rad} M$ and *diameter* $\operatorname{diam} M$ are given by $\operatorname{rad} M = \min_p \max_q \operatorname{dist}(p,q) \leq \max_{p,q} \operatorname{dist}(p,q) = \operatorname{diam} M$. Since $\operatorname{inj} M \geq \frac{\pi}{2}$ for 1-connected Riemannian manifolds M with $1 \leq \sec M \leq 4$, we also have $\operatorname{diam} M \geq \operatorname{rad} M \geq \frac{\pi}{2}$ for such manifolds. In particular,

$$\begin{aligned}\{M \mid 1 \leq \sec M \leq 4, \pi_1(M) = \{1\}\} &\subset \{M \mid 1 \leq \sec M, \operatorname{rad} M \geq \frac{\pi}{2}\} \\ &\subset \{M \mid 1 \leq \sec M, \operatorname{diam} M \geq \frac{\pi}{2}\} .\end{aligned}$$

For the largest of these classes we have the following diameter sphere Theorem [GS] a homotopy version of which was first proved in [B2] and its associated rigidity theorem [GG1,2].

Theorem 1.2. (Gromoll-Grove-Shiohama) — *Let M be a closed Riemannian manifold with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2}$. Then, either*

(i) M is a twisted sphere, with the possible exception that $H^*(M) \simeq H^*(CaP^2)$,

(ii) M is isometric to one of

(a) a rank 1 symmetric space,

(b) $\mathbb{C}P^{\mathrm{odd}}/\mathbb{Z}_2$,

(c) $S^n/\Gamma, \Gamma \subset O(n+1)$ acts reducibly on $\mathbb{R}^{n+1}$.

The principal new tool discovered in the proof of this sphere theorem was a "critical point theory" for nonsmooth distance functions. This signaled the beginning of intense investigations of manifolds with a lower curvature bound only (for surveys, cf. [C], [Gro]).

Aside from trying to deal with the exceptional case of the Cayley plane in the above result, the most obvious questions related to the theorem are

Problem 1.3.

(i) Are there any exotic spheres M with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2}$?

(ii) Are there "new" manifolds M with $\sec M \geq 1$ and $\operatorname{diam} M \geq \frac{\pi}{2} - \epsilon$, which are not on the list of the above theorem ?

At this moment it appears to be too ambitious to answer these questions at the level of generality at which they were posed (cf. the discussion in the next section).

It turns out, however, that it is possible to give answers if one assumes the existence of multiple points with large distances. The techniques used for this, however, come somewhat surprisingly from "outside" in both cases, i.e., it does not suffice to work exclusively within the manifold itself.

2. DEVELOPMENTS TO AND FROM THE OUTSIDE

When Gromov extended the classical Hausdorff distance to arbitrary pairs of compact (separable) metric spaces, [Gr], a new powerful tool became available, conceptually as well as technically. Its utility, however, is balanced between the facts that on the one hand large classes of Riemannian manifolds are precompact relative to this so-called *Gromov-Hausdorff topology*, but on the other hand only a limited amount of structure is transferred onto spaces in their closure.

One of the first striking applications of these ideas to Riemannian geometry is due to Berger. He showed that, for each even integer $2n$, there is an $\epsilon = \epsilon(2n)$ such that any closed 1-connected Riemannian $2n$-manifolds M which satisfies $1 \leq \sec M \leq 4+\epsilon$ is diffeomorphic to a projective space or homeomorphic to the $2n$-sphere, [B3].

The main difference between even and odd dimensions in this problem is that, under the given assumptions, $\operatorname{inj} M \geq \frac{\pi}{2}$ when $\dim M$ is even [K1]. Just very recently, Abresch and Meyer have extended this to odd dimensions in the remarkable paper [AM]. This yields a sphere theorem for below 1/4-pinched simply connected odd dimensional manifolds.

When the assumptions of simple connectivity and upper curvature bound are replaced by a lower bound for the diameter, one arrives at question 1.3(ii) raised at the end of the previous section. A natural approach to this problem is to investigate a situation, where a sequence $\{M_i\}$ of closed riemannian n-manifolds are given such that $\sec M_i \geq 1$ and $\frac{\pi}{2} > \operatorname{diam} M_i \geq \frac{\pi}{2} - \frac{1}{i},\ i = 1, \cdots$. By Gromov's precompactness theorem [Gr], a subsequence of $\{M_i\}$ will converge to an inner metric space X with

$\operatorname{diam} X = \frac{\pi}{2}$. Moreover, as observed in [GP1]: The Hausdorff dimension of X satisfies $\dim X \leq n$, and $\operatorname{curv} X \geq 1$ in the sense that the global Alexandrov-Toponogov distance comparison theorem holds in X. It has proven adventageous to consider these properties abstractly, and we adopt the terminology *Alexandrov space* for any finite Hausdorff dimensional inner metric space X for which $\operatorname{curv} X \geq k$ in distance comparison sense. In this framework one therefore asks a classification question.

Problem 2.1. Classify all n-dimensional Alexandrov spaces X with $\operatorname{curv} X \geq 1$ and $\operatorname{diam} X = \frac{\pi}{2}$.

In analogy to the case of Riemannian manifolds, this should be compared with the situation where $\operatorname{curv} X \geq 1$ and $\operatorname{diam} X > \frac{\pi}{2}$. Here, a completely satisfactory solution has been given by Perelman [P]. In fact, any such X is homeomorphic to the suspension $\sum E$, where E can be any Alexandrov space with $\operatorname{curv} E \geq 1$ and $\dim E = \dim X - 1$.

Once it has been established that critical point theory for distance functions extends to Alexandrov spaces, the proof of the above diameter suspension theorem is identical to that of the diameter sphere theorem. To establish this, however, is deeply intertwined with understanding the local structure of Alexandrov spaces (cf. [P]).

Here we give only a brief account on the structure of Alexandrov spaces X. First of all, the curvature assumption implies that geodesics are unique in the sense that they cannot bifurcate. This implies in particular that for every $p \in X$ there is a well defined space of geodesic directions (germs) at p. Moreover, the curvature assumption yields a natural notion of angle between geodesics emanating at p. The space of directions at p, S_pX is now simply the completion of the space of geodesic directions at p relative to the angle metric. This space is again an Alexandrov space and $\operatorname{curv} S_pX \geq 1$, $\dim S_pX = \dim X - 1$, [BGP], [Pl]. In fact, the euclidean cone $C_0\, S_pX = T_pX$ on S_pX, is the pointed Gromov-Hausdorff limit of the scaled spaces $(\lambda X, p)$, $\lambda \to \infty$, [BGP]. The basic local structure result is proved in [P]:

Theorem 2.2. (Perelman) — *Let X be an Alexandrov space. Then, any $p \in X$ has an open neighborhood which is (bi-Lipschitz) homeomorphic to the tangent cone, $T_pX = C_0S_pX$ to X at p.*

It is easy to see that this result yields a global stratification of X into topological manifolds, a fact which is used in the proof of 2.5 below. The following stability theorem is another key result proved in [P].

Theorem 2.3. (Perelman) — *Given a compact n-dimensional Alexandrov space X with* $\operatorname{curv} X \geq k$. *There is an $\epsilon = \epsilon(X)$ such that any other n-dimensional Alexandrov space Y with* $\operatorname{curv} Y \geq k$ *and Gromov-Hausdorff distance $d_{GH}(X,Y) < \epsilon$ is (bi-Lipschitz) homeomorphic to X.*

The key to both of these results is a general local fibration theorem for m-tuples of distance type functions, $1 \leq m \leq n$, near a "regular point", (see [P]). The ingenious proof is carried out via inverse induction on m. The idea for the induction anchor, i.e., when $m = n$ was in essence first used in a Riemannian setting in the paper [OSY], and then in [BGP].

To get an idea of the difficulty of Problem 2.1 we point out that there is an abundance of examples (cf. [GP2] and [GM] for more details).

Example 2.4. Let A and B be Alexandrov spaces with $\operatorname{curv} A \geq 1$ and $\operatorname{curv} B \geq 1$. Define a metric on the join $A * B$, such that the join $\alpha * \beta \subset A * B$ of segments $\alpha \subset A, \beta \subset B$ is isometric to $\alpha * \beta \subset S^3_1$ when α and β are identified with segments of the unit circle S^1. It is easy to see that $\operatorname{curv} A * B \geq 1$, in fact $A * B = S_{(a,b)}(C_0 A \times C_0 B)$ where $a \in C_0 A$ and $b \in C_0 B$ are the cone points. Moreover, $\operatorname{diam} A * B = \max\{\operatorname{diam} A, \operatorname{diam} B, \frac{\pi}{2}\}$.

Now, suppose G is a compact Lie group which acts by isometries on A and on B. Obviously, these actions extend to an isometric G action on $A * B$ and $\operatorname{diam}(A * B/G) \geq \frac{\pi}{2}$.

A special case occurs if $B = S^\circ$ is the two-point space with diameter π. Then $A * S^\circ$ is nothing but the spherical suspension $\Sigma_1 A$ of A, and $\operatorname{diam} \Sigma_1 A = \pi$. All Alexandrov spaces X with $\operatorname{curv} X \geq 1$ and $\operatorname{diam} X = \pi$ are of this type.

Another special case of interest occurs when $B = G$, i.e., $G = S^\circ, S^1$ or S^3. The space $A * G/G$ is then simply the spherical mapping cone $C_1(A \to A/G)$.

These examples show that even replacing the assumption $\operatorname{diam} X = \frac{\pi}{2}$ with $\operatorname{rad} X = \frac{\pi}{2}$ in (2.1) is ambitious. The corresponding inequality $\operatorname{rad} X > \frac{\pi}{2}$ has, however, a very satisfactory and optimal solution obtained in [GP2] and independently by Petrunin.

Radius Sphere Theorem 2.5. — *Any n-dimensional Alexandrov space X with* $\operatorname{curv} X \geq 1$ *and* $\operatorname{rad} X > \frac{\pi}{2}$ *is homeomorphic to S^n.*

Another natural strengthening of the assumptions in (2.1) is obtained by replacing the diameter assumption by having multiple points with distances $\geq \frac{\pi}{2}$. To be precise, let pack_q denote the *$q'th$ packing radius*, i.e.,

$$2 \operatorname{pack}_q X = \max_{(p_1,\cdots,p_q)} \min_{i<j} \operatorname{dist}(p_i, p_j) \ .$$

Then obviously

$$\frac{1}{2} \operatorname{diam} X = \operatorname{pack}_2 X \geq \cdots \geq \operatorname{pack}_q X \geq \cdots \ ,$$

and thus, $\operatorname{pack}_q X \geq \frac{\pi}{4},\ q \geq 2$, provides a natural filtration of the class, $\operatorname{diam} X \geq \frac{\pi}{2}$. A simple comparison argument shows that

$$\operatorname{pack}_q X \leq \operatorname{pack}_q S_1^n$$

for all q and any n-dimensional Alexandrov space X with $\operatorname{curv} X \geq 1$.

Using critical point theory (cf. [P]) and the global rigidity distance comparison theorem from [GM], one gets the following join theorem:

Theorem 2.6. ([GW]) — *Let X be an n-dimensional Alexandrov space with* $\operatorname{curv} X \geq 1$, *and let $q \leq n$. Then*

(i) $\operatorname{pack}_{q+2} X = \operatorname{pack}_{q+2} S_1^n$ *if and only if X is isometric to $S_1^q * E$ for some $n-q-1$ dimensional Alexandrov space E with* $\operatorname{curv} E \geq 1$ *;*

(ii) if $\operatorname{pack}_{q+2} X > \frac{\pi}{4}$, *then X is homeomorphic to $S_1^q * E$.*

This result is essentially optimal both with respect to the inequality and the chosen number of points.

The equality discussion $\operatorname{pack}_q X = \frac{\pi}{4}$ is very difficult in general. However, for $q = n+1$ and $\operatorname{diam} X = \frac{\pi}{2}$ we have the following partial answer to 2.1.

Theorem 2.7. ([GM]) — *Let X be an n-dimensional Alexandrov space with* $\operatorname{curv} X \geq 1$. *Then* $\operatorname{diam} X = 2 \operatorname{pack}_{n+1} X = \frac{\pi}{2}$ *if and only if X is isometric to S^n_1/H, where H is a finite group of commuting isometric involutions acting without fixed points on the unit sphere, S^n_1.*

Among this fairly large class of spaces, the only manifolds are $\mathbb{R}P^n_1$ and spaces homeomorphic to S^n_1 (see [GM]). As a consequence of the stability Theorem 2.3, and Yamaguchi's fibration theorem [Y] we thus have the following partial answer to 1.3(ii):

Corollary 2.8. ([GM]) — *For every integer $n \geq 2$, there is an $\epsilon = \epsilon(n)$ such that any closed Riemannian n-manifold M with* $\sec M \geq 1$ *and* $\operatorname{pack}_{n+1} M \geq \frac{\pi}{4} - \epsilon$ *is homeomorphic to S^n or diffeomorphic to $\mathbb{R}P^n$.*

Note that if in the above Corollary, $2 \operatorname{pack}_{n+1} M = \operatorname{diam} M = \frac{\pi}{2}$, then M is isometric to $\mathbb{R}P^n_1$. As a corresponding partial answer to 1.3(i) we have:

Theorem 2.9. ([GW]) — *If M is a closed Riemannian n-manifold with* $\sec M \geq 1$ *and* $\operatorname{pack}_{n+1} M > \frac{\pi}{4}$, *then M is diffeomorphic to S^n.*

The proof of this result involves yet another idea from "outside of M". In fact, the global Riemannian problem is changed to a local problem in Alexandrov geometry. It is shown that M can be smoothly embedded in $\mathbb{R}^{n+1}$ by establishing that a deleted neighborhood of one of the cone points in the spherical suspension $X^{n+1} = \Sigma_1 M$ is diffeomorphic to an open subset of $\mathbb{R}^{n+1}$. This is done by exhibiting the deleted neighborhood as a union of a 1-dimensional line bundle and a 1-dimensional annulus bundle. The fibers consists of points, where n smoothed distance functions take a given value. By appealing to the results of Smale [Sm] and Hatcher [H] on the diffeomorphism groups of S^2 and S^3 respectively, the technique can be pushed to yield the same conclusion with the weaker assumption $\operatorname{pack}_{n-1} M > \frac{\pi}{4}$ (see [GW]). The reason why that is a significantly stronger result is that there are metrics on $M = S^n$ with $\sec M \geq 1, \operatorname{pack}_{n-1} M$ arbitrarily close to $\operatorname{pack}_{n-1} S^n_1$ and yet with vol M arbitrarily small. To be concrete, a Gromov-Hausdorff limit space can be chosen to be the $(n-1)$-dimensional hemisphere, whereas in all previous differentiable sphere theorems the limiting object in the extreme case is S^n_1.

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