Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 4 (1998)

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# GEOMETRIC CONTROL APPROACH TO SYNTHESIS THEORY

## 1. Introduction

In this paper we describe the approach used in geometric control theory to deal with optimization problems. The concept of synthesis, extensively discussed in [20], appears to be the right mathematical object to describe a solution to general optimization problems for control systems.

Geometric control theory proposes a precise procedure to accomplish the difficult task of constructing an optimal synthesis. We illustrate the strength of the method and indicate the weaknesses that limit its range of applicability.

We choose a simple class of optimal control problems for which the theory provides a complete understanding of the corresponding optimal syntheses. This class includes various interesting controlled dynamics appearing in Lagrangian systems of mathematical physics. In this special case the structure of the optimal synthesis is completely described simply by a couple of integers, (cfr. Theorem 3). This obviously provides a very simple classification of optimal syntheses. A more general one, for generic plane control-affine systems, was developed in [18, 10].

First we give a definition of optimal control problem. We discuss the concepts of solution for this problem and compare them. Then we describe the geometric control approach and finally show its strength using examples.

## 2. Basic definitions

Consider an optimal control problem $(\mathcal{P})$ in Bolza form:

$$\dot{x} = f(x,u), \qquad x \in M,\ u \in U$$
$$\min\left(\int L(x,u)\,dt + \varphi(x_{term})\right)$$
$$x_{in} = x_0, \qquad x_{term} \in N \subset M$$

where $M$ is a manifold, $U$ is a set, $f : M \times U \to TM$, $L : M \times U \to \mathbb{R}$, $\varphi : M \to \mathbb{R}$, the minimization problem is taken over all admissible trajectory-control pairs $(x, u)$, $x_{in}$ is the initial point and $x_{term}$ the terminal point of the trajectory $x(\cdot)$. A solution to the problem $(\mathcal{P})$ can be given by an open loop control $u : [0, T] \to U$ and a corresponding trajectory satisfying the boundary conditions.

One can try to solve the problem via a feedback control, that is finding a function $u : M \to U$ such that the corresponding ODE $\dot{x} = f(x, u(x))$ admits solutions and the solutions to the Cauchy problem with initial condition $x(0) = x_0$ solve the problem $(\mathcal{P})$. Indeed, one explicits the dependence of $(\mathcal{P})$ on $x_0$, considers the family of problems $\mathcal{P} = (\mathcal{P}(x_0))_{x_0 \in M}$ and tries to solve them via a unique function $u : M \to U$, that is to solve the family $\mathcal{P}$ of problems all together.

A well-known approach to the solution of $\mathcal{P}$ is also given by studying the value function, that is the function $V : M \to \mathbb{R}$ assuming at each $x_0$ the value of the minimum for the corresponding problem $\mathcal{P}(x_0)$, as solution of the Hamilton-Jacobi-Bellmann equation, see [5, 13]. In general $V$ is only a weak solution to the HJB equation but can be characterized as the unique "viscosity solution" under suitable assumptions.

Finally, one can consider a family $\Gamma$ of pairs trajectory-control $(\gamma_{x_0}, \eta_{x_0})$ such that each of them solves the corresponding problem $\mathcal{P}(x_0)$. This last concept of solution, called synthesis, is the one used in geometric control theory and has the following advantages with respect to the other concepts:

**1)** Generality

**2)** Solution description

**3)** Systematic approach

Let us now describe in details the three items.

**1)** Each feedback gives rise to at least one synthesis if there are solutions to the Cauchy problems. The converse is not true, that is a synthesis is not necessarily generated by a feedback even if in most examples one is able to reconstruct a piecewise smooth control.

If one is able to define the value function this means that each problem $\mathcal{P}(x_0)$ has a solution and hence there exists at least one admissible pair for each $\mathcal{P}(x_0)$. Obviously, in this case, the converse is true that is every optimal synthesis defines a value function. However, to have a viscosity solution to the HJB equation one has to impose extra conditions.

**2)** Optimal feedbacks usually lack of regularity and generate too many trajectories some of which can fail to be optimal. See [20] for an explicit example. Thus it is necessary to add some structure to feedbacks to describe a solution. This is exactly what was done in [11, 22].

Given a value function one knows the value of the minimum for each problem $\mathcal{P}(x_0)$. In applications this is not enough, indeed one usually needs to drive the system along the optimal trajectory and hence to reconstruct it from the value function. This is a hard problem, [5]. If one tries to use the concept of viscosity solutions then in the case of Lagrangians having zeroes the solution is not unique. Various interesting problems (see for example [28]) present Lagrangians with zeroes. Recent results to deal with this problem can be found in [16].

**3)** Geometric control theory proposes a systematic way towards the construction of optimal syntheses, while there are not general methods to construct feedbacks or viscosity solutions for the HJB equation. We describe in the next session this systematic approach.

## 3. Optimal synthesis

The approach to construct an optimal synthesis can be summarized in the following way:

$$
\begin{array}{ll}
a) & \text{MP + geometric techniques} \\
 & \Downarrow \\
b) & \text{Properties of extremal trajectories} \\
 & \Downarrow \\
c) & \text{Construction of extremal synthesis} \\
 & \Downarrow \\
d) & \text{Optimal synthesis}
\end{array}
$$

We now explain each item of the picture for a complete understanding of the scheme.

**a)** The Maximum Principle remains the most powerful tool in the study of optimal control problems forty years after its first publication, see [21]. A big effort has been dedicated to generalizations of the MP in recent years, see [6], [23], and references therein.

Since late sixties the study of the Lie algebra naturally associated to the control system has proved to be an efficient tool, see [15]. The recent approach of simplectic geometry proposed by Agrachev and Gamkrelidze, see [1, 4], provides a deep insight of the geometric properties of extremal trajectories, that is trajectories satisfying the Maximum Principle.

**b)** Making use of the tools described in **a)** various results were obtained. One of the most famous is the well known Bang-Bang Principle. Some similar results were obtained in [8] for a special class of systems. For some planar systems every optimal trajectory is not bang-bang but still a finite concatenation of special arcs, see [19, 24, 25].

Using the theory of subanalytic sets Sussmann proved a very general results on the regularity for analytic systems, see [26]. The regularity, however, in this case is quite weak and does not permit to drive strong conclusions on optimal trajectories.

Big improvements were recently obtained in the study of Sub-Riemannian metrics, see [2, 3]. In particular it has been showed the link between subanalyticity of the Sub-Riemannian sphere and abnormal extremals.

**c)** Using the properties of extremal trajectories it is possible in some cases to construct an extremal synthesis. This construction is usually based on a finite dimensional reduction of the problem: from the analysis of **b)** one proves that all extremal trajectories are finite concatenations of special arcs. Again, for analytic systems, the theory of subanalytic sets was extensively used: [11, 12, 22, 27].

However, even simple optimization problems like the one proposed by Fuller in [14] may fail to admit such a kind of finite dimensional reduction. This phenomenon was extensively studied in [17, 28].

**d)** Finally, once an extremal synthesis has been constructed, it remains to prove its optimality. Notice that no regularity assumption property can ensure the optimality (not even local) of a single trajectory. But the contrary happens for a synthesis. The classical results of Boltianskii and Brunovsky, [7, 11, 12], were recently generalized to be applicable to a wider class of systems including Fuller's example (see [20]).

## 4. Applications to second order equations

Consider the control system:

$$\dot{x} = F(x) + uG(x),\ x \in \mathbb{R}^2,\ F, G \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2),\ F(0) = 0,\ |u| \leq 1 \tag{1}$$

and let $\mathcal{R}(\tau)$ be the reachable set within time $\tau$ from the origin. In the framework of [9, 19, 18], we are faced with the problem of reaching from the origin (under generic conditions on $F$ and $G$) in minimum time every point of $\mathcal{R}(\tau)$. Given a trajectory $\gamma : [a, b] \to \mathbb{R}^2$, we define the time along $\gamma$ as $T(\gamma) = b - a$.
A trajectory $\gamma$ of (1) is *time optimal* if, for every trajectory $\gamma'$ having the same initial and terminal points, one has $T(\gamma') \geq T(\gamma)$. A *synthesis* for the control system (1) at time $\tau$ is a family $\Gamma = \{(\gamma_x, u_x)\}_{x\in\mathcal{R}(\tau)}$ of trajectory-control pairs s.t.:

**(a)** for each $x \in \mathcal{R}(\tau)$ one has $\gamma_x : [0, b_x] \to \mathbb{R}^2$, $\gamma_x(0) = 0$, $\gamma_x(b_x) = x$;

**(b)** if $y = \gamma_x(t)$, where $t$ is in the domain of $\gamma_x$, then $\gamma_y = \gamma_x|_{[0,t]}$.

A synthesis for the control system (1) is *time optimal* if, for each $x \in \mathcal{R}(\tau)$, one has $\gamma_x(T(x)) = x$, where $T$ is the minimum time function $T(x) := \inf\{\tau : x \in \mathcal{R}(\tau)\}$. We indicate by $\Sigma$ a control system of the type (1) and by $Opt(\Sigma)$ the set of optimal trajectories. If $\gamma_1$, $\gamma_2$ are two trajectories then $\gamma_1 * \gamma_2$ denotes their concatenation. For convenience, we define also the vector fields: $X = F - G$, $Y = F + G$. We say that $\gamma$ is an $X$-trajectory and we write $\gamma \in \text{Traj}\,(X)$ if it corresponds to the constant control $-1$. Similarly we define $Y$-trajectories. If a trajectory $\gamma$ is a concatenation of an $X$-trajectory and a $Y$-trajectory, then we say that $\gamma$ is a $Y * X$-trajectory. The time $t$ at which the two trajectories concatenate is called $X$-$Y$ switching time and we say that the trajectory has a $X$-$Y$ switching at time $t$. Similarly we define trajectories of type $X * Y$, $X * Y * X$, etc.
In [19] it was shown that, under generic conditions, the problem of reaching in minimum time every point of the reachable set for the system (1) admits a regular synthesis. Moreover it was shown that $\mathcal{R}(\tau)$ can be partitioned in a finite number of embedded submanifolds of dimension 2, 1 and 0 such that the optimal synthesis can be obtained from a feedback $u(x)$ satisfying:

- on the regions of dimension 2, we have $u(x) = \pm 1$,
- on the regions of dimension 1, called frame curves (in the following FC), $u(x) = \pm 1$ or $u(x) = \varphi(x)$ (where $\varphi(x)$ is a feedback control that depends on $F, G$ and on their Lie bracket $[F, G]$, see [19]). The frame curves that correspond to the feedback $\varphi$ are called *turnpikes*. A trajectory that corresponds to the control $u(t) = \varphi(\gamma(t))$ is called a $Z$-trajectory.

The submanifolds of dimension 0 are called frame points (in the following FP). In [18] it was provided a complete classification of all types of FP and FC.
Given a trajectory $\gamma \in \Gamma$ we denote by $n(\gamma)$ the smallest integer such that there exist $\gamma_i \in \text{Traj}\,(X) \cup \text{Traj}\,(Y) \cup \text{Traj}\,(Z)$, $(i = 1, \ldots, n(\gamma))$, satisfying $\gamma = \gamma_{n(\gamma)} * \cdots * \gamma_1$.

The previous program can be used to classify the solutions of the following problem.

**Problem:** *Consider an autonomous ODE in* $\mathbb{R}$*:*

$$\ddot{y} = f(y, \dot{y}), \tag{2}$$

$$f \in \mathcal{C}^3(\mathbb{R}^2), \qquad f(0,0) = 0 \tag{3}$$

*that describes the motion of a point under the action of a force that depends on the position and on the velocity (for instance due to a magnetic field or a viscous fluid). Then let apply an external force, that we suppose bounded (e.g. $|u| \leq 1$):*

$$\ddot{y} = f(y, \dot{y}) + u\,. \tag{4}$$

*We want to reach in minimum time a point in the configuration space $(y_0, v_0)$ from the rest state $(0, 0)$.*

First of all observe that if we set $x_1 = y$, $x_2 = \dot{y}$, (4) becomes:

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = f(x_1, x_2) + u\,, \tag{6}$$

that can be written in our standard form $\dot{x} = F(x) + uG(x)$, $x \in \mathbb{R}^2$ by setting $x = (x_1, x_2)$, $F(x) = (x_2,\ f(x))$, $G(x) = (0, 1) := G$.
A deep study of those systems was performed in [9, 10, 19, 18]. From now on we make use of notations introduced in [19]. A key role is played by the functions $\Delta_A$, $\Delta_B$:

$$\Delta_A(x) = \det(F(x), G(x)) = x_2 \tag{7}$$

$$\Delta_B(x) = \det(G(x), [F(x), G(x)]) = 1\,. \tag{8}$$

From these it follows:

$$\Delta_A^{-1}(0) = \{x \in \mathbb{R}^2 : x_2 = 0\} \tag{9}$$

$$\Delta_B^{-1}(0) = \emptyset\,. \tag{10}$$

The analysis of [19] has to be completed in the following way.
Lemma 4.1 of [19] has to be replaced by the following (see [19] for the definition of $Bad(\tau)$ and $\tan_A$):

LEMMA 1. *Let $x \in Bad\,(\tau)$ and $G(x) \neq 0$ then:*

**A.** $x \in (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)) \quad \Rightarrow \quad x \in \tan_A$*;*

**B.** $x \in \tan_A, \quad X(x), Y(x) \neq 0 \quad \Rightarrow \quad x \in (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0))$.

*Proof.* The proof of **A.** is exactly as in [19]. Let us prove **B**. Being $G(x) \neq 0$ we can choose a local system of coordinates such that $G \equiv (1, 0)$. Then, with the same computations of [19], we have:

$$\Delta_B(x) = -\partial_1 F_2(x)\,. \tag{11}$$

From $x \in \tan_A$ it follows $x \in \Delta_A^{-1}(0)$, hence $F(x) = \alpha G$ ($\alpha \in \mathbb{R}$). Assume that $X(x)$ is tangent to $\Delta_A^{-1}(0)$, being the other case entirely similar. This means that $\nabla\Delta_A(x) \cdot X(x) = (\alpha - 1)\nabla\Delta_A(x) \cdot G = 0$. From $X(x) \neq 0$ we have that $\alpha \neq 1$, hence $\nabla\Delta_A \cdot G = 0$. This implies $\partial_1 F_2 = 0$ and using (11) we obtain $\Delta_B(x) = 0$, i.e. $x \in \Delta_B^{-1}(0)$.

□

Now the proof of Theorem 4.2 of [19] is completed considering the following case:

$$(4) \quad G(x) \neq 0, \qquad X(x) = 0 \text{ or } Y(x) = 0\,.$$

Note that (4) implies $x \in \tan_A$. In this case we assume the generic condition ($(P_1), \ldots, (P_8)$ were introduced in [19]):

$$(P_9) \qquad \Delta_B(x) \neq 0\,.$$

Suppose $X(x) = 0$ and $Y(x) \neq 0$. The opposite case is similar. Choose a new local system of coordinates such that $x$ is the origin, $Y = (0, -1)$ and $\Delta_A^{-1}(0) = \{(x_1, x_2) : x_2 = 0\}$.
Take $U = B(0, r)$, the ball of radius $r$ centered at 0, and choose $r$ small enough such that:

- 0 is the only bad point in $U$;
- $\Delta_B(x) \neq 0$ for every $x \in U$;
- for every $x \in U$ we have:

$$(12) \qquad |X(x)| \ll 1\,.$$

Let $U_1 = U \cap \{(x_1, x_2) : x_2 > 0\}$, $U_2 = U \cap \{(x_1, x_2) : x_2 < 0\}$. We want to prove the following:

THEOREM 1. *If $\gamma \in Opt\,(\Sigma)$ and $\{\gamma(t) : t \in [b_0, b_1]\} \subset U$ then we have a bound on the number of arcs, that is $\exists N_x \in \mathbb{N}$ s.t. $n(\gamma|_{[b_0,b_1]}) \leq N_x$.*

In order to prove Theorem 1 we will use the following Lemmas.

LEMMA 2. *Let $\gamma \in Opt\,(\Sigma)$ and assume that $\gamma$ has a switching at time $t_1 \in \mathrm{Dom}\,(\gamma)$ and that $\Delta_A(\gamma(t_1)) = 0$. Then $\Delta_A(\gamma(t_2)) = 0$, $t_2 \in \mathrm{Dom}\,(\gamma)$, iff $t_2$ is a switching time for $\gamma$.*

*Proof.* The proof is contained in [10].

□

LEMMA 3. *Let $\gamma : [a, b] \to U$ be an optimal trajectory such that $\gamma([a, b]) \subset U_1$ or $\gamma([a, b]) \subset U_2$, then $n(\gamma) \leq 2$.*

*Proof.* It is a consequence of Lemma 3.5 of [19] and of the fact that every point of $U_1$ (respectively $U_2$) is an ordinary point i.e. $\Delta_A(x) \cdot \Delta_B(x) \neq 0$.

□

LEMMA 4. *Consider $\gamma \in Opt\,(\Sigma)$, $\{\gamma(t) : t \in [b_0, b_1]\} \subset U$. Assume that there exist a $X$-$Y$ switching time $\bar{t} \in (b_0, b_1)$ for $\gamma$ and $\gamma(\bar{t}) \in U_1$. Then $\gamma|_{[\bar{t}, b_1]}$ is a $Y$-trajectory.*

*Proof.* Assume by contradiction that $\gamma$ switches at time $t' \in (b_0, b_1)$, $t' > \bar{t}$. If $\gamma(t') \in U_1$ then this contradicts the conclusion of Lemma 3. If $\gamma(t') \in \Delta_A^{-1}(0)$ then this contradicts Lemma 2. Assume $\gamma(t') \in U_2$. From $\mathrm{sgn}\,\Delta_A(\gamma(\bar{t})) = -\mathrm{sgn}\,\Delta_A(\gamma(t'))$ we have that $\frac{1}{2}X(\gamma(\bar{t})) \wedge Y = -\frac{1}{2}(X(\gamma(t')) \wedge Y)$. This means that:

$$(13) \qquad \mathrm{sgn}\,(X_2(\gamma(\bar{t})) = -\mathrm{sgn}\,(X_2(\gamma(t'))\,,$$

where $X_2$ is the second component of $X$. Choose $t_0 \in (b_0, \bar{t})$ and define the trajectory $\bar{\gamma}$ satisfying $\bar{\gamma}(b_0) = \gamma(b_0)$ and corresponding to the control $\bar{u}(t) = -1$ for $t \in [b_0, t_0]$ and $\bar{u}(t) = 1$ for $t \in [t_0, b_1]$. From (13) there exists $t_1 > t_0$ s.t. $\bar{\gamma}(t_1) = \gamma(t) \in U_2$. Using (12) it is easy to prove that $t_1 < t$. This contradicts the optimality of $\gamma$ (see fig. 1).

□

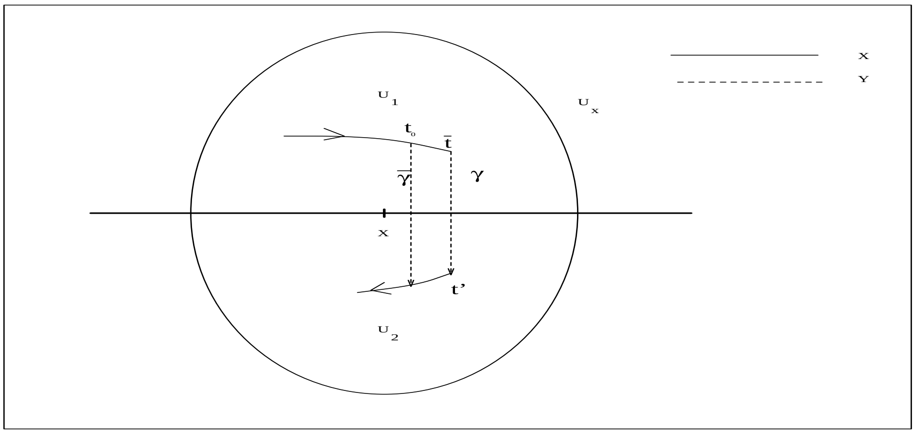


Figure 1:

*of Theorem 1.* For sake of simplicity we will write $\gamma$ instead of $\gamma|_{[b_0,b_1]}$.

Assume first that no switching happens on $\Delta_A^{-1}(0)$. We have the following cases (see fig. 2 for some of these):

**(A)** $\gamma$ has no switching; $n(\gamma) = 1$;

**(B)** for some $\varepsilon > 0$, $\gamma|_{[b_0,b_0+\varepsilon]}$ is an $X$-trajectory, $\gamma(b_0) \in U_1$, $n(\gamma) > 1$;

- **(B1)** if $\gamma$ switches to $Y$ in $U_1$, by Lemma 4, $n(\gamma) = 2$;
- **(B2)** if $\gamma$ crosses $\Delta_A^{-1}(0)$ and switches to $Y$ in $U_2$, by Lemma 3, $\gamma$ does not switch anymore. Hence $n(\gamma) = 2$;

**(C)** for some $\varepsilon > 0$, $\gamma|_{[b_0,b_0+\varepsilon]}$ is an $X$-trajectory, $\gamma(b_0) \in U_2$, $n(\gamma) > 1$;

- **(C1)** if $\gamma$ switches to $Y$ before crossing $\Delta_A^{-1}(0)$ then, by Lemma 3, $n(\gamma) = 2$;
- **(C2)** if $\gamma$ reaches $U_1$ without switching, then we are in the **(A)** or **(B)** case, thus $n(\gamma) \leq 2$;

**(D)** for some $\varepsilon > 0$, $\gamma|_{[b_0,b_0+\varepsilon]}$ is a $Y$-trajectory, $\gamma(b_0) \in U_1$, $n(\gamma) > 1$;

- **(D1)** if $\gamma$ switches to $X$ in $U_1$ and never crosses $\Delta_A^{-1}(0)$ then by Lemma 3 $n(\gamma) = 2$;
- **(D2)** if $\gamma$ switches to $X$ in $U_1$ (at time $t_0 \in [b_0, b_1]$) and then it crosses $\Delta_A^{-1}(0)$, then $\gamma|_{[t_0,b_1]}$ satisfies the assumptions of **(A)** or **(C)**. Hence $n(\gamma) \leq 3$;
- **(D3)** if $\gamma$ switches to $X$ in $U_2$ at $t_0 \in [b_0, b_1]$ and then it does not cross $\Delta_A^{-1}(0)$, we have $n(\gamma) = 2$;
- **(D4)** if $\gamma$ switches to $X$ in $U_2$ and then it crosses $\Delta_A^{-1}(0)$ we are in cases **(A)** or **(B)** and $n(\gamma) \leq 3$;

**(E)** for some $\varepsilon > 0$, $\gamma|_{[b_0,b_0+\varepsilon]}$ is a $Y$-trajectory, $\gamma(b_0) \in U_2$, $n(\gamma) > 1$.

- **(E1)** if $\gamma$ switches to $X$ in $U_2$ and it does not cross $\Delta_A^{-1}(0)$ then $n(\gamma) = 2$;

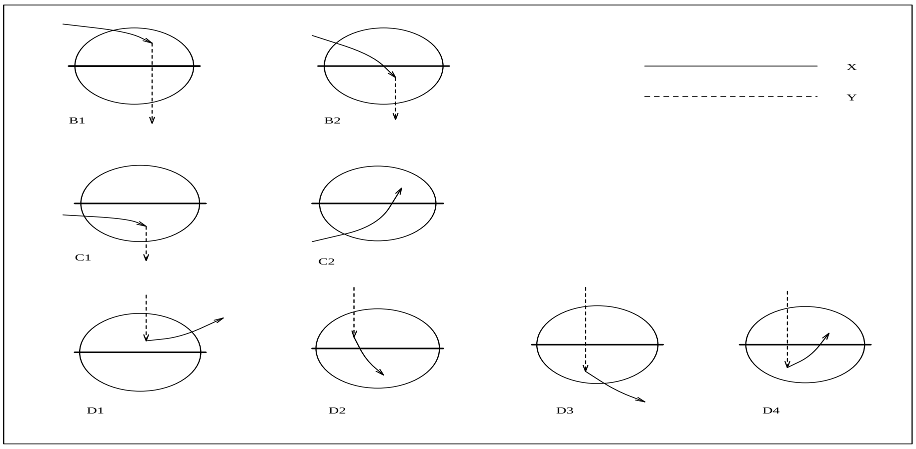


Figure 2:

**(E2)** if $\gamma$ switches to $X$ in $U_2$ and then it crosses $\Delta_A^{-1}(0)$, we are in case **(A)** or **B**. Hence $n(\gamma) \leq 3$.

If $\gamma$ switches at $\Delta_A^{-1}(0)$, by Lemma 2 all the others switchings of $\gamma$ happen on the set $\Delta_A^{-1}(0)$. Moreover, if $\gamma$ switches to $Y$ it has no more switchings. Hence $n(\gamma) \leq 3$.

The Theorem is proved with $N_x = 3$.

□

By direct computations it is easy to see that the *generic* conditions $P_1, \ldots, P_9$, under which the construction of [19] holds, are satisfied under the condition:

$$f(x_1, 0) = \pm 1 \quad \Rightarrow \quad \partial_1 f(x_1, 0) \neq 0 \tag{14}$$

that obviously implies $f(x_1, 0) = 1$ or $f(x_1, 0) = -1$ only in a finite number of points.
In the framework of [9, 19, 18] we will prove that, for our problem (5), (6), with the condition (14), the "shape" of the optimal synthesis is that shown in fig. 3. In particular the partition of the reachable set is described by the following

THEOREM 2. *The optimal synthesis of the control problem (5) (6) with the condition (14), satisfies the following:*

**1.** *there are no turnpikes;*

**2.** *the trajectory $\gamma^{\pm}$ (starting from the origin and corresponding to constant control $\pm 1$) exits the origin with tangent vector $(0, \pm 1)$ and, for an interval of time of positive measure, lies in the set $\{(x_1, x_2) : x_1, x_2 \geq 0\}$ respectively $\{(x_1, x_2) : x_1, x_2 \leq 0\}$;*

**3.** *$\gamma^{\pm}$ is optimal up to the first intersection (if it exists) with the $x_1$-axis. At the point in which $\gamma^{+}$ intersects the $x_1$-axis it generates a switching curve that lies in the half plane $\{(x_1, x_2) : x_2 \geq 0\}$ and ends at the next intersection with the $x_1$-axis (if it exists). At that*

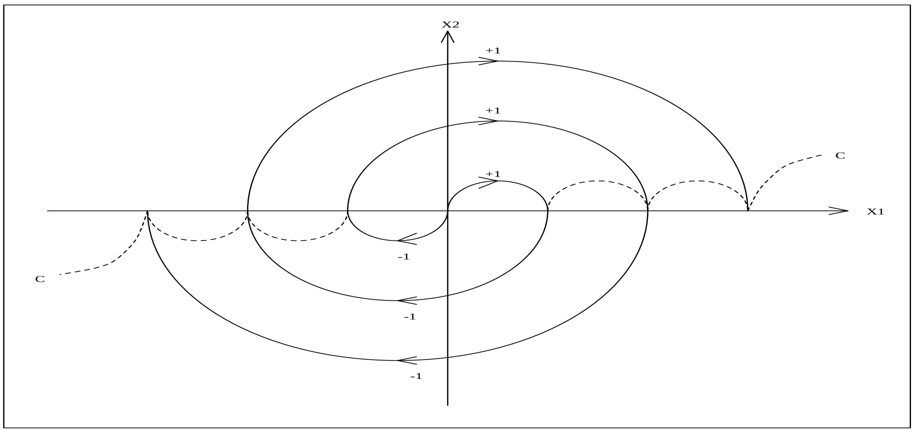


Figure 3: The shape of the optimal synthesis for our problem.

*point another switching curve generates. The same happens for $\gamma^-$ and the half plane $\{(x_1, x_2) : x_2 \leq 0\}$;*

**4.** *let $y_i$, for $i = 1, \ldots, n$ (n possibly $+\infty$) (respectively $z_i$, for $i = 1, \ldots, m$ (m possibly $+\infty$)) be the set of boundary points of the switching curves contained in the half plane $\{(x_1, x_2) : x_2 \geq 0\}$ (respectively $\{(x_1, x_2) : x_2 \leq 0\}$) ordered by increasing (resp. decreasing) first components. Under generic assumptions, $y_i$ and $z_i$ do not accumulate. Moreover:*

- *For $i = 2, \ldots, n$, the trajectory corresponding to constant control $+1$ ending at $y_i$ starts at $z_{i-1}$;*
- *For $i = 2, \ldots, m$, the trajectory corresponding to constant control $-1$ ending at $z_i$ starts at $y_{i-1}$.*

REMARK 1. The union of $\gamma^\pm$ with the switching curves is a one dimensional $\mathcal{C}^0$ manifold $M$. Above this manifold the optimal control is $+1$ and below is $-1$.

REMARK 2. The optimal trajectories turn clockwise around the origin and switch along the switching part of $M$. They stop turning after the last $y_i$ or $z_i$ and tend to infinity with $x_1(t)$ monotone after the last switching.

From **4.** of Theorem 2 it follows immediately the following:

THEOREM 3. *To every optimal synthesis for a control problem of the type (5) (6) with the condition (14), it is possible to associate a couple $(n, m) \in (\mathbb{N} \cup \infty)^2$ such that one of the following cases occurs:*

**A.** *$n = m$, n finite;*

**B.** *$n = m + 1$, n finite;*

**C.** *$n = m - 1$, n finite;*

**D.** $n = \infty, m = \infty$.

*Moreover, if* $\Gamma_1$, $\Gamma_2$ *are two optimal syntheses for two problems of kind (5), (6), (14), and* $(n_1, m_1)$ *(resp.* $(n_2, m_2)$*) are the corresponding couples, then* $\Gamma_1$ *is equivalent to* $\Gamma_2$ *iff* $n_1 = n_2$ *and* $m_1 = m_2$.

REMARK 3. In Theorem 3 the equivalence between optimal syntheses is the one defined in [9].

*of Theorem 3.* Let us consider the synthesis constructed by the algorithm described in [9]. The stability assumptions (SA1), ... ,(SA6) holds. The optimality follows from Theorem 3.1 of [9].

**1.** By definition a turnpike is a subset of $\Delta_B^{-1}(0)$. From (8) it follows the conclusion.

**2.** We leave the proof to the reader.

**3.** Let $\gamma_2^{\pm}(t) = \pi_2(\gamma^{\pm}(t))$, where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_2(x_1, x_2) = x_2$, and consider the adjoint vector field $v : \mathbb{R}^2 \times \text{Dom}\,(\gamma^{\pm}) \times \text{Dom}\,(\gamma^{\pm}) \to \mathbb{R}^2$ associated to $\gamma^{\pm}$ that is the solution of the Cauchy problem:

$$\text{(15)} \qquad \begin{cases} \dot{v}(v_0, t_0; t) & = & (\nabla F \pm \nabla G)(\gamma^{\pm}(t)) \cdot v(v_0, t_0; t) \\ v(v_0, t_0; t_0) & = & v_0\,, \end{cases}$$

We have the following:

LEMMA 5. *Consider the* $\gamma^{\pm}$ *trajectories for the control problem (5), (6). We have that* $v(G, t; 0)$ *and* $G$ *are parallel iff* $\Delta_A(\gamma^{\pm}(t)) = 0$ *(i.e.* $\gamma_2^{\pm}(t) = 0$*).*

*Proof.* Consider the curve $\gamma^+$, the case of $\gamma^-$ being similar. From (9) we know that $\Delta_A(\gamma^+(t)) = 0$ iff $\gamma_2^+(t) = 0$. First assume $\Delta_A(\gamma^+(t)) = 0$. We have that $G$ and $(F+G)(\gamma^+(t))$ are collinear that is $G = \alpha(F+G)(\gamma^+(t))$ with $\alpha \in \mathbb{R}$. For fixed $t_0, t$ the map:

$$\text{(16)} \qquad f_{t_0,t} : v_0 \mapsto v(v_0, t_0; t)$$

is clearly linear and injective, then using (15) and $\dot{\gamma}^+(t) = (F+G)(\gamma^+(t))$, we obtain $v(G, t; 0) = \alpha v\left((F+G)(\gamma^+(t)), t; 0\right) = \alpha(F+G)(0) = \alpha G$.
Viceversa assume $v(G, t; 0) = \alpha G$, then (as above) we obtain $v(G, t; 0) = \alpha v((F + G)(\gamma^+(t)), t; 0)$. From the linearity and the injectivity of (16) we have $G = \alpha(F + G)(\gamma^+(t))$ hence $\Delta_A(\gamma^+(t)) = 0$.

□

LEMMA 6. *Consider the trajectory* $\gamma^+$ *for the control problem (5), (6). Let* $\bar{t} > 0$ *(possibly* $+\infty$*) be the first time such that* $\gamma_2^+(\bar{t}) = 0$*. Then* $\gamma^+$ *is extremal exactly up to time* $\bar{t}$*. And similarly for* $\gamma^-$*.*

*Proof.* In [19] it was defined the function $\theta(t) = \arg\left(G(0), v\left(G(\gamma^+(t)), t, 0\right)\right)$. This function has the following properties:

**(i)** $\text{sgn}\,(\dot{\theta}(t)) = \text{sgn}(\Delta_B(\gamma(t)))$, that was proved in Lemma 3.4 of [19]. From (8) we have that $\text{sgn}\,(\dot{\theta}(t)) = 1$ so $\theta(t)$ is strictly increasing;

**(ii)** $\gamma^+$ is extremal exactly up to the time in which the measure of the range of $\theta$ is $\pi$ i.e. up to the time:

$$t^+ = \min\{t \in [0, \infty] : |\theta(s_1) - \theta(s_2)| = \pi, \text{ for some } s_1, s_2 \in [0, t^+]\}, \tag{17}$$

under the hypothesis $\dot\theta(t^+) \neq 0$. This was proved in Proposition 3.1 of [9].

From Lemma 5 we have that $\Delta_A(\gamma^+(t)) = 0$ iff there exists $n \in \mathbb{N}$ satisfying:

$$\theta(G, v(G, t, 0)) = n\pi . \tag{18}$$

In particular (18) holds for $t = \bar t$ and some $n$. From the fact that $\bar t$ is the first time in which $\gamma_2^+(\bar t) = 0$ and hence the first time in which $\Delta_A(\gamma^+(\bar t)) = 0$, we have that $n = 1$.

From $\theta(\bar t) = \pi$ and sgn $(\dot\theta(\bar t)) = 1$ the Theorem is proved with $t^+ = \bar t$.

□

From Lemma 6, $\gamma^\pm$ are extremal up to the first intersection with the $x_1$-axis.

Let $\underline{t}$ be the time such that $\gamma^-(\underline{t}) = z_1$, defined in **4** of Theorem 2. The extremal trajectories that switch along the $C$-curve starting at $y_1$ (if it exists), are the trajectories that start from the origin with control $-1$ and then, at some time $t' < \underline{t}$, switch to control $+1$. Since the first switching occurs in the orthant $\{(x_1, x_2) : x_1, x_2 < 0\}$, by a similar argument to the one of Lemma 6, the second switch has to occur in the half space $\{(x_1, x_2) : x_2 > 0\}$, because otherwise between the two switches we have meas (range $(\theta(t))) > \pi$. This proves that the switching curves never cross the $x_1$-axis.

**4.** The two assertions can be proved separately. Let us demonstrate only the first, being the proof of the second similar. Define $y_0 = z_0 = (0, 0)$. By definition the $+1$ trajectory starting at $z_0$ reaches $y_1$. By Lemma 2 we know that if an extremal trajectory has a switching at a point of the $x_1$-axis, then it switches iff it intersects the $x_1$-axis again. This means that the extremal trajectory that switches at $y_i$ has a switching at $z_j$ for some $j$. By induction one has $j \geq i - 1$. Let us prove that $j = i - 1$. By contradiction assume that $j > i - 1$, then there exists an extremal trajectory switching at $z_{i-1}$ that switches on the $C$ curve with boundary points $y_{i-1}, y_i$. This is forbidden by Lemma 2.

□

EXAMPLES 1. In the following we will show the qualitative shape of the synthesis of some physical systems coupled with a control. More precisely we want to determine the value of the couple $(m, n)$ of Theorem 3.

**Duffin Equation**
The Duffin equation is given by the formula $\ddot y = -y - \varepsilon(y^3 + 2\mu\dot y)$, $\varepsilon, \mu > 0$, $\varepsilon$ small. By introducing a control term and transforming the second order equation in a first order system, we have:

$$\dot x_1 = x_2 \tag{19}$$

$$\dot x_2 = -x_1 - \varepsilon(x_1^3 + 2\mu x_2) + u . \tag{20}$$

From this form it is clear that $f(x) = -x_1 - \varepsilon(x_1^3 + 2\mu x_2)$.
Consider the trajectory $\gamma^+$. It starts with tangent vector $(0, 1)$, then, from (19), we see that it

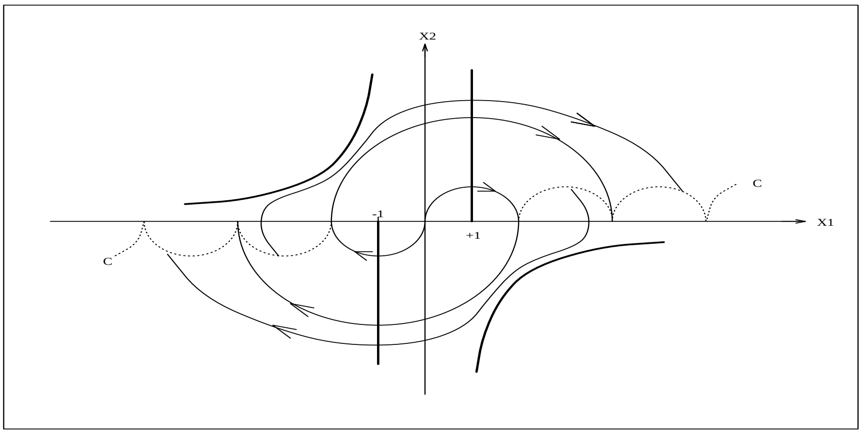


Figure 4: The synthesis for the Van Der Pol equation.

moves in the orthant $\Omega := \{(x_1, x_2),\ x_1, x_2 > 0\}$. To know the shape of the synthesis we need to know where $(F + G)_2(x) = 0$. If we set $a = \frac{1}{2\varepsilon\mu}$, this happens where

$$x_2 = a(1 - x_1 - \varepsilon x^3)\,. \tag{21}$$

From (19) and (20) we see that, after meeting this curve, the trajectory moves with $\dot\gamma_1^+ > 0$ and $\dot\gamma_2^+ < 0$. Then it meets the $x_1$-axis because otherwise if $\gamma^+(t) \in \Omega$ we necessarily have (for $t \to \infty$) $\gamma_1^+ \to \infty$, $\dot\gamma_2^+ \to 0$, that is not permitted by (20). The behavior of the trajectory $\gamma^-$ is similar.

In this case, the numbers $(n, m)$ are clearly $(\infty, \infty)$ because the $+1$ trajectory that starts at $z_1$ meets the curve (21) exactly one time and behaves like $\gamma^+$. So the $C$-curve that starts at $y_1$ meets again the $x_1$ axis. The same happens for the $-1$ curve that starts at $y_1$. In this way an infinite sequence of $y_i$ and $z_i$ is generated.

**Van der Pol equation**

The Van der Pol equation is given by the formula $\ddot y = -y + \epsilon(1-y^2)\dot y + u$, $\varepsilon > 0$ and small. The associated control system is: $\dot x_1 = x_2$, $\dot x_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$. We have $(F + G)_2(x) = 0$ on the curves $x_2 = -\frac{1}{\varepsilon(x_1+1)}$ for $x_1 \neq \pm 1$, $x_1 = -1$. After meeting these curves, the $\gamma^+$ trajectory moves with $\dot\gamma_1^+ > 0$ and $\dot\gamma_2^+ < 0$ and, for the same reason as before, meets the $x_1$-axis. Similarly for $\gamma^-$. As in the Duffin equation, we have that $m$ and $n$ are equal to $+\infty$. But here, starting from the origin, we cannot reach the regions: $\left\{(x_1, x_2) : x_1 < -1, x_2 \geq -\frac{1}{\varepsilon(x_1+1)}\right\}$, $\left\{(x_1, x_2) : x_1 > -1, x_2 \leq -\frac{1}{\varepsilon(x_1-1)}\right\}$ (see fig. 4).

**Another example**

In the following we will study an equation whose synthesis has $n, m < \infty$. Consider the equa-

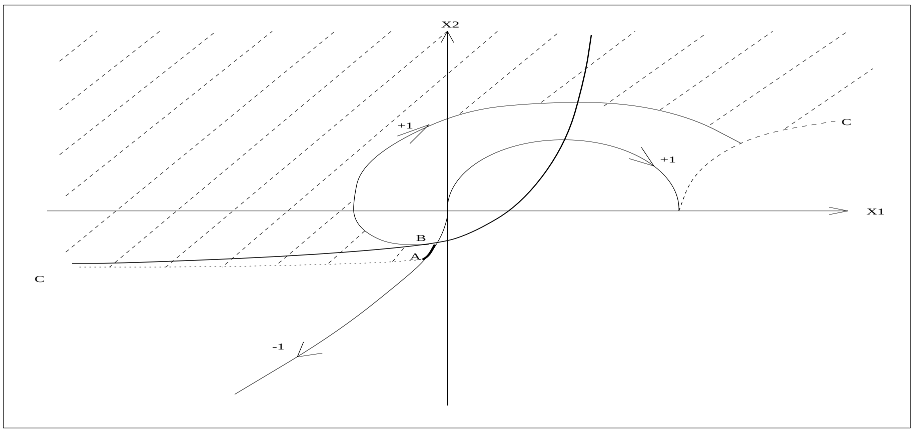


Figure 5: The synthesis for the control problem (22), (23). The sketched region is reached by curves that start from the origin with control $-1$ and then switch to $+1$ control between the points $A$ and $B$.

tion: $\ddot{y} = -e^y + \dot{y} + 1$. The associated control system is:

$$\dot{x}_1 = x_2 \tag{22}$$

$$\dot{x}_2 = -e^{x_1} + x_2 + 1 + u \tag{23}$$

We have $\dot{\gamma}_2^+ = 0$ on the curve $x_2 = e^{x_1} - 2$. After meeting this curve, the $\gamma^+$ trajectory meets the $x_1$-axis.

Now the synthesis has a different shape because the trajectories corresponding to control $-1$ satisfy $\dot{\gamma}_2 = 0$ on the curve:

$$x_2 = e^{x_1} \tag{24}$$

that is contained in the half plane $\{(x_1, x_2) : x_2 > 0\}$. Hence $\gamma^-$ never meets the curve given by (24) and this means that $m = 0$. Since we know that $n$ is at least 1, by Remark 4, we have $n = 1$, $m = 0$. The synthesis is drawn in fig. 5.

### 5. Optimal syntheses for Bolza Problems

Quite easily we can adapt the previous program to obtain information about the optimal syntheses associated (in the previous sense) to second order differential equations, but for more general minimizing problems.

We have the well known:

LEMMA 7. *Consider the control system:*

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad F, G \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2), \quad F(0) = 0, \quad |u| \le 1\,. \tag{25}$$

*Let* $L : \mathbb{R}^2 \to \mathbb{R}$ *be a* $\mathcal{C}^3$ *bounded function such that there exists* $\delta > 0$ *satisfying* $L(x) > \delta$ *for any* $x \in \mathbb{R}^2$.
*Then, for every* $x_0 \in \mathbb{R}^2$*, the problem:* $\min \int_0^\tau L(x(t))\, dt$ *s.t.* $x(0) = 0$, $x(\tau) = x_0$, *is equivalent to the* minimum time problem *(with the same boundary conditions) for the control system* $\dot{x} = F(x)/L(x) + uG(x)/L(x)$.

By this lemma it is clear that if we have a second order differential equation with a bounded-external force $\ddot{y} = f(y, \dot{y}) + u$, $f \in \mathcal{C}^3(\mathbb{R}^2)$, $f(0, 0) = 0$, $|u| \leq 1$, then the problem of reaching a point in the configuration space $(y_0, v_0)$ from the origin, minimizing $\int_0^\tau L(y(t), \dot{y}(t))\, dt$, (under the hypotheses of Lemma 7) is equivalent to the minimum time problem for the system: $\dot{x}_1 = x_2/L(x)$, $\dot{x}_2 = f(x)/L(x) + 1/L(x)u$. By setting: $\alpha : \mathbb{R}^2 \to ]0, 1/\delta[$, $\alpha(x) := 1/L(x)$, $\beta : \mathbb{R}^2 \to \mathbb{R}$, $\beta(x) := f(x)/L(x)$, we have: $F(x) = (x_2\alpha(x), \beta(x))$, $G(x) = (0, \alpha(x))$. From these it follows: $\Delta_A(x) = x_2\alpha^2$, $\Delta_B(x) = \alpha^2(\alpha + x_2\partial_2\alpha)$.
The equations defining turnpikes are: $\Delta_A \neq 0$, $\Delta_B = 0$, that with our expressions become the differential condition $\alpha + x_2\partial_2\alpha = 0$ that in terms of $L$ is:

$$L(x) - x_2\partial_2 L(x) = 0 \tag{26}$$

REMARK 4. Since $L > 0$ it follows that the turnpikes never intersect the $x_1$-axis. Since (26) depends on $L(x)$ and not on the control system, all the properties of the turnpikes depend only on the Lagrangian.

Now we consider some particular cases of Lagrangians.

**L**=$L(y)$ In this case the Lagrangian depends only on the position $y$ and not on the velocity $\dot{y}$ (i.e. $L = L(x_1)$). (26) is never satisfied so there are no turnpikes.

**L**=$L(\dot{y})$ In this case the Lagrangian depends only on velocity and the turnpikes are horizontal lines.

**L**=$V(y) + \frac{1}{2}\dot{y}^2$ In this case we want to minimize an energy with a kinetic part $\frac{1}{2}\dot{y}^2$ and a positive potential depending only on the position and satisfying $V(y) > 0$. The equation for turnpikes is $(x_2)^2 = 2V(x_1)$.

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**AMS Subject Classification: ???.**

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