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EXISTENCE OF SOLUTIONS FOR SOME NONLINEAR BEAM EQUATIONS *

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Abstract: We study the existence of solutions for some nonlinear ordinary differential equations under a nonlinear boundary condition which arise on beam theory. Assuming suitable conditions we prove the existence of at least one solution applying topological methods.

1 – Introduction

This work is devoted to the study of the existence of solutions for some nonlinear ordinary differential equations under a nonlinear boundary condition. In 1995 Rebelo and Sanchez [9] have considered the second order problem

$$\begin{cases} u'' + g(t,u) = 0 \qquad 0 < t < T \\ u'(0) = -f(u(0)) \\ u'(T) = f(u(T)) \end{cases} \tag{1}$$

with $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ for g satisfying a sign condition or either nondecreasing with respect to u, and $f \in C(\mathbb{R},\mathbb{R})$ continuous and strictly nondecreasing. This equation may be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam, with two nonlinear elastic springs acting at the extremities according to the law $u'(0) = -f(u(0))$, $u'(\pi) = f(u(\pi))$, and the total force exerted by the nonlinear spring undergoing the displacement u given by $g(t,u)$.

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On the other hand, the following fourth order problem for the deflection of a beam resting on elastic bearings was considered, among other authors, by Grossinho and Ma (see [3], [6], and also [4] for asymmetric boundary conditions):

$$\begin{cases} u^{(4)} + g(t,u) = 0 \qquad 0 < t < T \\ u''(0) = u''(T) = 0 \\ u'''(0) = -f(u(0)) \\ u'''(T) = f(u(T)) \ . \end{cases} \tag{2}$$

In section 2 we study (1) for $g = g(t,u,u')$. We remark that in this more general situation the problem is no longer variational; for this reason we shall apply instead topological methods. On the other hand, in order to find a priori bounds for the derivative we shall assume as in [2] the following Nagumo type condition:

$$|g(t,u,v)| \le \psi(|v|) \qquad \forall\, (t,u,v) \in \mathcal{E} \ . \tag{3}$$

Here $\mathcal{E}$ is a subset of $[0,T] \times \mathbb{R}^2$ to be specified, and $\psi : [0,+\infty) \to (0,+\infty)$ is a continuous function satisfying the inequality

$$\int_r^M \frac{1}{\psi(s)}\, ds > T$$

for some constants M and r to be specified. Under these assumptions we shall prove the existence of solutions by the method of upper and lower solutions.

Moreover, in section 3 we obtain an existence result under Landesman–Lazer type conditions (see e.g. [8]) applying topological degree methods [7].

Finally, in section 4 we consider the fourth order problem (2) for $g = g(t,u,u', u'',u''')$. More precisely, we prove the existence of symmetric solutions, i.e. such that $u(t) = u(T-t)$, under appropriate Landesman–Lazer and Nagumo type conditions.

2 – The second order case. Upper and lower solutions

In this section we prove an existence result for the following second order problem:

$$\begin{cases} u'' + g(t,u,u') = 0 \quad 0 < t < T \\ u'(0) = -f(u(0)) \\ u'(T) = f(u(T)) \ . \end{cases} \tag{4}$$

We shall assume the existence of an ordered couple of a lower and an upper solution. Namely, we shall assume there exist $\alpha,\beta:[0,T]\to\mathbb{R}$ such that $\alpha(t)\le\beta(t)$,

$$\alpha''(t)+g(t,\alpha,\alpha')\ \ge\ 0\ , \tag{5}$$

$$\beta''(t)+g(t,\beta,\beta')\ \le\ 0\ , \tag{6}$$

and

$$\begin{cases}\alpha'(0)\ge -f(\alpha(0)), & \alpha'(T)\le f(\alpha(T))\\ \beta'(0)\le -f(\beta(0)), & \beta'(T)\ge f(\beta(T))\ .\end{cases} \tag{7}$$

In this context, set

$$r\ =\ \min\left\{\max\left\{\frac{|\alpha(0)-\beta(T)|}{T},\frac{|\alpha(T)-\beta(0)|}{T}\right\},\ \max_{\alpha(0),\alpha(T)\le s\le\beta(0),\beta(T)}|f(s)|\right\}\ ,$$

fix a constant $M>r$ such that

$$M\ \ge\ \max\Big\{\|\alpha'\|_{C([0,T]}, \ \|\beta'\|_{C([0,T]}\Big\}$$

and define

$$\mathcal{E}\ =\ \Big\{(t,u,v)\in[0,T]{\times}\mathbb{R}^2\colon\ \alpha(t)\le u\le\beta(t),\ |v|\le M\Big\}\ .$$

Theorem 2.1. *With the previous notations, assume there exists an ordered couple of a lower and an upper solution of (4). Furthermore, assume that g satisfies the Nagumo condition (3). Then the boundary value problem (4) admits at least one solution u, with*

$$\alpha(t)\le u(t)\le\beta(t)\ ,\qquad |u'(t)|<M\quad\forall\,t\in[0,T]\ .$$

Proof: Set $\lambda>0$ and consider the functions $P:[0,T]\times\mathbb{R}\to\mathbb{R}$, $Q:\mathbb{R}\to\mathbb{R}$ given by

$$P(t,x)=\begin{cases}x & \alpha(t)\le x\le\beta(t)\\ \beta(t) & x>\beta(t)\\ \alpha(t) & x<\alpha(t)\ ,\end{cases}$$

$$Q(x)=\begin{cases}x & -M\le x\le M\\ M & x>M\\ -M & x<-M\ .\end{cases}$$

We define a compact fixed point operator $\phi : C^1([0,T]) \to C^1([0,T])$ in the following way: for each $v \in C^1([0,T])$, let $u = \phi(v)$ be the unique solution of the linear Neumann problem

$$u'' - \lambda u = g\big(t, P(t,v), Q(v')\big) - \lambda P(t,v) ,$$

$$u'(0) = -f\big(P(0, v(0))\big) , \qquad u'(T) = f\big(P(T, v(T))\big) .$$

By standard results, ϕ is well defined and compact. Moreover, multiplying the previous equation by u it follows that

$$-\int_0^T (u'' - \lambda u)\, u \;\le\; C\, \|u\|_{L^2}$$

for some constant C. Hence

$$\|u'\|^2_{L^2} + \lambda \|u\|^2_{L^2} \;\le\; C\, \|u\|_{L^2} + f\big(P(T, v(T))\big)\, u(T) + f\big(P(0, v(0))\big)\, u(0) ,$$

and it follows that $\|u\|_{H^1} \le C$ for some constant C. We conclude that $\|u\|_{C^1} \le C$ for some constant C, and by a straightforward application of Schauder Theorem it follows that ϕ has a fixed point u. We claim that

$$\alpha(t) \le u(t) \le \beta(t) , \qquad |u'(t)| < M \quad \forall\, t \in [0,T] ,$$

and hence u is a solution of the problem. Indeed, if for example $(u-\beta)(t_0) > 0$ for some $t_0 \in (0,T)$ maximum, then $P(t_0, u(t_0)) = \beta(t_0)$, $u'(t_0) = \beta'(t_0)$, and

$$\begin{aligned}(u-\beta)''(t_0) - \lambda(u-\beta)(t_0) \;\ge\;& g\big(t_0, P(t_0, u(t_0)), Q(u'(t_0))\big) - \lambda\, P(t_0, u(t_0)) \\ &-\Big[g\big(t_0, \beta(t_0), \beta'(t_0)\big) - \lambda\beta(t_0)\Big] \;=\; 0 ,\end{aligned}$$

a contradiction. Now, if $u-\beta$ attains an absolute positive maximum for example at $t=0$, then $(u-\beta)'(0) \le 0$. Moreover, as $P(0,u(0)) = \beta(0)$ we deduce that $(u-\beta)'(0) = -f(P(0,u(0))) - \beta'(0) \ge 0$, and hence $(u-\beta)'(0) = 0$. On the other hand, in a neighborhood of 0 we have that $u(t) > \beta(t)$ and then

$$\begin{aligned}(u-\beta)'' - \lambda(u-\beta) \;\ge\;& g\big(t, P(t,u), Q(u')\big) - \lambda P(t,u) - \big[g(t,\beta,\beta') - \lambda\beta\big] \\ =\;& g(t,\beta,Q(u')) - g(t,\beta,\beta') .\end{aligned}$$

As $u'(0) = \beta'(0) \in [-M, M]$, the right-hand term vanishes at $t=0$, meanwhile $u(0) - \beta(0) > 0$. It follows that $(u-\beta)'' \ge \lambda(u-\beta) + g(t,\beta,Q(u')) - g(t,\beta,\beta') > 0$ in $(0,\delta)$ for some $\delta > 0$, which contradicts the fact that 0 is an absolute maximum of $u-\beta$. In the same way, it follows that $u-\beta$ cannot attain a positive absolute maximum at T. We deduce in a similar way that $u(t) \geq \alpha(t)$ for every $t \in [0,T]$.

Next, assume for example that $u'(t_0) = M$ for some t_0.

If $r = \max_{\alpha(0),\alpha(T) \leq s \leq \beta(0),\beta(T)} |f(s)|$, then $|u'(0)|, |u'(T)| \leq r$; otherwise there exists $\tilde{t}$ such that

$$u'(\tilde{t}) \;=\; \frac{u(T)-u(0)}{T} \;\leq\; \frac{\beta(T)-\alpha(0)}{T} \;\leq\; r\;.$$

In both cases, we deduce the existence of t_1 such that $u'(t_1) = r$. We may assume that $r < u'(t) < M$ for any t between t_1 and t_0, and hence

$$T \;<\; \int_r^M \frac{1}{\psi(s)}\,ds \;=\; \int_{t_1}^{t_0} \frac{u''(t)}{\psi(u'(t))}\,dt \;\leq\; \left|\int_{t_1}^{t_0} \frac{g(t,u,u')}{\psi(u'(t))}\,dt\right| \;\leq\; |t_0-t_1|\;,$$

a contradiction. The proof is analogous if $u'(t_0) = -M$. ∎

Remark 2.2. In particular, the conditions of the previous theorem hold if there exist two constants $\alpha < \beta$ such that

$$g(t,\alpha,0) \;\geq\; 0 \;\geq\; g(t,\beta,0)$$

and

$$f(\alpha) \;\geq\; 0 \;\geq\; f(\beta)$$

provided that g satisfies $|g(t,u,v)| \leq \psi(|v|)$ for $\alpha \leq u \leq \beta$, $|v| < M$ and $\int_0^M \frac{1}{\psi(s)}\,ds > T$. □

Remark 2.3. When f is nondecreasing, a more general result is proved in [1]. □

3 – Landesman–Lazer type conditions

In this section we prove the existence of solutions of (4) under Landesman–Lazer type conditions. We shall assume that f is one-side globally bounded, i.e. $f \leq r$ or $f \geq -r$ for some positive constant r, and that g satisfies the Nagumo condition (3) over the set

$$\mathcal{E} \;=\; \Big\{(t,u,v) \in [0,T]\times\mathbb{R}^2\colon\; |v| \leq M\Big\}$$

for some $M > r$.

Moreover, we shall assume that the limits

$$\limsup_{u\to\pm\infty} g(t,u,v) \;:=\; g_s^{\pm}(t)$$

and

$$\liminf_{u\to\pm\infty} g(t,u,v) \;:=\; g_i^{\pm}(t)$$

exist, and that they are uniform for $|v|<M$. We also define the (possibly infinite) quantities

$$\limsup_{u\to\pm\infty} f(u) \;:=\; f_s^{\pm}$$

and

$$\liminf_{u\to\pm\infty} f(u) \;:=\; f_i^{\pm}\;.$$

Then we have:

Theorem 3.1. *Under the previous assumptions, problem (4) admits at least one solution, provided that one of the following conditions holds:*

$$2\,f_s^{+} + \int_0^T g_s^{+}(t)\,dt \;<\; 0 \;<\; 2\,f_i^{-} + \int_0^T g_i^{-}(t)\,dt \tag{8}$$

$$2\,f_s^{-} + \int_0^T g_s^{-}(t)\,dt \;<\; 0 \;<\; 2\,f_i^{+} + \int_0^T g_i^{+}(t)\,dt\;. \tag{9}$$

Remark 3.2. Conditions of this kind are known in the literature as Landesman–Lazer type conditions after the pioneering paper of E. Landesman and A. Lazer [5]. In particular, taking $f=0$ in Theorem 3.1 we obtain standard Landesman–Lazer conditions for the Neumann problem. □

For the sake of completeness, we summarize the main aspects of coincidence degree theory. Let $\mathbb{V}$ and $\mathbb{W}$ be real normed spaces, $L:\mathrm{Dom}(L)\subset\mathbb{V}\to\mathbb{W}$ a linear Fredholm mapping of index 0, and $N:\mathbb{V}\to\mathbb{W}$ continuous. Moreover, set two continuous projectors $\pi_{\mathbb{V}}:\mathbb{V}\to\mathbb{V}$ and $\pi_{\mathbb{W}}:\mathbb{W}\to\mathbb{W}$ such that $\mathrm{R}(\pi_{\mathbb{V}})=\mathrm{Ker}(L)$ and $\mathrm{Ker}(\pi_{\mathbb{W}})=\mathrm{R}(L)$, and an isomorphism $J:\mathrm{R}(\pi_{\mathbb{W}})\to\mathrm{Ker}(L)$. It is readily seen that

$$L_{\pi_{\mathbb{V}}} \;:=\; L|_{\mathrm{Dom}(L)\cap\mathrm{Ker}(\pi_{\mathbb{V}})}:\; \mathrm{Dom}(L)\cap\mathrm{Ker}(\pi_{\mathbb{V}}) \;\to\; \mathrm{R}(L)$$

is one-to-one; denote its inverse by $K_{\pi_{\mathbb{V}}}$. If Ω is a bounded open subset of $\mathbb{V}$, N is called L-compact on Ω if $\pi_{\mathbb{W}}N(\Omega)$ is bounded and $K_{\pi_{\mathbb{V}}}(I-\pi_{\mathbb{W}})N:\Omega\to\mathbb{V}$ is compact.

The following continuation theorem is due to Mawhin [7]:

Theorem 3.3. *Let L be a Fredholm mapping of index zero and N be L-compact on a bounded domain $\Omega\subset\mathbb{V}$. Suppose that:*

1. *$Lx\neq\lambda Nx$ for each $\lambda\in(0,1]$ and each $x\in\partial\Omega$.*
2. *$\pi_{\mathbb{W}}Nx\neq 0$ for each $x\in Ker(L)\cap\partial\Omega$.*
3. *$d(J\pi_{\mathbb{W}}N,\Omega\cap Ker(L),0)\neq 0$, where d denotes the Brouwer degree.*

Then the equation $Lx=Nx$ has at least one solution in $Dom(L)\cap\Omega$. ∎

Proof of Theorem 3.1: Set $\mathbb{V}=C^1([0,T])$, $\mathbb{W}=L^2(0,T)\times\mathbb{R}^2$, and the operators $L:H^2(0,T)\to\mathbb{W}$, $N:\mathbb{V}\to\mathbb{W}$ given by

$$Lu=\big(u'',u'(0),u'(T)\big)\ ,\qquad Nu=\big(-g(\cdot,u,u'),-f(u(0)),f(u(T))\big)\,.$$

It is easy to verify that

$$\mathrm{Ker}(L)=\mathbb{R}\ ,\qquad \mathrm{R}(L)=\left\{(\varphi,A,B)\in\mathbb{W}\colon\ \overline{\varphi}=\frac{B-A}{T}\right\}\,,$$

where $\overline{\varphi}$ denotes the usual average given by $\overline{\varphi}=\frac{1}{T}\int_0^T\varphi(t)\,dt$. Then, we may define $\pi_{\mathbb{V}}(X)=\overline{u}$, $\pi_{\mathbb{W}}(\varphi,A,B)=(\overline{\varphi}-\frac{B-A}{T},0,0)$, and $J:\mathrm{R}(\pi_{\mathbb{W}})\to\mathbb{R}$ given by $J(C,0,0)=C$. In this case, for $(\varphi,A,B)\in\mathrm{R}(L)$, the function $U=K_{\pi_{\mathbb{V}}}(\varphi,A,B)$ is defined as the unique solution of the problem

$$U''=\varphi\ ,\qquad U'(0)=A,\quad U'(T)=B$$

that satisfies $\overline{U}=0$. Writing $U'(t)=A+\int_0^t\varphi$ and using Wirtinger inequality, L-compactness of N follows.

We claim there exists a constant R such that if $Lu=\lambda Nu$ with $0<\lambda\leq 1$ then $\|u\|_{C^1}\leq R$. Indeed, suppose by contradiction that $Lu_n=\lambda_nNu_n$, with $0<\lambda_n\leq 1$ and $\|u_n\|_{C^1}\to\infty$. As $u_n''=-\lambda_ng(t,u_n,u_n')$ and $u_n'(0)=-\lambda_nf(u_n(0))$, $u_n'(T)=\lambda_nf(u_n(T))$, by the Nagumo condition and using the fact that

$$\min\big\{u_n'(0),u_n'(T)\big\}\leq r\qquad\text{and}\qquad\max\big\{u_n'(0),u_n'(T)\big\}\geq -r\ ,$$

it follows as in the previous section that $\|u_n'\|_{C([0,T])}<M$ for every n. Hence $\|u_n\|_{C([0,T])}\to\infty$, and $\|u_n-\overline{u}_n\|_{C([0,T])}\leq C$ for some constant C. Taking

a subsequence, assume for example that $\overline{u}_n \to +\infty$ and that (8) holds; then integrating the equation we obtain the equality

$$f(u_n(T)) + f(u_n(0)) \;=\; -\int_0^T g(t, u_n, u_n')\,dt\;,$$

and thus

$$0 \;\le\; \limsup_{n\to\infty} f(u_n(T)) + \limsup_{n\to\infty} f(u_n(0)) + \int_0^T g_s^+(t)\,dt \;<\; 0$$

a contradiction. The proof is similar for the other cases; hence, taking $\Omega = B_R(0)$ for R large enough, the first condition in Theorem 3.3 is fulfilled.

Further, the function $J\pi_{\mathbb{W}}N|_{\overline{\Omega}\cap\mathrm{Ker}(L)} = [-R, R]$ is given by

$$J\pi_{\mathbb{W}}N(s) \;=\; -\frac{1}{T}\left(\int_0^T g(t,s,0)\,dt \;+\; 2\,f(s)\right)\;,$$

and in the same way as before it follows that for R large enough

$$J\pi_{\mathbb{W}}N(R)\,J\pi_{\mathbb{W}}N(-R) \;<\; 0\;.$$

Thus, $\deg\big(J\pi_{\mathbb{W}}N,\,\Omega\cap\mathrm{Ker}(L),\,0\big) = \pm 1$, and the proof is complete. ∎

4 – Symmetric solutions for the general fourth order case

In this section we study the existence of symmetric solutions for the problem

$$\text{(10)} \qquad \begin{cases} u^{(4)} + g(t,u,u',u'',u''') = 0 \qquad 0<t<T \\ u''(0) = u''(T) = 0 \\ u'''(0) = -f(u(0)) \\ u'''(T) = f(u(T))\;. \end{cases}$$

We shall assume that g is symmetric with respect to t, namely:

$$\text{(11)} \qquad g(t,u,v,w,x) \;=\; g(T-t,u,v,w,x)\;.$$

Our Nagumo condition for this problem reads:

$$\text{(12)} \qquad \big|g(t,u,v,w,x)\big| \le \psi(|x|) \qquad \forall\,(t,u,v,w,x)\in\mathcal{E}$$

with $\mathcal{E} = [0,T]\times\mathbb{R}^3\times[-M,M]$, and $\psi : [0,+\infty) \to (0,+\infty)$ continuous, with

$$\int_0^M \frac{1}{\psi(s)}\,ds > T\ .$$

Moreover, assume that the limits

$$\limsup_{s\to\pm\infty} g(t,s,v,w,x) := g_s^\pm(t)$$

and

$$\liminf_{s\to\pm\infty} g(t,s,v,w,x) := g_i^\pm(t)$$

exist, and that they are uniform over the set

$$\mathcal{C} = \left\{(v,w,x)\in\mathbb{R}^3\colon\ |v|<\frac{T^2}{4}M,\ |w|<\frac{T}{2}M\ \text{ and }\ |x|<M\right\}.$$

The quantities $f_s^\pm$ and $f_i^\pm$ are defined as before. Then we have:

Theorem 4.1. *Under the previous assumptions, problem (10) admits at least one symmetric solution, provided that one of the conditions (8) or (9) holds.*

Proof: We proceed as in the proof of Theorem 3.1. Let

$$\mathbb{V} = \left\{u\in C^3([0,T])\colon\ u(t)=u(T-t),\ u''(0)=0\right\}\ ,$$

$$\mathbb{W} = \left\{u\in L^2(0,T)\colon\ u(t)=u(T-t)\right\}\times\mathbb{R}$$

and define the operators $L : H^4(0,T)\cap\mathbb{V}\to\mathbb{W}$, $N : \mathbb{V}\to\mathbb{W}$ by

$$Lu = \big(u^{(4)}, u'''(0)\big)\ ,\qquad Nu = -\big(g(\cdot,u,u',u'',u'''), f(u(0))\big)\,.$$

Again, it is easy to verify that

$$\mathrm{Ker}(L)=\mathbb{R}\ ,\qquad \mathrm{R}(L) = \left\{(\varphi,c)\in\mathbb{W}:\int_0^T \varphi(t)\,dt\,+\,2c\,=\,0\right\}.$$

Then, we may define $\pi_{\mathbb{V}}(u)=\overline{u}$, $\pi_{\mathbb{W}}(\varphi,c)=(\overline{\varphi}+2c,0)$, and $J:\mathrm{R}(\pi_{\mathbb{W}})\to\mathbb{R}$ given by $J(C,0)=C$. For $(\varphi,c)\in\mathrm{R}(L)$, the function $U=K_{\pi_{\mathbb{V}}}(\varphi,c)$ is defined as the unique solution of the problem

$$\begin{cases} U^{(4)}=\varphi \\ U''(0)=0,\ \ U'''(0)=c \\ U(t)=U(T-t) \\ \overline{U}=0\ . \end{cases}$$

As before, it is easy to prove that N is L-compact. Next, if $Lu_n = \lambda_n N u_n$, with $0 < \lambda_n \leq 1$ and $\|u_n\|_{C^3} \to \infty$, by the Nagumo condition and using the fact that $u_n'''(\frac{T}{2}) = 0$, it follows that $\|u_n'''\|_{C([0,T])} < M$ for every n. Moreover, for $t \leq \frac{T}{2}$ we have:

$$|u_n''| \leq \int_0^t |u_n'''| < \frac{T}{2} M$$

and

$$|u_n'| \leq \left| \int_t^{\frac{T}{2}} u_n'' \right| < \frac{T^2}{4} M \ .$$

As u_n is symmetric, we conclude that $(u_n'(t), u_n''(t), u_n'''(t)) \in \mathcal{C}$ for every $t \in [0,T]$. Then $\|u_n\|_{C([0,T])} \to \infty$, and $\|u_n - \overline{u}_n\|_{C^3([0,T])} \leq C$ for some constant C. The rest of the proof follows as in the second order case. ∎

Some examples and remarks

Example 4.2. As an example of Theorem 4.1 we may consider a symmetric function g such that

$$g(t,u,v,w,x) = g_0(t,u) + \gamma(u)\, g_1(t,u,v,w,x) \ ,$$

where g_0 is bounded, $|g_1(t,u,v,w,x)| \leq A + B|x|$ and $\gamma(u) \to 0$ as $|u| \to \infty$.

Then $|g(t,u,v,w,x)| \leq C + D|x|$ for some positive constants C and D and the Nagumo condition is satisfied taking $\psi(x) = C + Dx$ and M large enough. Moreover,

$$\limsup_{u\to\pm\infty} g_0(t,u) = g_s^{\pm}(t) \ , \qquad \liminf_{u\to\pm\infty} g_0(t,u) = g_i^{\pm}(t) \ ,$$

and the assumptions of Theorem 4.1 are fulfilled if (8) or (9) holds. For example, it suffices to assume that

$$\lim_{|u|\to\infty} f(u)\ \mathrm{sgn}(u) = +\infty \quad \text{ or } \quad \lim_{|u|\to\infty} f(u)\ \mathrm{sgn}(u) = -\infty \ . \ \square$$

Remark 4.3. In the situation of Theorem 4.1, if $g_s^{\pm} = g_i^{\pm} := g^{\pm}$ and $f_s^{\pm} = f_i^{\pm} := f^{\pm}$, integrating the equation it follows that if for example

$$g^+(t) \leq g \leq g^-(t) \quad \text{ and } \quad f^+ < f < f^-$$

or

$$g^-(t) \leq g \leq g^+(t) \quad \text{ and } \quad f^- < f < f^+$$

then the respective conditions (8) and (9) are also necessary. □

Remark 4.4. The Nagumo condition (12) can be dropped if we assume that g has a linear growth of the type

$$|g(t,u,v,w,x)| \;\le\; A + B|u| + C|v| + D|w| + E|x|$$

(with B, C, D and E small enough), and that the limits $g_i^{\pm}$ and $g_s^{\pm}$ are uniform on $\mathbb{R}^3$. Indeed, in this case if $Lu_n = \lambda_n N u_n$, with $0 < \lambda_n \le 1$, then using the fact that $u_n''' = \lambda_n \int_{\frac{T}{2}}^{t} g(s, u_n, u_n', u_n'', u_n''')ds$, we deduce:

$$\left(1 - \frac{TE}{2}\right) \|u_n'''\|_{C([0,T])} \;\le\; \frac{T}{2}\left(A + B\|u_n\|_{C([0,T])} + C\|u_n'\|_{C([0,T])} + D\|u_n''\|_{C([0,T])}\right).$$

Integrating twice, as E, D and C are small, we obtain:

$$\|u_n'\|_{C([0,T])} \;\le\; \delta\big(A + B\|u_n\|_{C([0,T])}\big)$$

for some constant δ. By the mean value theorem, for $B < \delta$ we conclude that if for example $\overline{u}_n \to +\infty$ then $\inf_{t\in[0,T]} u_n(t) \to +\infty$, and the rest of the proof follows as before. In particular, for $g = g(t,u)$ it suffices to take $B < \frac{16}{T^4}$. □

Remark 4.5. In [3], Theorem 2, it is proved by variational methods that if $g = g(t,u)$ is symmetric on t, and f, $g(t,\cdot)$ are nondecreasing, then problem (10) admits a symmetric solution if and only if

$$2\,f(a) + \int_0^T g(t,a)\,dt \;=\; 0 \qquad \text{for some} \quad a \in \mathbb{R}\,.$$

By monotonicity, this condition is equivalent to (9), unless $f(u) \equiv f(a)$ and $g(t,u) \equiv g(t,a)$ for all $u \ge a$ or for all $u \le a$. Note that, in this last case, existence of solutions can be easily proved; thus, taking into account the previous remarks 4.3 and 4.4, when $|g(t,u)| \le A + B|u|$ (with $B < \frac{16}{T^4}$) we may conclude that Theorem 4.1 is essentially equivalent to Theorem 2 in [3].

Moreover, without the monotonicity condition the authors prove (see [3], Theorem 5) the existence of a symmetric solution of (10) for g and f sublinear, i.e.

$$\frac{g(t,u)}{u} \to 0 \qquad \text{as} \quad |u| \to \infty$$

uniformly in t, and

$$\frac{f(u)}{u} \to 0 \qquad \text{as} \quad |u| \to \infty\,,$$

assuming a growth condition for f and g, and that one of the following hypotheses holds:

i) $g(t,u) \to \pm\infty$ as $u \to \pm\infty$ uniformly in t and f bounded by below.

ii) $f(u) \to \pm\infty$ as $u \to \pm\infty$ and g bounded by below.

It is clear that the sublinearity condition implies that $|g(t,u)| \leq A + B|u|$ for some $B < \frac{16}{T^4}$ and some A, and that if i) or ii) holds then the second inequality in condition (9) is fulfilled. Thus, some cases of Theorem 5 in [3] are covered by Theorem 4.1; in particular, if f is bounded by above for $u < 0$ in i) or if g is bounded by above for $u < 0$ in ii).

However, the first inequality in (9) does not necessarily hold under assumptions i) or ii): one may consider for instance the (sublinear) functions $f(u) = |u|^{1/2}$ and $g(t,u) = u^{1/3}$. □

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