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ON A CLASS OF MONGE–AMPÉRE PROBLEMS WITH NON-HOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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Abstract: We assume in the plane that Ω is a strictly convex domain, with its boundary $\partial\Omega$ sufficiently regular. We consider the Monge–Ampére equations in its general form $\det u_{ij} = g(|\nabla u|^2)h(u)$, where u_{ij} denotes the Hessian of u, and g, h are some given functions. This equation is subject to the non-homogeneous Dirichlet boundary condition $u = f$. A sharp necessary condition of solvability for this equation is given using the maximum principle in $\mathbb{R}^2$. We note that this maximum principle is extended to the N-dimensional case and two different applications have been given to illustrate this principle.

1 – Introduction

Let u be a classical solution of the following Monge–Ampére equations

$$\det(u_{,ij}) = F(x, u, |\nabla u|^2) \quad \text{in} \ \ \Omega \ , \tag{1}$$

where Ω is assumed to be a bounded domain, strictly convex. In this note, we derive a new maximum principle for the general Monge–Ampére equations (1) with $F(x, u, |\nabla u|^2) = g(|\nabla u|^2)h(u)$ in $\mathbb{R}^N$, $N \geq 2$, which generalizes a recent result of Ma [11] (the particular case when $g.h = \text{const.}$ in Ω).

In order to prove this maximum principle, we assume in the sequel that the functions g and h are subject to some appropriate conditions. These conditions lead to some differential inequality, which will be investigated in Section 2.

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Then employing the second maximum principle of E. Hopf [10], we conclude that the corresponding maximum value is attained on the boundary $\partial\Omega$ of Ω.

For the first application, we shall treat the following non-homogeneous Dirichlet boundary condition

$$u = f \quad \text{on} \quad \partial\Omega\ , \tag{2}$$

where $\partial\Omega$ denotes the boundary of Ω sufficiently regular and f is a positive function of class C^1. Monge–Ampére equations in conjunction with Dirichlet and Neumann conditions were investigated in [1,2,4,5,6,7,8]. For the second application, we consider the particular Dirichlet case $f = 0$. Ma [11] showed that the combination $P = |\nabla u|^2 - 2\sqrt{c}u$ is a constant, u is radial and Ω is a ball. We extend this result for more general combination and prove for some particular values of g, that Ω is an N-ball and u is radial.

Some applications are given involving different situations, where various bounds for u and its gradient $|\nabla u|$ are obtained. The maximum principle for Monge–Ampére equations was already used by Ma [11, 12] and Safoui [13].

In the case of the Neumann boundary condition

$$\frac{\partial u}{\partial n} = \cos(\theta(x,u))\,(1+|\nabla u|^2)^{\frac{1}{2}} \quad \text{on} \quad \partial\Omega\ , \tag{3}$$

where $\mathbf{n}$ is the outward normal vector and the wetting angle θ is an element of $(0, \frac{\pi}{2})$, Ma in [11], proved the following result, by assuming that the bounded domain Ω is strictly convex, the constant c is positive and, the angle θ is an element of $(0, \frac{\pi}{2})$

Theorem 1. *Under the above hypotheses on Ω, c, θ_0, if u is a strictly convex solution of* (1), (3) *then the following relation is satisfied*

$$k_0 \le \max\Big\{c^{\frac{1}{2}}\cos(\theta_0),\ c^{\frac{1}{2}} tan(\theta_0)\Big\}\ , \tag{4}$$

where $k_0 := \min\limits_{x\in\partial\Omega} k(x)$ and $k(x)$ is the curvature of the boundary $\partial\Omega$ of Ω at x.

In the case when $F :=$ const. and $f(x) = 0$, he showed the following theorem (see [11])

Theorem 2. *Under the above hypotheses on the domain Ω and constant c, if u is a strictly convex solution for the boundary value problems (1)–(2) then we have the following estimates*

$$\max_{x\in\bar\Omega} |\nabla u|^2 \le \frac{c}{{k_0}^2}\ , \tag{5}$$

$$-\frac{\sqrt{c}}{2k_0} \leq u \leq 0 \quad \text{in } \bar{\Omega} , \tag{6}$$

where $k_0 := \min_{x\in\partial\Omega} k(x)$, $k(x)$ *is the curvature of* $\partial\Omega$ *at* x.

For the proof of Theorem 2, he used the maximum principle [9,10] in $\mathbb{R}^2$ for the following combination

$$\Phi := |\nabla u|^2 - 2\, c^{\frac{1}{2}} u , \tag{7}$$

and the expression of the Monge–Ampére equations (1) in normal coordinates (see Section 3, (40)).

The purpose of this paper is, firstly, to generalize this maximum principle in $\mathbb{R}^N$ for a general combination of the form

$$\Phi := g(|\nabla u|^2) + h(u) , \tag{8}$$

where g and h are supposed to be positive. Secondly, to consider a more general equation

$$\det(u_{,ij}) = g(|\nabla u|^2)\, h(u) \quad \text{in } \bar{\Omega} , \tag{9}$$

with non-homogeneous boundary condition (2). This generalization gives us an upper bound for u and its gradient $|\nabla u|$ in function of the geometry of Ω and the first and second derivatives of f.

Throughout the paper, we shall be concerned with a bounded domain Ω of $\mathbb{R}^N$, strictly convex. A comma will be used to denote differentiation. We make use the summation convention with repeated Latin indices running from 1 to N.

$$\begin{aligned} u_{,i} &:= \frac{\partial u}{\partial x_i} , \\ u_{,ij}u_{,ij} &:= \sum_{i=1}^{i=N}\sum_{j=1}^{j=N}\left[\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right]^2, \\ u_n &:= \frac{\partial u}{\partial n} , \\ u_s &:= \frac{\partial u}{\partial s} , \\ (u_s)_n &:= \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial s}\right) , \\ (u_s)_n &:= (u_n)_s - K u_s , \\ u_{nn} &:= \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n}\right) , \\ u_{ss} &:= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) . \end{aligned} \tag{10}$$

2 – On a maximum principle

Hereafter, we shall assume that the solution u of the Monge–Ampère equations defined by (8) is at least of class $C^2(\bar\Omega)\cap C^3(\Omega)$ in a bounded domain Ω described in Section 1. In this Section, we will show, that the maximum principle of the combination Φ defined by the following equation

$$\Phi := g(u_{,i}u_{,i}) + h(u) \quad \text{in } \bar\Omega , \tag{11}$$

attains its maximum value on the boundary $\partial\Omega$, where the functions h and g are subject to some conditions. For the differential equation of the form

$$\Delta u + f(u) = 0 \quad \text{in } \bar\Omega ,$$

the corresponding function constructed for this type of equation depends essentially on the dimension N and the imposed boundary conditions, for which in general the treatment in $\mathbb{R}^2$ differs from that of $\mathbb{R}^N$, where $N \geq 3$, since some differential equalities are valid in $\mathbb{R}^2$ and unfortunately not valid in $\mathbb{R}^N$, as

$$|\nabla u|^2 u_{,ij} u_{,ij} = |\nabla u|^2 (\Delta u)^2 + u_{,i} u_{,ik} u_{,j} u_{,jk} - 2(\Delta u) u_{,i} u_{,j} u_{,ij} .$$

It is already known that, the combination Φ attains its maximum principle at three different places for an arbitrary g and f (see R. Sperb [14]). Assuming that Φ is nonconstant, the corresponding maximum is attained on the boundary $\partial\Omega$ as first possibility, at a critical point as second possibility and finally at an interior point of the domain Ω. In our context, we choose g and h such that, the elliptic differential inequality formed is strictly positive.

Theorem 3. *Let u be a strictly convex solution of* (9) *and Φ the combination defined by* (8)*, then*

$$\frac{1}{2} u^{ij} \Phi_{,ij} + \cdots = g'(|\nabla u|^2)\left(-\frac{h'}{g} + \frac{h'}{h} - \frac{h' g''}{(g')^2} - \frac{h''}{h'}\right) + 2 g' \Delta u + N h' \tag{12}$$

where the dots stand for terms of the form $V_{,k}\Phi_{,k}$ with specific vector fields $V_{,k}$ which are bounded except at critical points of u.

To start the proof of Theorem 3, we construct an appropriate differential inequality for Φ, except at a critical value of the solution u. Let Φ defined by (8) then

$$\Phi_{,i} = 2 u_{,ik} u_{,k} g' + u_{,i} h' , \tag{13}$$

$$\Phi_{,ij} = 2 g'(u_{,ijk} u_{,k} + u_{,jk} u_{,ik}) + 4(u_{,jl} u_{,l} u_{,ik} u_{,k}) g'' + u_{,ij} h' + u_{,i} u_{,j} h'' . \tag{14}$$

Let u_{ij} be the inverse of the Hessian matrix $H := u^{ij}$. As u is strictly convex solution of (9), the matrix u^{ij} is positive definite and consequently by computing

$$u^{ij}\Phi_{,ij} = 2\,g'(u^{ij}u_{,ijk}\,u_{,k} + u^{ij}u_{,jk}\,u_{,ik}) + 4(u^{ij}u_{,jl}\,u_{,l}\,u_{,ik}\,u_{,k})\,g'' + u^{ij}u_{,ij}\,h' + u^{ij}u_{,i}\,u_{,j}h'' \,, \tag{15}$$

we claim that $u^{ij}\Phi_{,ij}$ is strictly positive in $\bar{\Omega}$. Knowing that the following identities $u^{ij}u_{,ij}\,u_{,jl} = \Delta u$, $u^{ij}u_{,ij} = N$, $u^{ij}u_{,il}\,u_{,l}\,u_{,jk}\,u_{,k} = u_{,kl}\,u_{,k}\,u_{,l}$ and $(gh)[u^{ij}u_{,ijk}\,u_{,k}] = (gh)_{,j}u_{,j}$ are valid in $\mathbb{R}^N$, then we are able to prove that Φ satisfies an appropriate differential inequality. For this, we compute

$$u^{ij}\Phi_{,i}\,u_{,j} = u^{ij}\{2\,u_{,j}u_{,ik}\,u_{,k}\,g' + u_{,i}u_{,j}h'\} \,, \tag{16}$$

$$u_i\Phi_{,i} = 2\,g'u_{,i}\,u_{,ik}\,u_{,k} + u_{,i}\,u_{,i}\,h' \,. \tag{17}$$

From (16) and (17), we obtain

$$-\,u^{ij}u_{,i}\,u_{,j}h' + u^{ij}\Phi_{,i}\,u_{,j} = 2\,g'u^{ij}u_{,j}\,u_{,ik}\,u_{,k} = 2\,u_{,i}\,u_{,i}\,g' \,, \tag{18}$$

$$2\,u_{,ij}\,u_{,j}\,u_{,i}\,g' - u_{,i}\,\Phi_{,i} = -u_{,i}\,u_{,i}\,h' \,. \tag{19}$$

Hence by (18) and (19), we conclude that

$$u^{ij}\Phi_{,ij} + \cdots = 2\,g'\Big(|\nabla u|^2\Big)\Big(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}\Big) + 2\,g'\Delta\,u + Nh' \,. \tag{20}$$

Using the following arithmetic-geometric inequality

$$\Delta\,u \ \geq\ N(gh)^{\frac{1}{N}}$$

(or simply $\Delta\,u > 0$ since g' is positive), we obtain

$$u^{ij}\Phi_{,ij} + \cdots \ \geq\ 2\,g'\Big(|\nabla u|^2\Big)\Big(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}\Big) + 2\,g'N(gh)^{\frac{1}{N}} + Nh' \,, \tag{21}$$

where g, h, g' and h' satisfy the following conditions

$$g' > 0\,, \qquad h' > 0 \,, \tag{22}$$

and

$$-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'} \ >\ 0 \,. \tag{23}$$

Then the maximum of Φ is attained on the boundary $\partial\Omega$ of Ω at some point P. If inequalities (22) and (23) are reversed, then we conclude that the minimum value of Φ occurs on the boundary $\partial\Omega$, or at a critical point of u.

We have then established the following theorem which extend the result of Ma [11,12] to the N dimensional case.

Theorem 4. *Let Φ be defined by (8) where g, h, g' and h' satisfy (22), (23) and u supposed to be strictly convex. Then the maximum principle of the combination Φ is attained on the boundary $\partial\Omega$ of Ω at some point P.*

3 – Estimates of the solution u and its gradient $|\nabla u|$ for the Dirichlet boundary condition

In this Section, we investigate in dimension 2 the following result which illustrates Theorem 4. The bounds obtained for u and its gradient $|\nabla u|$ seems appear for the first time in the non-homogeneous Dirichlet case.

Theorem 5. *We assume that u is a classical solution of the non-homogeneous Dirichlet problem (2), (9), strictly convex, at least of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$. Let Ω be a bounded domain, convex in $\mathbb{R}^2$. Then we have*

$$\max_{\bar{\Omega}} |\nabla u|^2 \le \frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\} , \tag{24}$$

$$-h(u_{\min}) + h(f) \le g\left(\frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\}\right) \quad \text{in } \bar{\Omega} . \tag{25}$$

For the proof of Theorem 5, we need to use in dimension 2 some differential equality valid in $\mathbb{R}^2$. This fact consists on the computation of the normal derivative of the combination Φ in function of the mean curvature K, the first u_s and second u_{ss} tangential derivatives of u in the plane. This result will be established as follows.

We begin by computing the normal derivative of Φ in $\mathbb{R}^2$

$$\begin{aligned}
\frac{\partial\Phi}{\partial n} &= 2\,g'\Big\{u_{,1}(u_{,11}n_1 + u_{,12}n_2) + u_{,2}(u_{,21}n_1 + u_{,22}n_2)\Big\} + h'u_n \\
&= 2\,g'u_n\Big\{\Delta u + u_{,2}\frac{\partial}{\partial s}u_{,1} - u_{,1}\frac{\partial}{\partial s}u_{,2}\Big\} + h'u_n \\
&= 2\,g'u_n\Big\{\Delta u + u_s u_{ns} - u_{ss} - K|\nabla u|^2\Big\} + h'u_n , \qquad (26)
\end{aligned}$$

where s denotes differentiation in the tangential direction on the boundary $\partial\Omega$ and K stands for curvature of $\partial\Omega$ at some point $\hat{P}$.

In the terms $\frac{\partial}{\partial s}u_{,1}$ and $\frac{\partial}{\partial s}u_{,2}$ we have broken $u_{,1}$ and $u_{,2}$ into normal and tangential derivative components and used the identities

$$\frac{\partial u_{,1}}{\partial s} = -Kn_2 \quad \text{and} \quad \frac{\partial u_{,2}}{\partial s} = Kn_1 . \tag{27}$$

Since the maximum of the combination Φ defined by (11) is attained on the boundary $\partial\Omega$ at $\hat{P}$, we must have

$$\begin{aligned} \frac{\partial \Phi}{\partial s}(\hat{P}) &= g' \frac{\partial}{\partial s}(|\nabla u|^2) + h' u_s \\ &= 2\, g'(u_n u_{ns} + u_s u_{ss}) + h' u_s = 0 \, . \end{aligned} \tag{28}$$

Now we need to use the differential equality (28) in order to eliminate the product $u_n u_{ns}$ in (26). In fact, involving (28) we deduce

$$u_n u_{ns} = \frac{h' u_s}{2g'} - u_s u_{ss} \, . \tag{29}$$

The Monge–Ampére equations (9) can be rewritten in $\mathbb{R}^2$ as

$$u_{nn}(K u_n + u_{ss}) = gh + [u_{sn}]^2 \, . \tag{30}$$

In this case, by using (26), (28), (29), and making use of the following inequality

$$u_{ns} = u_{sn} - K u_s \, , \tag{31}$$

we obtain

$$\max_{\bar{\Omega}} |\nabla u|^2 \le \frac{1}{K}\left\{ \frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2} {f_s}^2 + |f_{ss}| \right\} , \tag{32}$$

where M and $\tilde{M}$ are two positive bounds of Laplace u and the mixed derivative u_{sn} since u is assumed to be of class C^2.

For this last differential inequality (32), we have only considered the case when the normal derivative of the solution u is non-equal to zero. Since for the nullity case, we are conducted to the triviality of the solution u. We are concerned now with the estimation of the solution u, which will be illustrated by applying the statement of Theorem 4. We know that

$$- h(u) \le g(A) - h(u) + g(|\nabla u|^2) \, , \tag{33}$$

where A is defined by

$$A := \frac{1}{K}\left\{ \frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2} {f_s}^2 + |f_{ss}| \right\} , \tag{34}$$

At a critical point of u, we obtain

$$0 < -h(u_{\min}) \le g(A) - h(f) \, , \tag{35}$$

from which we deduce

$$-h(u_{\min}) + h(f) \le g(A) . \tag{36}$$

Finally, we have explicitly

$$-h(u_{\min}) + h(f) \le g\left(\frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\}\right) . \tag{37}$$

4 – On an over-determined Monge–Ampére problem

Ma in [11] proved in $\mathbb{R}^2$ the following result

Theorem 6. *Under the same hypothesis of* c, Ω *and* $u(x)$ *as in Theorem* 2, *if* $P(x) := |\nabla u|^2 - 2\sqrt{c}u$ *attains its maximum in* Ω, *then*

$$\Omega = B_R(0) , \tag{38}$$

$$u = \frac{\sqrt{c}\,(x_1^2 + x_2^2)}{2} - \frac{\sqrt{c}\,R^2}{2} , \tag{39}$$

$$P = c\,R^2 , \tag{40}$$

where R *is a positive constant.*

Our goal is to extend this result to the N-dimensional space for more general Monge–Ampére equations (9). In the next theorem, we establish our result.

Theorem 7. *We assume that* u *is a classical solution of* (2), (9) *with* $f = 0$.

If $\Phi = g(|\nabla u|^2) + h(u)$, *where* $gg' = 1$ *and* $h(0) > 0$, *attains its maximum on the boundary* $\partial\Omega$, *then*

$$\Omega = B_r(0) , \tag{41}$$

$$u = \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}} \Big(|x|^2 - h(0)\Big)^{\frac{N-1}{2}} - \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}} \Big(r^2 - h(0)\Big)^{\frac{N-1}{2}} , \tag{42}$$

$$\Phi = \left(\frac{2h(0)}{h'(0)K}\right)^{\frac{2}{N-1}} + h(0) = const. \quad on \ \ \partial\Omega , \tag{43}$$

where r *is a positive constant.*

In order to establish this statement, we compute the normal and tangential derivatives of Φ and we use the fact that its maximum is attained on the boundary $\partial\Omega$, we then obtain

$$\frac{\partial \Phi}{\partial n} \;=\; 2\, g'\Big(u_n u_{nn} + (\nabla_s u)(\nabla_s u)_n\Big) + h' u_n \;=\; 0 \;, \tag{44}$$

where $\nabla_s u$ denotes the tangential gradient of u on the boundary $\partial\Omega$.

From (44), we deduce that the second normal derivative of u can be evaluated explicitly on the boundary $\partial\Omega$ as

$$u_{nn} = -\frac{h'}{2g'} \;, \tag{45}$$

which is non-positive by the hypothesis on h' and g' (see (22)).

The general Monge Ampère equations (9) with $f = 0$ takes the form

$$K(P)\, |\nabla\, u|^{N-1}\, u_{nn}(P) \;=\; g({u_n}^2)\, h(0) \;. \tag{46}$$

In fact, this is due to the following Lemma investigated by Safoui in [13]

Lemma 1 (Lemma 1.5 p:16 [13])**.** *Let u be a function of class C^2 strictly convex in $\bar\Omega$ and constant on $\partial\Omega$, and let P_0 be an element of $\partial\Omega$ where $|\nabla u|^2$ realizes its maximum.*

We have then at this point the relation

$$\det(u_{,ij}) \;= \Gamma(P_0)\, u_n^{N-1}\, u_{nn} \;,$$

where $\Gamma(P_0)$ denotes the curvature of Gauss of $\partial\Omega$ at the point P_0.

This last differential equality (46) becomes in view of Lemma 2

$$u_{nn} = \frac{g({u_n}^2)\, h(0)}{K |\nabla\, u|^{N-1}} \;. \tag{47}$$

Combining (45) and (47), we get

$$\frac{\partial u}{\partial n} = \left(\frac{2\, h(0)}{h'(0) K}\right)^{\frac{1}{N-1}} \;. \tag{48}$$

Then we have

$$\Phi \;=\; \left(\frac{2\, h(0)}{h'(0) K}\right)^{\frac{2}{N-1}} + h(0) \;=\; \text{const.} \quad \text{on} \;\; \partial\Omega \;. \tag{49}$$

From (49), we obtain the value of the mean curvature K of the boundary $\partial\Omega$ as follows

$$K = \left(\frac{h'(0)}{2h(0)}\right)\left(r^2 - h(0)\right)^{\frac{N-1}{2}}\,, \tag{50}$$

where r is a positive constant.

To this end, the solution u takes the form

$$u = \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}}\left(|x|^2 - h(0)\right)^{\frac{N-1}{2}} - \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}}\left(r^2 - h(0)\right)^{\frac{N-1}{2}}\,, \tag{51}$$

which achieves the proof of our theorem.

We remark that the statement of Theorem 7 is also valid if we have $gg' = A.|\nabla u|^{2N}$, where A is a positive constant. In the special case when $A = N = 1$, we obtain the result of Ma [11] in $\mathbb{R}^2$.

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