PORTUGALIAE MATHEMATICA
Vol. 61 Fasc. 3 – 2004
Nova Série

FREE GROUPS OF SEMIGROUPS IN SEMI-SIMPLE LIE GROUPS

Osvaldo Germano do Rocio * and Alexandre J. Santana *

Recommended by Jorge Almeida

Abstract: Let G be a Lie group and $S \subset G$ a Lie semigroup. Neeb [*Glasg. Math. J.*, 34 (1992), 379–394] studied the free group on a generating Lie semigroup (S, G) using the image $i_*(\pi_1(S))$, where $i : S \to G$ is the inclusion mapping. Now, take G a noncompact semi-simple Lie group, $G = KAN$ its Iwasawa decomposition and S a subsemigroup which contains a large Lie semigroup. With these assumptions, San Martin–Santana [*Monatsh. Math.*, 136, (2002), 151–173] showed that the homotopy groups $\pi_n(S)$ and $\pi_n(K(S))$ are isomorphic, where $K(S) \subset K$ is a compact and connected subgroup. Here, using the technique developed in the above papers we extend the study of free group $G(S)$ and prove that the results of Neeb can be applied for semigroups containing a ray semigroup.

1 – Introduction

Let G be a Lie group and take $S \subset G$ a generating Lie semigroup, that is, a closed subsemigroup which is generated by one-parameter semigroups. The free group $G(S)$, the largest covering group of G into which S lifts, was studied in Neeb [5]. Consider the homomorphism $i_* : \pi_1(S) \to \pi_1(G)$ induced by the inclusion mapping $i : S \to G$, in the reference [5] Neeb proved that the image $i_*(\pi_1(S))$ is the fundamental group of $G(S)$ and that this subsemigroup S satisfies

Received: December 18, 2002; *Revised*: October 16, 2003.
AMS Subject Classification: 20M20, 22E46, 57T99.
Keywords: semigroups; Lie groups; homotopy groups; free group and flag manifolds.
* Partially supported by PROCAD/CAPES grant n° 0186/00-7.

the hypothesis for the existence of a universal covering semigroup $\tilde{S}$. Moreover several other properties regarding the covering space $\tilde{S}$ were given.

In this paper, we study the results of [5] with the weaker assumption that S is a connected semigroup containing a ray semigroup T with nonempty interior. This extends the theory to a larger class of semigroups including several ones (like e.g. $\mathrm{Sl}^+(n,\mathbb{R})$) which are not Lie semigroups. Moreover, we give a description of the free group on S. An application of our results is a quick computation of the universal covering of $\mathrm{Sl}^+(n,\mathbb{R})$ and of semigroups in rank one groups.

More precisely, consider a connected noncompact semi-simple Lie group G with finite center and take $S \subset G$ a connected subsemigroup containing a ray semigroup with nonempty interior. Denote by $G = KAN$ the Iwasawa decomposition of G. With these assumptions, San Martin–Santana [10] proved that the homotopy groups $\pi_n(S)$ and $\pi_n(K(\Theta))$ are isomorphic, where $K(\Theta) \subset G$ is a compact subgroup of K. We first extend the results of [5]. After this, we get a description of the free group $G(S)$ using the isomorphism $\pi_n(S) \simeq \pi_n(K(\Theta))$ shown in [10]. In this direction, we have the result that provides us the free group $G(S)$ from $K(\Theta) \subset S$. And finally, we compute some free groups over semigroups.

2 – Preliminaries

In this section, we establish our notations and recall some background results. We work in the context of [10] and follow closely the notation of Warner [12].

Let G be any connected noncompact Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. We can describe the flag manifolds of G directly from the simple roots of $\mathfrak{g}$. Choose an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let Π be the set of roots of the pair $(\mathfrak{g},\mathfrak{a})$ and Π^+ [respectively Σ] be the set of positive [respectively simple] roots. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The standard minimal parabolic subalgebra of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent subalgebra

$$\mathfrak{n} = \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha \ ,$$

with $\mathfrak{g}_\alpha$ standing for the α-root space. We denote by $\mathbb{B}$ the maximal flag manifold of G and it is defined by the set of subalgebras $\mathrm{Ad}(G)\,\mathfrak{p}$, where Ad stands for the adjoint representation of G in $\mathfrak{g}$. There is an identification of $\mathbb{B}$ with G/P where P is the normalizer of $\mathfrak{p}$ in G. Furthermore, the subgroup P is equal to MAN, with $A = \exp\mathfrak{a}$, $N = \exp\mathfrak{n}$ and M as the centralizer of A in $K = \exp\mathfrak{k}$.

Given a subset $\Theta \subset \Sigma$, let $\mathfrak{n}^-(\Theta)$ be the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle\Theta\rangle$. We denote by $\mathfrak{p}_\Theta$ the parabolic subalgebra

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p} ,$$

where $\langle\Theta\rangle$ is the set of positive roots generated by Θ. The set of parabolic subalgebras conjugate to $\mathfrak{p}_\Theta$ is identified with the homogenous space G/P_Θ, where P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \left\{g \in G : \operatorname{Ad}(g)\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\right\} .$$

Note that this construction yields the flag manifold $\mathbb{B}_\Theta = G/P_\Theta$.

Let

$$\mathfrak{a}^+ = \left\{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\right\}$$

be the Weyl chamber associated to Σ. We say that $X \in \mathfrak{g}$ is split-regular in case $X = \operatorname{Ad}(g)(H)$ for some $g \in G$, $H \in \mathfrak{a}^+$. Analogously, $x \in G$ is said to be split-regular in case $x = ghg^{-1}$ with $h \in A^+ = \exp\mathfrak{a}^+$, that is, $x = \exp X$, with X split-regular in $\mathfrak{g}$.

Given two subsets $\Theta_1 \subset \Theta_2 \subset \Sigma$, the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $G/P_{\Theta_1} \to G/P_{\Theta_2}$, with $gP_{\Theta_1} \mapsto gP_{\Theta_2}$. Alternatively, the fibration assigns to the parabolic subalgebra $\mathfrak{q} \in \mathbb{B}_{\Theta_1}$ the unique parabolic subalgebra in $\mathbb{B}_{\Theta_2}$ containing $\mathfrak{q}$. In particular, $\mathbb{B} = \mathbb{B}_\emptyset$ projects onto every flag manifold $\mathbb{B}_\Theta$.

Recall that the fiber P_Θ/P of $\mathbb{B} \to \mathbb{B}_\Theta$ is obtained from the structure of the parabolic subgroup P_Θ. In order to state the Bruhat–Moore decomposition we need to fix some notation. Denote by $\mathfrak{a}_\Theta$ the annihilator of Θ in $\mathfrak{a}$:

$$\mathfrak{a}_\Theta = \left\{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta\right\} .$$

Put L_Θ as the centralizer of $\mathfrak{a}_\Theta$ in G and $M_\Theta(K) = L_\Theta \cap K$ as the centralizer of $\mathfrak{a}_\Theta$ in K. The Lie algebra $\mathfrak{l}_\Theta$ of L_Θ decomposes as $\mathfrak{l}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta$ with $\mathfrak{m}_\Theta$ reductive. Let M_Θ^0 be the connected subgroup whose Lie algebra is $\mathfrak{m}_\Theta$ and put $M_\Theta = M_\Theta(K)\, M_\Theta^0$, it follows that the identity component of M_Θ is M_Θ^0. With this we can state the Bruhat–Moore Theorem (see [12, Thm.1.2.4.8] for details):

1. $P_\Theta = M_\Theta A_\Theta N_\Theta$, where $A_\Theta = \exp\mathfrak{a}_\Theta$ and N_Θ is the unipotent radical of P_Θ.

2. $P_\Theta = M_\Theta(K)\, AN$.

2.1. Semigroups and homotopy

In this subsection, we recall some basic facts from the general theory of semigroups and their action on flag manifolds. We use the control sets created by this action to study the homotopy type of the semigroup. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A subsemigroup $S \subset G$ is called ray semigroup if there exists a subset $U \subset \mathfrak{g}$ such that S is generated by the one-parameter semigroups $\exp(tX)$, $X \in U$, $t \geq 0$, that is,

$$S = \langle \exp(\mathbb{R}^+ U) \rangle .$$

In this case, S is said to be generated by U (see e.g. Hilgert–Hofmann–Lawson [2]).

A semigroup is said to be a Lie semigroup (or infinitesimally generated semigroup) provided it is the closure of a ray semigroup (see e.g. [3] and [5]). Here, as in [10], it is not necessary to ask for S to be closed.

We restrict our attention to semigroups which have nonempty interior. If S is generated by $U \subset \mathfrak{g}$, this condition holds if and only if $\mathfrak{g}$ is generated by U. Furthermore, in case U is generating, $\operatorname{int} S$ is dense in S (see Hofmann–Ruppert [4, Thm. 2.8]).

Now, let G be a semi-simple Lie group with finite center. Take S a subsemigroup of G with $\operatorname{int} S \neq \emptyset$. Consider the action of S in the flag manifolds of G. It was proved in San Martin–Tonelli [11, Thm. 6.2] that S is not transitive in $\mathbb{B}_\Theta$ unless $S = G$. Moreover, there exists just one closed invariant subset $C_\Theta \subset \mathbb{B}_\Theta$ such that Sx is dense in C_Θ for all $x \in C_\Theta$. This subset is called the invariant control set of S in C_Θ. Since S is not transitive, $C_\Theta \neq \mathbb{B}_\Theta$.

The fact that Sx is dense in C_Θ for all $x \in C_\Theta$ implies the existence of an open subset $C_\Theta^0 \subset C_\Theta$ such that for all $x, y \in C_\Theta^0$ there exists $g \in S$ with $gx = y$. Moreover, C_Θ^0 is dense in C_Θ. This subset C_Θ^0 is called the set of transitivity of C_Θ, and it is given by $C_\Theta^0 = (\operatorname{int} S)\, x \cap (\operatorname{int} S)^{-1} x$, for all $x \in C_\Theta$. In case S is a ray semigroup, it follows that $C_\Theta^0 = \operatorname{int} C_\Theta$ (see [11, §2]).

We introduce here the notion of parabolic type of a semigroup. This concept distinguishes the semigroups according to the geometry of their invariant control sets. Precisely, there exists $\Theta \subset \Sigma$ such that $\pi_\Theta^{-1}(C_\Theta) \subset \mathbb{B}$ is the invariant control set in the maximal flag manifold. Among the subsets Θ satisfying this property, there is one which is maximal, in the sense that it contains all the others. We denote this subset by $\Theta(S)$ and say that it is the parabolic type of S. We also denote this type of S by the corresponding flag manifold $\mathbb{B}(S) = \mathbb{B}_{\Theta(S)}$ (see San Martin [8] and [11] for further discussions about this).

We end this section with the theorem that gives the homotopy type of S (see [10], for a complete study of this). But before, it is necessary recall the concept of reversibility and of homotopy group .

A subsemigroup T of a group L is left reversible if for any finite subset $\{h_1, ..., h_k\} \subset L$, $k \geq 1$

$$(h_1 T) \cap \cdots \cap (h_k T) \neq \emptyset \ .$$

The n-th homotopy group $\pi_n(X, x_0)$ of a space based at $x_0 \in X$ is the set of homotopy classes of pointed maps $\gamma \colon (\mathbb{S}^n, s_0) \to (X, x_0)$ where $\mathbb{S}^n$ stands for the n sphere and s_0 is a base point in $\mathbb{S}^n$, for example, $s_0 = (1, 0, ..., 0)$.

In the remainder of this subsection consider G a noncompact semi-simple Lie group with finite center and $S \subset G$ a subsemigroup with nonempty interior in G. Let C be an invariant control set of S in G/AN and C_0 its set of transitivity.

Fix $x \in C_0$ and denote by e_x (or simply by e) the map $e \colon S \to C_0$ given by $e(g) = gx$. It was proved in [10] that the induced homomorphisms e_* between the homotopy groups are isomorphisms. We sketch here the proof: firstly, it was proved that C_0 is diffeomorphic to $C^0_\Theta \times F_0$ where $F_0 = P^0_\Theta/AN$ is the identity component of $F = P_\Theta/AN$ and secondly it was shown that C^0_Θ is contractible. Then, any cycle in C_0 is homotopic to one in F_0. Hence in order to prove the surjectivity of e_* it is enough to show the existence of a cross section $\sigma : F \to \operatorname{int} S$ for the evaluation map. The injectivity of e_* is obtained as follows. Fix basic points $x \in C_0$ and $g_0 \in \operatorname{int} S$ such that $g_0 x = x$. Assuming that $\gamma : (\mathbb{S}^n, s_0) \to (S, g_0)$ satisfies $e_*[\gamma] = [e \circ \gamma] = 1$, one proves that $[\gamma] = 1$, that is, there is a homotopy based at g_0 carrying γ into g_0. This homotopy can be constructed from homotopies inside S carrying γ successively into smaller groups until it reaches $A_\Theta N_\Theta$. Using reversibility properties of $S \cap A_\Theta N_\Theta$, one obtains that there exists an unbased homotopy between γ and a constant cycle g_1. Hence a standard argument shows that $[\gamma] = 1$.

Now, with the next definition we can summarize the above remarks in a theorem.

Definition 2.1. Let $T_1 \subset T_2$ be subsemigroups with nonempty interior in G. Given a flag manifold $\mathbb{B}_\Theta = G/P_\Theta$, we say that T_1 is Θ-large (or $\mathbb{B}_\Theta$-large) in T_2 provided the invariant control set for both T_1 and T_2 on $\mathbb{B}_\Theta$ coincide. Also, T_1 is large in T_2 in case T_1 is Θ-large for every Θ. □

Theorem 2.2. *Assume that S is connected and contains a $\Theta(S)$-large ray semigroup T with nonempty interior. Let C be an S-i.c.s. in G/AN and C_0 its interior. Then the homomorphism $e_* \colon \pi_n(S) \to \pi_n(C_0)$ induced by an evaluation* *map $e\colon S \to C_0$, $e(g) = gx$, $x \in C_0$, is an isomorphism. The same statement is true with e defined in $\operatorname{int} S$ instead of in S.* ∎

Remark 2.3. This theorem implies that the homotopy groups of S and $\operatorname{int} S$ are isomorphic to the homotopy groups of the compact group $K(\Theta(S)) \approx F_0$. In other words, e is a weak homotopy equivalence. □

Remark 2.4. It is also important to note that under the assumptions of last theorem, there exists $z \in \operatorname{int} S$ such that $K(\Theta(S))\, z \subset \operatorname{int} S$. Furthermore, for any z satisfying this condition, the coset $K(\Theta(S))\, z$ is a deformation retract of $\operatorname{int} S$ (see [10, Thm. 4.15]). □

Now, consider the image of the fundamental group of S under the map $i_* : \pi_1(S) \to \pi_1(G)$. As a consequence of the Remark 2.4, the image $i_*(\pi_1(S))$ is described by the inclusion of the subgroup $K(\Theta(S))$ in G. In fact, the following proposition, shown in [10], holds for all the homotopy groups. We will repeat the proof because this result has a good insertion here.

Proposition 2.5. *Assume that $S \subset G$ admits a $\Theta(S)$-large ray semigroup with nonempty interior. Then the image $i_*\pi_n(S)$ in $\pi_n(G)$ coincides with the image $j_*\pi_n(K(\Theta(S)))$ where $j : K(\Theta(S)) \to G$ is the inclusion.*

Proof: By Proposition 2.3 of [10], the homomorphism induced by the inclusion $\operatorname{int} S \hookrightarrow S$ is an isomorphism of homotopy groups. Hence it is enough to prove the claim with $\operatorname{int} S$ in place of S. Since there exists $z \in \operatorname{int} S$ such that $K(\Theta(S))\, z \subset \operatorname{int} S$ is a deformation retract of $\operatorname{int} S$ (see [10, Thm. 4.15]), it follows that the inclusion $K(\Theta(S))\, z \hookrightarrow \operatorname{int} S$ induces an isomorphism of homotopy groups. Hence $i_*\pi_n(S)$ is the image of the homomorphism induced by $K(\Theta(S))\, z \hookrightarrow G$. By right translation this image coincides with $j_*\pi_n(K(\Theta(S))$. ∎

3 – Covering semigroup

In this section, we show that the same results in [3] Section 3 (see also [5]) hold under weaker assumptions. In [3] and [5] Lie semigroups were studied. However, here we assume S a connected semigroup containing a ray semigroup with nonempty interior.

If S contains a ray semigroup, then the Theorem 2.1 of [4] ensures the existence of an analytical path $\phi : [0,1] \to G$ such that $\phi(0) = 1$ and $\phi(t) \in \operatorname{int}(S)$ for all

$t \in (0,1]$. Moreover, it is possible to prove that S and $\operatorname{int} S$ are path connected. This fact is proved, for example in [5], taking S as a generating Lie semigroup. Hence, using a similar technique of [3], [4] and [5], we extend the Proposition 1.2 of [5] to a connected semigroup that contains a ray semigroup with nonempty interior.

Lemma 3.1. *If S is path connected and contains a ray semigroup with nonempty interior then $\operatorname{int}(S)$ is also path connected.*

Proof: As S contains a ray semigroup then $\operatorname{int}(S)$ and $\operatorname{int}(S^{-1})$ are nonempty. Let $a, b \in \operatorname{int}(S)$ and take $U = a(\operatorname{int}(S^{-1})) \cap b(\operatorname{int}(S^{-1}))$. Then U is an open subset in G containing 1 in its interior. Hence, $U \cap \operatorname{int}(S) \neq \emptyset$, because $1 \in S$ and $\operatorname{int}(S)$ is dense in S. Now, taking $s_0 \in U \cap \operatorname{int}(S)$ it follows that $a, b \in s_0(\operatorname{int}(S))$. However, $s_0 S$ is path connected and it is contained in $\operatorname{int}(S)$. Therefore, a and b can be linked by a path in $\operatorname{int}(S)$. ∎

Lemma 3.2. *Let C be a path component in S and suppose that S contains a ray semigroup with nonempty interior. Then $\operatorname{int}(C)$ is path connected and dense in C.*

Proof: Consider $a, b \in C$ and $\phi : [0,1] \to S$ an analytical curve such that $\phi(0) = 1$ and $\phi(t) \in \operatorname{int}(S)$ with $t \in (0,1]$. If $\gamma : [0,1] \to S$ is an arbitrary path linking a and b, then $t \to \gamma(t)\phi(1)$ is a path linking $a' = a\phi(1)$ and $b' = b\phi(1)$ inside $\operatorname{int}(S)$. This implies that a' and b' lie in the same path component of the open manifold $\operatorname{int}(S)$. But, the path $t \to a\phi(t)$ links a and a' and the path $t \to b\phi(t)$ links b and b'. Hence, there exists a path connecting a and b in $\operatorname{int}(S)$. Therefore $\operatorname{int}(C)$ is path connected and it is dense in C. ∎

Now, we can show that S and $\operatorname{int}(S)$ are path connected. In the remainder of this section, we assume S connected and containing a generating ray semigroup with nonempty interior.

Proposition 3.3. *Let S be a connected subsemigroup of a Lie group G containing a generating ray semigroup with nonempty interior. Then S and $\operatorname{int}(S)$ are path connected.*

Proof: It is enough to prove that S is path connected. Consider C a path component of S.

Denote by $\mathfrak{g}$ the Lie algebra of G. Let $T_0 = \langle \exp\mathbb{R}^+E\rangle$ be the semigroup of S generated by a generating subset $E \subset \mathfrak{g}$. Since E generates $\mathfrak{g}$ it follows that $\operatorname{int}(T_0) \neq \emptyset$. Hence, $U = \operatorname{int}(T_0)\operatorname{int}(T_0^{-1})$ is an open subset of G containing 1.

Now, taking $p \in C$ we see that $pU \cap S$ is an open subset of S containing p. On the other hand, if $q \in pU \cap S$ then $\operatorname{int}(T_0)$ is path connected (as T_0 is path connected this follows from Lemma 3.1). Hence, $q\operatorname{int}(T_0)$ and $p\operatorname{int}(T_0)$ are path connected subsets of S such that $q\operatorname{int}(T_0) \cap p\operatorname{int}(T_0) \neq \emptyset$. Therefore, this subsets are contained in the same path component of S. Now considering $\operatorname{cl}(\operatorname{int}(T_0))$, the above assertions are also true, and as $1 \in \operatorname{cl}(\operatorname{int}(T_0))$ then $q \in C$, showing that $pU \cap S \subset C$ is an open set in S and it contains p.

In order to show that C is closed, take $a \in \operatorname{cl}(C)$. If S_1 denotes the path component of 1 then CS_1 is path connected and contains C. Hence $\operatorname{cl}(C)\operatorname{cl}(S_1) = \operatorname{cl}(C)$ and then aS_1 is a path connected subset of $\operatorname{cl}(C)$ containing the open subset $a(\operatorname{int}S_1)$. By Lemma 3.2, $\operatorname{int}(C)$ is dense in C and then $aS_1 \cap C \neq \emptyset$. As C is a path component we have $aS_1 \subset C$ and hence $a \in C$. Then, we conclude, finally, that C is closed. ∎

Hence we can summarize the above results in a theorem that extends the Proposition 1.2. of [5].

Theorem 3.4. *Take a Lie group G and consider $S \subset G$ a connected semigroup that contains a generating ray semigroup with nonempty interior and the element* 1. *Then the following assertions hold:*

1. *there exists an analytical path $\alpha : [0,1] \to G$ such that*

$$\alpha(0) = 1 \quad \text{and} \quad \alpha((0,1]) \subseteq \operatorname{int}(S)\ .$$

2. *the interior $\operatorname{int}(S)$ is a dense semigroup ideal.*

3. *S and $\operatorname{int}(S)$ are path connected.*

4. *S is locally path connected.*

5. *S is semi-locally simply connected.*

Proof: For (1), see the comments at the beginning of this section. The second item follows from [4, Thm. 2.1]. The Proposition 3.3 implies (3). Finally, in order to obtain (4) and (5), take $T = \overline{\langle \exp(\mathbb{R}^+E)\rangle}$ the subsemigroup generated by E, where $E \subset \mathfrak{g}$ is a subset that generates $\mathfrak{g}$ as Lie algebras and suppose that $T \subset S$. Let $s \in S$ and U be an open subset of G containing s. Then, $s^{-1}U$ is an open subset of G containing 1 and $s^{-1}U \cap T$ contains a path connected and simply connected neighborhood V of 1, with respect to T (cf. [5, Prop. 1.2]). But $s \in sV \subset sT \subset sS \subset S$ and $V \subset s^{-1}U$ imply that $sV \subseteq U \cap S$. Therefore sV is a path connected and simply connected neighborhood of s which is contained in $S \cap U$. ∎

Remark 3.5. Once we have the above theorem, the results in [3] section 3 can be proved in the same way, but with the assumptions on S weaker than originally taken. Hence, we will state here, without proof, some of those results that will be necessary for our goal. □

In the next proposition, we remark that the ideal I does not need to be dense in S, as assumed in [3].

Proposition 3.6. *Let I be a path connected semigroup ideal in the path connected topological monoid S. Suppose that there exists a path $\beta\colon [0,1] \to S$ such that*

$$\beta(0) = 1 \quad \text{and} \quad \beta(]0,1]) \subseteq I \ .$$

Then the inclusion $i : I \to S$ induces an isomorphism

$$i_* : \ \pi_1(I) \to \pi_1(S) \ . \ ∎$$

With this result and denoting by $\tilde{S}$ the universal covering group of S we have the following

Corollary 3.7. *The inclusions $i : \operatorname{int}(S) \to S$, $\tilde{i} : \operatorname{int}(\tilde{S}) \to \tilde{S}$ induce isomorphisms*

$$i_*\colon \pi_1(\operatorname{int}(S)) \to \pi_1(S) \quad \text{and} \quad \tilde{i}_*\colon \pi_1(\operatorname{int}(\tilde{S})) \to \pi_1(\tilde{S}) \ .$$

Furthermore, $\pi_1(\operatorname{int}(\tilde{S})) = \{1\}$. ∎

Corollary 3.8. *Let $p : \tilde{S} \to S$ be the covering mapping, then the following assertions hold:*

1. *Let $\tilde{\gamma}$ denotes the lift of γ with $\tilde{\gamma}(0) = \tilde{1}$. Then the mapping $[\gamma] \mapsto \tilde{\gamma}(1)$, $\pi_1(S) \to D = p^{-1}(1)$ is an isomorphism of groups.*

2. *$D \subseteq Z(\tilde{S}) = \{s \in \tilde{S} :$ for all $t \in \tilde{S},\ st = ts\}$.*

3. *$\pi_1(S)$ is abelian.*

4. *The mapping $\tilde{S}/D \to S$ is an isomorphism of topological semigroups.* ∎

As in [3], we can study the multiplication mapping of $\tilde{S}$. Since the conclusions reached are similar, despite of our weaker assumptions, we avoid the description.

Theorem 3.9. *Let $D \subseteq Z(\tilde{S})$ be a discrete subgroup. Then the following assertions hold:*

1. *D acts properly on $\tilde{S}$.*

2. *The quotient mapping*

$$q\colon \tilde{S} \to S_D := \tilde{S}/D\,, \qquad s \mapsto sD$$

is a covering morphism of locally compact semigroups. ∎

Lemma 3.10. *Let $q : \tilde{G} \to G$ be the universal covering group of G, identify $\pi_1(G)$ with $\ker q$ and $\pi_1(S)$ with $p^{-1}(\mathbf{1})$. Then there exists a continuous homomorphism $\tilde{i} : \tilde{S} \to \tilde{G}$ such that $q \circ \tilde{i} = i \circ p, \tilde{i}|_{\pi_1(S)} = i_*$, and the image of $\tilde{i}$ is the path-component of $\mathbf{1}$ in $q^{-1}(S)$.* ∎

Theorem 3.11. *Let $j : H(S) \to S$ be the inclusion mapping and*

$$j_*\colon\ \pi_1(H(S)) \to \pi_1(S)$$

the induced homomorphism. Then $\ker j_* = \pi_1(H(\tilde{S}))$ and $\operatorname{im} j_* = H(\tilde{S})_\circ \cap \pi_1(S)$. ∎

Corollary 3.12. *The mapping $j_* : \pi_1(H(S)) \to \pi_1(S)$ is:*

1. *injective if and only if $H(\tilde{S})$ is simply connected.*

2. *surjective if and only if $H(\tilde{S})$ is connected.* ∎

4 – Free group

Consider the inclusion map $i : S \hookrightarrow G$ and take its induced homomorphism $i_* : \pi_1(S) \to \pi_1(G)$. In [3] and [5] it was proved that the largest covering group of G into which S lifts is isomorphic to $G(S)$ and it coincides with $\tilde{G}/\mathrm{im}(i_*)$. Hence $\operatorname{im} i_* = \pi_1(G(S))$ (where $\tilde{G}$ is the universal covering of G). By Theorem 2.2 one can find a better description of this largest covering group of G into which S lifts. In this section, we use the cited results to obtain the Theorem 4.5, which helps the computations of the examples in the next section. We start recalling the following definitions (see [1] for more details).

Definition 4.1. The pair (H,η) is called an S-group if H is a group and η is homomorphism $\eta : S \to H$ such that $\eta(S)$ is a set of group-generators of H. □

Definition 4.2. The pair (G,γ) is called a free group on S, and denoted by $G(S)$ if (G,γ) is an S-group and if for all S-groups (H,η), there exists a unique homomorphism $\theta : G \to H$ such that $\theta\gamma = \eta$. □

The next Theorem, proved in [3], identifies $\pi_1(G(S))$ as the image of i_*.

Theorem 4.3. $\operatorname{im} i_* = \pi_1(G(S))$.

Proof: See [3, Thm. 3.30]. ∎

In this situation, the next statement is also trivial.

Corollary 4.4. *If $\pi_1(S) = \{1\}$ then $G(S)$ is the universal covering of G.* ∎

In the rest of this section we are assuming G a non-compact semi-simple Lie group with finite center. Now we have the result that makes the computation of the free group of a semigroup S easy. Note that the Remark 2.4 and the Proposition 2.5 justify the following expression $i_*(\pi_1(K(\Theta(S))))$.

Theorem 4.5. *Consider $S \subset G$ a connected subsemigroup which contains a $\Theta(S)$-large ray semigroup with nonempty interior. Then $\pi_1(G(S)) = i_*(\pi_1(K(\Theta(S))))$, where $i : K(\Theta(S)) \to G$ is the inclusion mapping.*

Proof: The homotopy type of S is equal to that of $K(\Theta)$, hence the proof follows straight forward from Theorem 4.3. ∎

5 – Examples

There are many important consequences of the results in the last section. For example, it is possible to compute the fundamental group of the free group $G(S)$ and, in some cases, to compute easily.

5.1. $S = \mathrm{Sl}^{+}(n, \mathbb{R})$

Let $S = \mathrm{Sl}^{+}(n, \mathbb{R})$ be the semigroup of determinant one matrices having nonnegative entries. This is the compression semigroup of the positive orthant $\mathbb{R}_{+}^{n}$ in $\mathbb{R}^{n}$:

$$\mathbb{R}_{+}^{n} = \left\{ (x_1, ..., x_n) \colon \, x_i \geq 0 \right\} .$$

It turns out that the type of $\mathrm{Sl}^{+}(n, \mathbb{R})$ is the projective space $\mathbb{P}^{n-1}$, and the invariant control set in $\mathbb{P}^{n-1}$ is the set $[\mathbb{R}_{+}^{n}]$ of lines contained in $\mathbb{R}_{+}^{n}$. In our previous notation, $C_{\Theta} = [\mathbb{R}_{+}^{n}]$.

The semigroup $\mathrm{Sl}^{+}(n, \mathbb{R})$ is closed but it is not a Lie semigroup. Now, put

$$\mathcal{L}(S) = \left\{ X \in \mathfrak{sl}(n, \mathbb{R}) \colon \, \exp(tX) \in \mathrm{Sl}^{+}(n, \mathbb{R}) \text{ for all } t \geq 0 \right\}$$

for the Lie wedge of $\mathrm{Sl}^{+}(n, \mathbb{R})$. One checks easily that $\mathcal{L}(S) = \{ X = (x_{ij}) \colon \, x_{ij} \geq 0, i \neq j \}$. Put $S_{\mathrm{inf}} = \langle \exp \mathcal{L}(S) \rangle$ for the corresponding ray semigroup. Since $\mathcal{L}(S)$ generates $\mathfrak{sl}(n, \mathbb{R})$, S_{inf} has nonempty interior in $\mathrm{Sl}(n, \mathbb{R})$. We claim that the invariant control set of S_{inf} in $\mathbb{P}^{n-1}$ is also C_{Θ} and $C_{\Theta}^{0} = \mathrm{int}(C_{\Theta})$. In fact, consider matrices of the form $H = \mathrm{diag}\{n-1, -1, \cdots, -1\}$ with respect to a basis $B = \{f_1, \cdots, f_n\}$ such that $f_1 \in \mathbb{R}_{+}^{n}$ and $\mathrm{span}\{f_2, \cdots, f_n\} \cap \mathbb{R}_{+}^{n} = 0$. Take $\exp(tH), t \geq 0$. Since $H \in \mathcal{L}(S)$, any $x \in C_{\Theta}$ is the fixed point of some element of S_{inf}. Therefore, C_{Θ} is contained in the invariant control set of S_{inf}. On the other hand, since $S_{\mathrm{inf}} \subset S$ the other inclusion follows from the definition of invariant control set (see San Martin [7, §2]).

Therefore, $S = \mathrm{Sl}^{+}(n, \mathbb{R})$ contains a $\Theta(S)$-large ray semigroup. It was proved by Ribeiro–San Martin [6] that S is connected. Hence the isomorphism theorem holds for $\mathrm{Sl}^{+}(n, \mathbb{R})$. With the canonical choices, it is not difficult to check that P_{Θ}^{0}/AN is diffeomorphic to $\mathrm{SO}(n-1)$. It follows that the homotopy groups of $\mathrm{Sl}^{+}(n, \mathbb{R})$ are isomorphic to the homotopy groups of $\mathrm{SO}(n-1)$.

Now, in order to study the fundamental group of its free group we recall that for all covering maps p, the induced map p_* is injective. Consider $n > 3$, i.e., take the group $G = \mathrm{Sl}(n, \mathbb{R})$ and the semigroup $S = \mathrm{Sl}^{+}(n, \mathbb{R})$, for $n > 3$. By Cartan decomposition we have $\pi_1(\mathrm{Sl}(n, \mathbb{R})) = \pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$. And, as we saw above $\pi_1(\mathrm{Sl}^{+}(n, \mathbb{R})) = \pi_1(\mathrm{SO}(n-1)) = \mathbb{Z}_2$. Hence the image $\mathrm{im}\, i_*(\pi_1(S))$ has one or two elements, so $\pi_1(G(\mathrm{Sl}^{+}(n, \mathbb{R})))$ has at most two elements.

If we take $n = 3$, we have $\pi_1(G(\mathrm{Sl}^{+}(3, \mathbb{R})))$ discrete.

Finally, consider $n = 2$, that is, $G = \mathrm{Sl}(2, \mathbb{R})$ and the semigroup $S = \mathrm{Sl}^{+}(2, \mathbb{R})$, by Cartan decomposition we have $\pi_1(\mathrm{Sl}(2, \mathbb{R})) = \pi_1(\mathrm{SO}(2)) = \mathbb{Z}$. But, as we saw above $\pi_1(\mathrm{Sl}^{+}(2, \mathbb{R})) = \pi_1(\mathrm{SO}(1)) = \pi_1(\{1\}) = 1$, hence the image $\mathrm{im}\, i_*(\pi_1(S))$

is trivial. Therefore, $\pi_1(G(\mathrm{Sl}^+(2,\mathbb{R})))$ is trivial, so $G(\mathrm{Sl}^+(2,\mathbb{R}))$ is the universal covering of $\mathrm{Sl}(2,\mathbb{R})$.

Similarly, considering the covering mapping

$$p\colon\ G(\mathrm{Sl}^+(2,\mathbb{R})) \to \mathrm{Sl}(2,\mathbb{R})$$

we note that the induced homomorphism

$$p_*\colon\ \pi_1(G(\mathrm{Sl}^+(2,\mathbb{R}))) \to \pi_1(\mathrm{Sl}(2,\mathbb{R}))$$

is injective, or rather,

$$p_*\colon\ \pi_1(G(\mathrm{Sl}^+(2,\mathbb{R}))) \to \mathbb{Z}$$

is injective.

We will see that the example $\mathrm{Sl}(2,\mathbb{R})$ can be generalized to rank one groups.

5.2. Compression semigroup

The facts of the above example extend to the compression semigroup of a cone in $\mathbb{R}^n$. Let $W \subset \mathbb{R}$ be a pointed and generating cone and form the semigroup

$$S_W = \left\{ g \in \mathrm{Sl}(n,\mathbb{R})\colon\ gW \subset W \right\} .$$

It was proved in [6] that S_W is connected. Again the type of S_W is the projective space, and similar to the proof for $\mathrm{Sl}^+(n,\mathbb{R})$, the semigroup generated by $\mathcal{L}(S_W)$ is large in S_W, that is, the invariant control set of $\langle \exp\mathcal{L}(S_W)\rangle$ is the same of that S_W. Hence the homotopy type of S_W is also $\mathrm{SO}(n-1)$. Therefore, the computations of the fundamental group of the free group on S_W follow the case $\mathrm{Sl}^+(n,\mathbb{R})$.

5.3. Rank one groups

Suppose that G is a rank one group. Then there exists just one class of parabolic subgroups and hence just one flag manifold G/MAN. Then proper semigroups with nonempty interior in G all have the same type, namely $\Theta = \emptyset$. The subgroup $K(\Theta)$ is the identity component of MAN/AN, that is, $K(\Theta) = M_0$ so that every semigroup S in G admitting a large ray semigroup has the same homotopy groups (in particular, the fundamental group), and they are isomorphic to the homotopy groups of M_0. Moreover, $\operatorname{int} S$ can be continuously deformed into M_0. Summarizing, in rank one groups, M_0 gives the fundamental group of this semigroups.

For instance, if $G = \mathrm{Sl}\,(2, \mathbb{R})$ then $M_0 = \{1\}$. Hence the fundamental group of S is trivial and the fundamental group of the free group is computed as in the first example.

Consider the rank one group $\mathrm{SU}(1, p)$, the Iwasawa decomposition of its Lie algebra is $\mathfrak{s}u(1, p) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where:

$$\mathfrak{k} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\},$$

with α and β skew Hermitian and $\mathrm{tr}(\alpha + \beta) = 0$. A typical element of $\mathfrak{a}$ is given by the $(p + 1) \times (p + 1)$-matrix

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 & z \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \overline{z} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$z \in \mathbb{C}$. And $\mathfrak{n}$ is the correspondent nilpotent subalgebra.

Hence the subgroup M is found by calculating $XH - HX = 0$ with $X \in \mathfrak{k}$. Summarizing, $M = \mathrm{SU}(p - 1)$. But $\mathrm{SU}(1, p)$ is homeomorphic to $\mathrm{SU}(p) \times T \times \mathbb{R}^d$ with T being the multiplicative group of complex numbers of modulus 1. Hence

$$\pi_1(\mathrm{SU}(1, p)) \;=\; \pi_1(\mathrm{SU}(p)) \times \pi_1(T) \;=\; \pi_1(\mathrm{SU}(p)) \times \mathbb{Z}\ .$$

Hence, $\pi_1(G(S)) = i_*(\pi_1(\mathrm{SU}(p - 1)))$, where the map i_* is induced by inclusion map

$$i\colon\, S \hookrightarrow \mathrm{SU}(1, p)\ .$$

Then since $\pi_1(\mathrm{SU}(p))$ is trivial for all p, we conclude that $\pi_1(G(S))$ is also trivial. Therefore $G(S)$ is isomorphic to the universal covering $\tilde{G}$ of G.

Take now the real hyperbolic groups, the identity component is $G = \mathrm{SO}\,(1, p)_0$. In this case, similarly to the last group, $M_0 = \mathrm{SO}\,(p - 1)$. So it gives the fundamental group of the Lie semigroups in $\mathrm{SO}\,(1, p)$.

It is known that $\mathrm{SO}(1, p)$ is homeomorphic to the topological product of $\mathrm{SO}(1, p) \cap \mathrm{SO}(p+1)$ and $\mathbb{R}^d$ for some integer d. We have also that this intersection consists of all matrices of the form

$$\begin{pmatrix} \det B & 0 \\ 0 & B \end{pmatrix},$$

where B is an orthogonal matrix of order p.

Then

$$\pi_1(\mathrm{SO}(1,p)) = \pi_1(\mathrm{SO}(p)) \ .$$

Hence, $\pi_1(G(S)) = i_*(\pi_1(\mathrm{SO}(p{-}1)))$, where the map i_* is induced by the inclusion map

$$i\colon\ S \hookrightarrow \mathrm{SO}(1,p) \ .$$

Therefore as in the case of $G = \mathrm{Sl}\,(n,\mathbb{R})$, the knowledge of this free group depends on $\pi_1(\mathrm{SO}(p-1))$.

Moreover, it is possible to show that $G(S) = G$. In fact, knowing that the inclusions $\mathrm{SO}(p) \hookrightarrow \mathrm{SO}(p+1)$ induce surjections on the level of π_1, for all p, and noting that $\pi_1(S) = \pi_1(\mathrm{SO}(p-1))$ and $\pi_1(G) = \pi_1(\mathrm{SO}(p))$ we see that

$$i_*\colon\ \pi_1(S) \hookrightarrow \pi_1(G)$$

is sujective, and therefore, by Corollary 3.31 of [3], it follows that $G(S) = G$.

Finally, consider the connected rank one group $\mathrm{Sp}(1,p)$. Since this group is homeomorphic to $\mathrm{Sp}(p) \times \mathbb{R}^d$, the computation follows as in the case of rank one group above.

REFERENCES

[1] Clifford, A.H. and Preston, G.B. – The algebraic theory of semigroups, *American Math. Soc., Mathematical Surveys,* 7, Providence, Rhode Island, (1961).

[2] Hilgert, J.; Hofmann, K.H. and Lawson; J.D. – *Lie groups, Convex cones and Semigroups*, Oxford University Press, (1993).

[3] Hilgert, J. and Neeb, K.-H. – Lie semigroups and their applications, *Lecture Notes in Math.,* 1552 (Springer, Berlin 1993).

[4] Hofmann, K.H. and Ruppert, W.A.F. – On the interior of subsemigroups of Lie groups, *Trans. Amer. Math. Soc.,* 324 (1991), 169–179.

[5] Neeb, K.-H. – On the fundamental group of a Lie semigroup, *Glasg. Math. J.,* 34 (1992), 379–394.

[6] Ribeiro, G.J. and San Martin, L.A.B. – The compression semigroup of a cone is connected, *Portugal. Math.,* 60 (2003), 305–318.

[7] San Martin, L.A.B. – Invariant control sets on flag manifolds, *Math. Control Signals Systems,* 6 (1993), 41–61.

[8] San Martin, L.A.B. – Maximal semigroups in semi-simple Lie groups, *Trans. Amer. Math. Soc.,* 353 (2001), 5165–5184.

[9] San Martin, L.A.B. – Nonexistence of invariant semigroups in affine symmetric spaces, *Math. Ann.,* 321 (2001), 587–600.

[10] San Martin, L.A.B. and Santana, A.J. – The homotopy type of Lie semigroups in semi-simple Lie groups, *Monatsh. Math.,* 136 (2002), 151–173.

[11] San Martin, L.A.B. and Tonelli, P.A. – Semigroup actions on homogeneous spaces, *Semigroup Forum,* 50 (1995), 59–88.

[12] Warner, G. – *Harmonic analysis on semi-simple Lie groups I*, Springer, Berlin 1972.

Osvaldo Germano do Rocio,
Departamento de Matemática, Universidade Estadual de Maringá,
87020-900 Maringá Pr – BRASIL
E-mail: `rocio@uem.br`

and

Alexandre J. Santana,
Departamento de Matemática, Universidade Estadual de Maringá,
87020-900 Maringá Pr – BRASIL
E-mail: `ajsantana@uem.br`