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ON THE EXISTENCE OF MONOTONE SOLUTIONS FOR SECOND-ORDER NON-CONVEX DIFFERENTIAL INCLUSIONS IN INFINITE DIMENSIONAL SPACES

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Abstract: This paper is concerned with the existence of monotone solutions in an infinite dimensional Hilbert space for a second order differential inclusion and without the assumption of the convexity.

1 – Introduction

The existence of solutions for either first or second order differential inclustions or functional differential inclusions has been studied extensively in recent papers. For instance we refer to [1, 4, 6, 7, 8, 10, 12, 14, 16, 17, 18, 19, 20, 21, 22].

In order to explain our aim let H be an infinite dimensional Hilbert space, $Q = K\times\Omega$ be a subset of $H\times H$, F be a set-valued function defined on Q and its values are not necessary convex subsets of H. Consider the following differential inclusion:

$$(*)\qquad \begin{aligned} &x''(t) \in F(x(t), x'(t)), \quad \text{a.e. on } [0,T]\ , \\ &(x(t), x'(t)) = (x_\circ, y_\circ) \in Q = K\times\Omega\ . \end{aligned}$$

By a solution of $(*)$ we mean an absolutely continuous function $x\colon [0,T] \to K$ with absolutely continuous derivative such that $(*)$ is satisfied. A solution $x\colon [0,T]\to K$ of $(*)$ is called monotone if there is a set-valued function P defined from K to the family of nonempty subsets of K such that (i) $x \in P(x)$ for all $x\in K$, (ii) if $y\in P(x)$ then $P(y) \subseteq P(x)$ and (iii) if $t \le s$, $t,s \in [0,T]$

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then $x(s) \in P(x(t))$. In the particular case $H = \mathbb{R}$ (the set of real numbers) and $P(x) = [x, \infty) \cap K$ the monotoncity of the solution becomes, according to this definition, as follows if $t \leq s$ then $x(t) \leq x(s)$.

The purpose of this paper is to obtain conditions on the data that guarantee the existence of monotone solution for $(*)$.

We refer to in a recent paper V. Lupulescu [20] proved the existence of a solution wich is not necessary monotone and in the case when dimension H is finite. Also, in the above mentioned papers the monotoncity of the obtained solutions were not researched.

The paper will organize as follows: in section 2 we will recall briefly some basic definitions and preliminary facts which will be used throught the sequel. In section 3 we will establish the main result.

2 – Notations and preliminaries

In this section we give the notations and known facts that we will use throught the paper.

- H is an infinite dimensional seperable real Hilbert space.
- If $x \in H$ and $\delta > 0$, $B(x,\delta) = \{y \in H \colon \|y - x\| < \delta\}$ is the ball centered at x with radius δ and $\overline{B(x,\delta)}$ its closure.
- If A is a subset of H and $x \in H$, $d(x,A) = \inf\{\|y - x\| \colon y \in A\}$ is the distance from x to A.
- If A is a subset of H then $|A| = \sup\{\|a\| \colon a \in A\}$ is the excess of A over $\{0\}$ and $\operatorname{co} A$ is the convex hull of A.
- A function $u \colon [0,T] \to H$ is called Lebesgue–Bochner integrable if $t \to \|u(t)\|$ is Lebesgue integrable and u is strongly measurable, i.e. the a.e. limit of a sequence of step functions. The Banach space of equivalence class of such u will be denoted by $L^1([0,T],H)$. It's known that if $w \in L^1([0,T],H)$ then $\left(\int_0^t w(s)\,ds\right)' = w(t)$, a.e.
- $L^2([0,T],H)$ is the Banach space of all strongly measurable functions $u : [0,T] \to H$ such that $\int_0^T \|u(t)\|^2\,dt < \infty$.
- A function $u : [0,T] \to H$ is absolutely continuous if there is a function $v \in L^1([0,T],H)$ such that $u(t) = u(0) + \int_0^t v(s)\,ds$, for all $t \in [0,T]$.

- If X and Y are two topological spaces, a set-valued function $G\colon X \to Y$ is called upper semicontinuous (lower semicontinuous) at $x_\circ \in X$ if for any open set U of Y containing $G(x_\circ)$ ($G(x_\circ) \cap U \neq \emptyset$), the set $\{x \in X : G(x) \subseteq U\}$ ($\{x \in X : G(x) \cap U \neq \emptyset\}$) is a neighbourhood of $x_\circ$. G is upper semicontinuous (lower semicontinuous) if it's upper semicontinuous (lower semicontinuous) at each point in X.
- If E is a topological vector space, $f : E \to \mathbb{R}$ and $x_\circ \in E$, then $x' \in E'$, the topological dual of E, is said to be a subgradient of f at $x_\circ$ if for every $x \in E$,
$$f(x) - f(x_\circ) \;\geq\; \langle x',\, x - x_\circ \rangle \;.$$
The set of all subgradients of f at $x_\circ$ is called subdifferential and is denoted by $\partial f(x_\circ)$. It's known that if E is a Hausdorff locally convex space, then $\partial f(x_\circ)$ is closed and convex. (See for instance [13]).
- If K is a subset of H and $x \in K$, then the Bouligand's contingent cone of K at x is defined by:
$$T_K(x) \;=\; \left\{ y \in H\colon\;\; \liminf_{h \to 0^+} \frac{d(x + hy, K)}{h} = 0 \right\} .$$
It's known that if x is an interior point in K, then $T_K(x) = H$ and if K is closed and convex then $T_K(x) = \overline{\{\lambda(z - x)\colon \lambda \geq 0,\; z \in K\}}$. (See [3]).
- If K is a subset of H, $x \in K$ and $y \in H$ then the second order contingent cone of K at (x, y) is defined by (see [9]):
$$T_K^{(2)}(x, y) \;=\; \left\{ z \in H\colon\;\; \liminf_{h \to 0^+} \frac{d(x + hy + \frac{h^2}{2} z,\, K)}{\frac{h^2}{2}} = 0 \right\} .$$
We remark that if $T_K^{(2)}(x, y) \neq \emptyset$ then $y \in T_K(x)$.
- If B is a bounded set of a normed space E, then the Kuratowski's measure of noncompactness of B, $\alpha(B)$, is defined by $\alpha(B) = \inf\{d > 0\colon B = \bigcup_{i=1}^m B_i$ for some m and B_i with diameter less than or equal to $d\}$. In the following lemma we recall some useful properties for the measure of noncompactness α. For instance see Prop. 9.1 [15].

Lemma 2.1. *Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X.*

(**i**) $\alpha(D_1) = 0 \iff D_1$ *is relatively compact.*

(**ii**) $\alpha(\lambda D_1) = |\lambda|\, \alpha(D_1)$; $\lambda \in \mathbb{R}$.

(**iii**) $D_1 \subseteq D_2 \implies \alpha(D_1) \le \alpha(D_2)$.

(**iv**) $\alpha(D_1 + D_2) \le \alpha(D_1) + \alpha(D_2)$.

(**v**) *If* $x_\circ \in X$ *and* r *is a positive real number then* $\alpha(B(x_\circ, r)) = 2\,r$. ∎

For other properties of α we refer to ([5] and [15]), and for more details about set-valued function we refer to ([2], [3], [13], [15], [19]).

3 – Main result

In this section we give the main result. First we start by the following Lemma which plays an important role in the sequel. The proof will be based on the same technique that was used in Lemma 3.1 in [20].

Lemma 3.1. *Let* K, Ω *be two nonempty subsets of* H, P *be a lower semi-continuous set-valued function from* K *to the non-empty subsets of* K *and* F *be a set-valued function defined on* $Q = K \times \Omega$ *with non-empty subsets of* H.

Assume that:

(**i**) *For all* $x \in K$, $x \in P(x)$;

(**ii**) *For all* $(x, y) \in Q$, $F(x, y) \cap T^{(2)}_{P(x)}(x, y) \ne \emptyset$.

If $Q_\circ$ *is a compact subset of* Q *and* k *is a positive integer, then there is* $\eta_k > 0$ *such that for all* $(x_\circ, y_\circ) \in Q_\circ$ *there exist* $h_{\circ,k} \in [\eta_k, \frac{1}{k}]$, $u_{\circ,k}, v_{\circ,k} \in H$ *and* $(x_{j_\circ}, y_{j_\circ}) \in Q_\circ$ *such that:*

1. $z_\circ = x_\circ + h_{\circ,k}\, y_\circ + \frac{1}{2}\, h^2_{\circ,k}\, u_{\circ,k} \in P(x_\circ)$;

2. $v_{\circ,k} \in F(x_{j_\circ}, y_{j_\circ})$;

3. $d((x_\circ, y_\circ), (x_{j_\circ}, y_{j_\circ})) < \frac{1}{k}$;

4. $\|u_{\circ,k} - v_{\circ,k}\| < \frac{1}{k}$.

Proof: Let (x, y) be a fixed element in $Q = K \times \Omega$. By (ii) there is $v = v(x, y) \in F(x, y)$ such that:

$$\liminf_{h\to 0^+} \frac{d\Big(x + hy + \frac{h^2}{2} v,\ P(x)\Big)}{\frac{h^2}{2}} = 0\ .$$

Hence there is $h_k = h_k(x,y) \in (0, \frac{1}{k}]$ such that

$$(1) \qquad d\Big(x + h_k\, y + \frac{h_k^2}{2}\, v,\; P(x)\Big) < \frac{h_k^2}{4\,k}\;.$$

Since P is lower semicontinuous, Cor. 1.2.1 [3] yields that the function $(a,b) \to d(b, P(a))$ is upper semicontinuous. Consequently, the function $(a,b) \to d(a + h_k b + \frac{1}{2}\, h_k^2 v,\; P(a))$ is upper semicontinuous from $H \times H$ to $\mathbb{R}$. Thus the subset

$$N(x,y) \;=\; \Big\{(a,b)\colon\; d\Big(a + h_k\, b + \frac{1}{2}\, h_k^2\, v,\; P(a)\Big) < \frac{h_k^2}{4\,k}\Big\}$$

is open. By (1), $(x,y) \in N(x,y)$. Then there exists $r = r(x,y) \in (0, \frac{1}{k}]$ such that $B((x,y), r) \subset N(x,y)$.

Now $\{B((x,y), r)\colon\; (x,y) \in Q_\circ\}$ is an open cover for $Q_\circ$. Since $Q_\circ$ is compact, there exists a finite set $\{(x_i, y_i) \in Q_\circ\colon\; 1 \le i \le m\}$ such that:

$$Q_\circ \;\subseteq\; \bigcup_{i=1}^{m} B((x_i, y_i), r_i)\;.$$

Put $\eta_k = \min\{h_k(x_i, y_i)\colon 1 \le i \le m\}$. Since $(x_\circ, y_\circ) \in Q_\circ$ there is $j_\circ \in \{1, 2, ..., m\}$ such that:

$$(x_\circ, y_\circ) \;\in\; B((x_{j_\circ}, y_{j_\circ}), r_{j_\circ}) \;\subseteq\; N(x_{j_\circ}, y_{j_\circ})\,, \qquad (x_{j_\circ}, y_{j_\circ}) \in Q_\circ\;.$$

Denote by $h_{\circ,k} = h_k(x_{j_\circ}, y_{j_\circ})$, $v_{\circ,k} = v(x_{j_\circ}, y_{j_\circ}) \in F(x_{j_\circ}, y_{j_\circ})$. From the definition of the distance we can find $z_\circ \in P(x_\circ)$ such that:

$$\begin{aligned}
\frac{1}{\frac{h_{\circ,k}^2}{2}}\, d\Big(x_\circ + h_{\circ,k}\, y_\circ + \frac{h_{\circ,k}^2}{2}\, v_{\circ,k},\; z_\circ\Big) \;&\le\; \frac{d\Big(x_\circ + h_{\circ,k}\, y_\circ + \frac{h_{\circ,k}^2}{2}\, v_{\circ,k},\; P(x_\circ)\Big)}{\frac{h_{\circ,k}^2}{2}} + \frac{1}{2\,k} \\
&<\; \frac{h_{\circ,k}^2/4k}{h_{\circ,k}^2/2} + \frac{1}{2\,k} \\
&=\; \frac{1}{2\,k} + \frac{1}{2\,k} \\
&=\; \frac{1}{k}\;,
\end{aligned}$$

hence:

$$\left\|\frac{z_\circ - x_\circ - h_{\circ,k}\, y_\circ}{\frac{h_{\circ,k}^2}{2}} - v_{\circ,k}\right\| < \frac{1}{k}\;.$$

Let

$$u_{\circ,k} = \frac{z_\circ - x_\circ - h_{\circ,k}\, y_\circ}{\frac{h_{\circ,k}^2}{2}} \ .$$

Then

$$\|u_{\circ,k} - v_{\circ,k}\| < \frac{1}{k} \ ,$$

$$x_\circ + h_{\circ,k}\, y_\circ + \frac{h_{\circ,k}^2}{2}\, u_{\circ,k} \ = \ z_\circ \ \in \ P(x_\circ) \ ,$$

$$v_{\circ,k} \in F(x_{j_\circ}, y_{j_\circ}) \ ,$$

and

$$d\Big((x_\circ, y_\circ), (x_{j_\circ}, y_{j_\circ})\Big) < \frac{1}{k} \ . \ \blacksquare$$

Theorem 3.2. *Let K be a subset of H, Ω be an open subset of H such that $Q = K \times \Omega$ be a locally compact subset of $H \times H$, F be an upper semicontinuous set-valued function from Q to the family of non-empty compact subsets of H, and P be a lower semicontinuous set-valued function from K to the family of non-empty subsets of K with closed graph.*

Assume the following conditions:

(**H1**) (**i**) *For all $x \in K$, $x \in P(x)$,*

(**ii**) *For all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$.*

(**H2**) *For all $(x,y) \in Q$, $F(x,y) \cap T^{(2)}_{P(x)}(x,y) \neq \emptyset$.*

(**H3**) *There exist a proper convex and lower semicontinuous function $V: H \to \mathbb{R}$ such that:*

$$F(x,y) \subseteq \partial V(y) \,, \qquad \forall (x,y) \in Q \ ,$$

where $\partial V(y)$ is the subdifferential of V.

Then for all $(x_\circ, y_\circ) \in Q$ there exists $T > 0$ and an absolutely continuous function $x: [0,T] \to H$ with absolutely continuous derivative such that:

$$x''(t) \in F(x(t), x'(t)) \quad \text{a.e. on} \ \ [0,T] \ ,$$

$$x(s) \in P(x(t)) \quad \text{for all} \ \ t \in [0,T] \ \ \text{and all} \ \ s \in [t,T] \ ,$$

$$x(0) = x_\circ \,, \qquad x'(0) = y_\circ \ .$$

Proof: Let $(x_\circ, y_\circ) \in Q$. Since Q is locally compact, each component K and Ω is locally compact. Because $x_\circ \in K$ we can find $\delta_1 > 0$ such that $\overline{B(x_\circ, \delta_1)} \cap K$ is compact in H. Also, since $y_\circ \in \Omega$ and Ω is open we can find $\delta_2 > 0$ such that $\overline{B(y_\circ, \delta_2)} \subseteq \Omega$ is compact in H. Let $\delta = \min(\delta_1, \delta_2)$ and put $Q_\circ = (\overline{B(x_\circ, \delta)} \cap K) \times (\overline{B(y_\circ, \delta)})$. So, $Q_\circ$ is a compact subset of Q. Since F is upper semicontinuous, $F(Q_\circ)$ is compact subset of H. Then we can find $M > 0$ such that:

$$\sup\Big\{\|v\| \colon v \in F(Q_\circ)\Big\} \le M .$$

Put

$$T = \min\left\{\frac{\delta}{2(M+1)}, \sqrt{\frac{\delta}{M+1}}, \frac{\delta}{2(\|y_\circ\|+1)}\right\}. \tag{1}$$

Let k be a fixed positive integer. We are going to show that there are a positive real number η_k and a positive integer $m(k)$ such that for each $r \in \{0, 1, ..., m(k)-1\}$ there exist $h_{r,k} \in [\eta_k, \frac{1}{k}]$, $(x_{r,k}, y_{r,k}) \in Q_\circ$, $u_{r,k}, v_{r,k} \in H$ and $(x_{j_r}, y_{j_r}) \in Q_\circ$ with the following properties:

(i) $\sum_{r=0}^{m(k)-1} h_{r,k} \le T < \sum_{r=0}^{m(k)} h_{r,k}$.

(ii) $x_{\circ,k} = x_\circ$, $y_{\circ,k} = y_\circ$.

(iii) For all $r = 0, 1, 2, ..., m(k)-2$ we have

$$x_{r+1,k} = x_{r,k} + h_{r,k}\, y_{r,k} + \frac{1}{2}(h_{r,k})^2\, u_{r,k} \in P(x_{r,k})$$

and

$$y_{r+1,k} = y_{r,k} + h_{r,k}\, u_{r,k} . \tag{2}$$

(iv) For all $r = 0, 1, 2, ..., m(k)-1$ we have

$$v_{r,k} \in F(x_{j_r}, y_{j_r}), \qquad d\Big((x_{r,k}, y_{r,k}), (x_{j_r}, y_{j_r})\Big) < \frac{1}{k}$$

and

$$\|u_{r,k} - v_{r,k}\| < \frac{1}{k} .$$

By Lemma 3.1 there exist $\eta_k > 0$, $h_{\circ,x} \in [\eta_k, \frac{1}{k}]$, $u_{\circ,k}, v_{\circ,k} \in H$ and $(x_{j_\circ}, y_{j_\circ}) \in Q_\circ$, such that:

$$\begin{gathered} x_\circ + h_{\circ,k}\, y_\circ + \frac{1}{2}\,(h_{\circ,k})^2\, u_{\circ,k} \in P(x_\circ) \subset K , \\ v_{\circ,k} \in F(x_{j_\circ}, y_{j_\circ}) , \\ d\Big((x_\circ, y_\circ), (x_{j_\circ}, y_{j_\circ})\Big) < \frac{1}{k}, \qquad \|u_{\circ,k} - v_{\circ,k}\| < \frac{1}{k} . \end{gathered} \tag{3}$$

Define

$$x_{1,k} = x_\circ + h_{\circ,k}\, y_\circ + \frac{1}{2}\,(h_{\circ,k})^2\, u_{\circ,k}$$

and

$$y_{1,k} = y_\circ + h_{\circ,k}\, u_{\circ,k}\ .$$

Then $x_{1,k} \in P(x_\circ)$ and if $h_{\circ,k} < T$ we have

$$\begin{aligned}\|x_{1,k} - x_\circ\| &\leq h_{\circ,k}\,\|y_\circ\| + \frac{1}{2}\,(h_{\circ,k})^2\,\|u_{\circ,k}\| \\ &< h_{\circ,k}\,\|y_\circ\| + \frac{1}{2}\,(h_{\circ,k})^2\Big(\|v_{\circ,k}\| + \frac{1}{k}\Big) \\ &< T\,\|y_\circ\| + \frac{1}{2}\,T^2\,(M+1) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta\ ,\end{aligned}$$

and

$$\begin{aligned}\|y_{1,k} - y_\circ\| &\leq h_{\circ,k}\,\|u_{\circ,k}\| \\ &< h_{\circ,k}\Big(M + \frac{1}{k}\Big) \\ &< T(M+1) < \delta\ .\end{aligned}$$

Therefore $(x_{1,k}, y_{1,k}) \in Q_\circ$. Again by Lemma 3.1 there exist $h_{1,k} \in [\eta_k, \frac{1}{k}]$, $u_{1,k}, v_{1,k} \in H$ and $(x_{j_1}, y_{j_1}) \in Q_\circ$ such that

$$\begin{gathered} x_{1,k} + h_{1,k}\, y_{1,k} + \frac{1}{2}\,(h_{1,k})^2\, u_{1,k} \in P(x_{1,k}) \subseteq K\ , \\ v_{1,k} \in F(x_{j_1}, y_{j_1})\ , \\ d\Big((x_{1,k}, y_{1,k}), (x_{j_1}, y_{j_1})\Big) < \frac{1}{k}\,, \qquad \|u_{1,k} - v_{1,k}\| < \frac{1}{k}\ . \end{gathered} \tag{4}$$

If $h_{\circ,k} + h_{1,k} \geq T$ we set $m(k) = 1$, otherwise we define

$$x_{2,k} = x_{1,k} + h_{1,k}\, y_{1,k} + \frac{1}{2}\, h_{1,k}^2\, u_{1,k}$$

and

$$y_{2,k} = y_{1,k} + h_{1,k}\, u_{1,k}\ .$$

By (3) and (4) we obtain $x_{2,k} \in P(x_{1,k})$ and

$$\begin{aligned}\|x_{2,k} - x_\circ\| &= \Big\|x_\circ + h_{\circ,k}\, y_\circ + \frac{1}{2}\, h_{\circ,k}^2\, u_{\circ,k} + h_{1,k}(y_\circ + h_{\circ,k}\, u_{\circ,k}) + \frac{1}{2}\, h_{1,k}^2\, u_{1,k} - x_\circ\Big\| \\ &\leq (h_{\circ,k} + h_{1,k})\,\|y_\circ\| + \frac{1}{2}\, h_{\circ,k}^2\,\|u_{\circ,k}\| + h_{1,k}\, h_{\circ,k}\,\|u_{\circ,k}\| + \frac{1}{2}\, h_{1,k}^2\,\|u_{1,k}\| <\end{aligned}$$

$$
\begin{aligned}
&< (h_{\circ,k}+h_{1,k})\,\|y_\circ\| + \frac{1}{2}\,h_{\circ,k}^2(M+1) + h_{1,k}\,h_{\circ,k}(M+1) + \frac{1}{2}\,h_{1,k}^2(M+1)\\
&= (h_{\circ,k}+h_{1,k})\,\|y_\circ\| + \frac{1}{2}(M+1)\,(h_{\circ,k}+h_{1,k})^2\\
&< T\,\|y_\circ\| + \frac{1}{2}(M+1)\,T^2 \;<\; \frac{\delta}{2}+\frac{\delta}{2} \;=\; \delta\;.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|y_{2,k}-y_\circ\| &\le h_{\circ,k}\|u_{\circ,k}\| + h_{1,k}\|u_{1,k}\|\\
&< h_{\circ,k}(M+1) + h_{1,k}(M+1)\\
&= (h_{\circ,k}+h_{1,k})\,(M+1) \;<\; T(M+1) \;<\; \delta\;.
\end{aligned}
$$

Thus $(x_{2,k},y_{2,k})\in Q_\circ$. Invoking to Lemma 3.1, there exist $h_{2,k}\in[\eta_k,\frac{1}{k}]$, $u_{2,k},v_{2,k}\in H$ and $(x_{j_2},y_{j_2})\in Q_\circ$ such that

$$
\begin{aligned}
&z_{2,k} = x_{2,k} + h_{2,k}\,y_{2,k} + \frac{1}{2}\,h_{2,k}^2\,u_{2,k} \;\in\; P(x_{2,k})\;,\\
&v_{2,k}\in F(x_{j_2},y_{j_2})\;,\\
&d\Big((x_{2,k},y_{2,k}),(x_{j_2},y_{j_2})\Big) < \frac{1}{k}\,,\qquad \|u_{2,k}-v_{2,k}\| < \frac{1}{k}\;.
\end{aligned}
\tag{5}
$$

We reiterate this process. Since $h_{\circ,k}$, $h_{1,k}$, $h_{2,k}$, ... are in $[\eta_k,\frac{1}{k}]$ we are sure that there exists a positive integer $m(k)$ such that for each $r\in\{0,1,...,m(k)-1\}$ there exist $h_{r,k}\in[\eta_k,\frac{1}{k}]$, $(x_{r,k},y_{r,k})\in Q_\circ$, $u_{r,k},v_{r,k}\in H$ and $(x_{j_r},y_{j_r})\in Q_\circ$ with properties in (2).

Now let us set $t_k^\circ=0$ and $t_k^r = h_{\circ,k}+h_{1,k}+\cdots+h_{r-1,k}$; $r\in\{1,2,...,m(k)\}$. We remark that for all $r\in\{1,2,...,m(k)\}$ we have

$$
t_k^r - t_k^{r-1} < \frac{1}{k} \qquad\text{and}\qquad t_k^{m(k)-1}\le T < t_k^{m(k)}\;.
$$

We define a function $x_k\colon[0,T]\to H$ as follows. If $t\in[t_k^{r-1},t_k^r]$, $r\in\{1,2,...,m(k)\}$ we put

$$
x_k(t) \;=\; x_{r-1,k} + (t-t_k^{r-1})\,y_{r-1,k} + \frac{1}{2}\,(t-t_k^{r-1})^2\,u_{r-1,k}\;.
$$

Then we get

$$
x_k(t_k^{r-1}) = x_{r-1,k} \;\in\; P(x_{r-2,k})\,,\qquad r\in\{1,2,...,m(k)\}\;,\tag{6}
$$

$$
x_k{}'(t) = y_{r-1,k} + (t-t_k^{r-1})\,u_{r-1,k}\,,\qquad \forall\; t\in[t_k^{r-1},t_k^r],\; r\in\{1,2,...,m(k)\}\;,\tag{7}
$$

$$
x_k{}''(t) = u_{r-1,k}\,,\qquad \forall\; t\in[t_k^{r-1},t_k^r],\; r\in\{1,2,\ldots,m(k)\}\;.\tag{8}
$$

Hence by (2), for all $t \in [0,T]$ we obtain

$$
\begin{aligned}
\|x_k(t)\| &\leq \|x_{r-1,k}\| + \frac{1}{k}\|y_{r-1,k}\| + \frac{1}{2}\frac{1}{k^2}\|u_{r-1,k}\| \\
&\leq \|x_{r-1,k} - x_\circ\| + \|x_\circ\| + \|y_{r-1,k} - y_\circ\| \\
&\quad + \|y_\circ\| + \|u_{r-1,k} - v_{r-1,k}\| + \|v_{r-1,k}\| \\
&< 2\,\delta + \|x_\circ\| + \|y_\circ\| + \frac{1}{k} + M \\
&\leq 2\,\delta + \|x_\circ\| + \|y_\circ\| + 1 + M
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\|{x_k}'(t)\| &\leq \|y_{r-1,k}\| + \frac{1}{k}\|u_{r-1,k}\| \\
&\leq \|y_\circ\| + \delta + \frac{1}{k}\Big(\|u_{r-1,k} - v_{r-1,k}\| + \|v_{r-1,k}\|\Big) \\
&< \|y_\circ\| + \delta + \frac{1}{k}\Big(\frac{1}{k} + M\Big) \\
&\leq \|y_\circ\| + \delta + M + 1
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
\|{x_k}''(t)\| = \|u_{r-1,k}\| &\leq \|u_{r-1,k} - v_{r-1,k}\| + \|v_{r-1,k}\| \\
&< \frac{1}{k} + M \\
&\leq M + 1\ .
\end{aligned} \tag{11}
$$

Moreover, let t be a fixed point in $[0,T]$. Then there is $r \in \{1,2,...,m(k)\}$ such that $t \in [t_k^{r-1}, t_k^r]$. We have

$$
\begin{aligned}
\|x_k(t) - x_{j_{r-1}}\| &\leq \|x_{r-1,k} - x_{j_{r-1}}\| + \frac{1}{k}\|y_{r-1,k}\| \\
&\quad + \frac{1}{2}\frac{1}{k^2}\Big(\|u_{r-1,k} - v_{r-1,k}\| + \|v_{r-1,k}\|\Big) \\
&< \frac{1}{k} + \frac{1}{k}\Big(\delta + \|y_\circ\|\Big) + \frac{1}{2}\frac{1}{k^2}\Big(\frac{1}{k} + M\Big) \\
&\leq \frac{1}{k}\Big(2 + \delta + \|y_\circ\| + M\Big)\ ,
\end{aligned}
$$

$$
\begin{aligned}
\|{x_k}'(t) - y_{j_{r-1}}\| &\leq \|y_{r-1,k} - y_{j_{r-1}}\| + \frac{1}{k}\Big(\|u_{r-1,k} - v_{r-1,k}\| + \|v_{r-1,k}\|\Big) \\
&< \frac{1}{k} + \frac{1}{k}\Big(\frac{1}{k} + M\Big) \\
&\leq \frac{1}{k}(M+2)
\end{aligned}
$$

and

$$\|x_k''(t) - v_{r-1,k}\| = \|u_{r-1,k} - v_{r-1,k}\| < \frac{1}{k}\,.$$

Since $v_{r-1,k} \in F(x_{j_{r-1}}, y_{j_{r-1}})$, then we obtain

$$\Big(x_k(t), x_k'(t), x_k''(t)\Big) \in \text{ graph } F + \varepsilon_k\Big(B(0,1) \times B(0,1) \times B(0,1)\Big)\,, \tag{12}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. Since t is arbitrary point in $[0,T]$, the relation (12) is true for all $t \in [0,T]$.

By (10) and (11) the sequences (x_k) and (x_k') are equicontinuous. In order to apply Ascoli–Arzela theorem we are going to show that for every $t \in [0,T]$ the two sets $Z_1(t) = \{x_k(t) \colon k \geq 1\}$ and $Z_2(t) = \{x_k'(t) \colon k \geq 1\}$ are relatively compact in H. So, for every $k \geq 1$ let $\theta_k \colon [0,T] \to [0,T]$ defined by $\theta_k(0) = 0$, $\theta_k(t) = t_k^r$, $t \in\,]t_k^{r-1}, t_k^r]$. Also let $Q_{\circ,1} = \{x \colon (x,y) \in Q_\circ \text{ for some } y\}$, $Q_{\circ,2} = \{y \colon (x,y) \in Q_\circ \text{ for some } x\}$. Hence, each of $Q_{\circ,1}$ and $Q_{\circ,2}$ is compact in H. From the definition of (x_k) and (x_k') we have for all $k \geq 1$ and all $t \in [0,T]$, $x_k(\theta_k(t)) \in Q_{\circ,1}$, $x_k'(\theta_k(t)) \in Q_{\circ,2}$. Thus for all $t \in [0,T]$ the two sets $\{x_k(\theta_k(t)) \colon k \geq 1\}$ and $\{x_k'(\theta_k(t)) \colon k \geq 1\}$ are relatively compact in H. Now, for all $t \in [0,T]$

$$\begin{aligned}\alpha(Z_1(t)) &= \alpha\Big\{x_k(t) \colon k \geq 1\Big\}\\ &= \alpha\Big\{x_k(t) - x_k(\theta_k(t)) + x_k(\theta_k(t)) \colon k \geq 1\Big\}\,.\end{aligned}$$

From (iii) and (iv) of Lemma 2.1 we get

$$\alpha(Z_1(t)) \leq \alpha\Big\{x_k(t) - x_k(\theta_k(t)) \colon k \geq 1\Big\} + \alpha\Big\{x_k(\theta_k(t)) \colon k \geq 1\Big\}\,.$$

Since the set $\{x_k(\theta_k(t)) \colon k \geq 1\}$ is relatively compact, $\alpha\{x_k(\theta_k(t)) \colon k \geq 1\} = 0$ (Lemma 2.1 (i)). Then

$$\begin{aligned}\alpha(Z_1(t)) &\leq \alpha\Big\{x_k(t) - x_k(\theta_k(t)) \colon k \geq 1\Big\}\\ &= \alpha\left\{\int_t^{\theta_k(t)} x_k'(s)\,ds \colon k \geq 1\right\}.\end{aligned}$$

By relation (10) we obtain

$$\begin{aligned}\alpha(Z_1(t)) &\leq \alpha\left(B\Big(0, \frac{1}{k}\Big(\|y_\circ\| + \delta + M + 1\Big)\Big)\right)\\ &= \frac{2}{k}\Big(\|y_\circ\| + \delta + M + 1\Big)\,. \qquad \text{(By Lemma 2.1 (v))}\end{aligned}$$

Since $\frac{1}{k} \to 0$ as $k \to \infty$, $\alpha(Z_1(t)) = 0$. Hence $Z_1(t)$ is relatively compact. Similarly the set $Z_2(t)$ is relatively compact. By a corollary of Ascoli–Arzela theorem (see Th. 0.3.4 [3]) the sequence (x_k), $k \geq 1$ has a subsequence (again denoted by (x_k)) and absolutely continuous function $x\colon [0,T] \to H$ with absolutely continuous derivative x' such that (x_k) converges uniformly to x on $[0,T]$, $(x_k{}')$ converges uniformly to x' on $[0,T]$ and $(x_k{}'')$ converges weakly in $L^2([0,T],H)$ to x''. Invoking to the convergence theorem (see Th. 1.4.1 [3]) we get that

$$x''(t) \in \operatorname{co} F(x(t), x'(t)) \quad \text{a.e. on } [0,T] \,. \tag{13}$$

Note that here the values of F are not necessary convex. Now we use condition (H3) to show that

$$x''(t) \in F(x(t), x'(t)) \quad \text{a.e. on } [0,T] \,.$$

Since V is proper convex lower semicontinuous then by Lemma 3.3 in [11], we have

$$\frac{d}{dt} V(x'(t)) = \|x''(t)\|^2 \quad \text{a.e. on } [0,T] \,.$$

Then

$$V(x'(T)) - V(x'(0)) = \int_0^T \|x''(t)\|^2 \, dt \,. \tag{14}$$

From (2) for every integer $k \geq 1$ and every $r \in \{1, 2, ..., m(k)\}$ there exist $\alpha_{r-1,k}$, $\beta_{r-1,k}$, $\gamma_{r-1,k} \in B(0, \frac{1}{k})$ such that

$$\begin{aligned} u_{r-1,k} - \gamma_{r-1,k} = v_{r-1,k} &\in F\Big(x_{r-1,k} - \alpha_{r-1,k} \,,\ y_{r-1,k} - \beta_{r-1,k}\Big) \\ &\subseteq \partial V(y_{r-1,k} - \beta_{r-1,k}) \,. \end{aligned} \tag{15}$$

From the definition of the subdifferential ∂V, the last relation gets us

$$\begin{aligned} &V(y_{r,k} - \beta_{r,k}) - V(y_{r-1,k} - \beta_{r-1,k}) \geq \\ &\quad \geq \Big\langle u_{r-1,k} - \gamma_{r-1,k} \,,\ y_{r,k} - \beta_{r,k} - (y_{r-1,k} - \beta_{r-1,k})\Big\rangle \\ &\quad = \Big\langle u_{r-1,k} - \gamma_{r-1,k} \,,\ y_{r,k} - y_{r-1,k} + \beta_{r-1,k} - \beta_{r,k}\Big\rangle \\ &\quad = \Big\langle u_{r-1,k} - \gamma_{r-1,k} \,,\ x_k{}'(t_k^r) - x_k{}'(t_k^{r-1}) + \beta_{r-1,k} - \beta_{r,k}\Big\rangle \\ &\quad = \Big\langle u_{r-1,k} - \gamma_{r-1,k} \,,\ \int_{t_k^{r-1}}^{t_k^r} x_k{}''(t) \, dt\Big\rangle + \Big\langle u_{r-1,k} - \gamma_{r-1,k} \,,\ \beta_{r-1,k} - \beta_{r,k}\Big\rangle = \end{aligned}$$

$$\begin{aligned}
&= \Big\langle u_{r-1,k}\,,\ u_{r-1,k}(t_k^r - t_k^{r-1})\Big\rangle - \Big\langle \gamma_{r-1,k}\,,\ \int_{t_k^{r-1}}^{t_k^r} x''(t)\,dt \Big\rangle \\
&\quad + \Big\langle u_{r-1,k} - \gamma_{r-1,k}\,,\ \beta_{r-1,k} - \beta_{r,k}\Big\rangle \\
&= (t_k^r - t_k^{r-1})\,\|u_{r-1,k}\|^2 - \Big\langle \gamma_{r-1,k}\,,\ \int_{t_k^{r-1}}^{t_k^r} x''(t)\,dt \Big\rangle \\
&\quad + \Big\langle u_{r-1,k} - \gamma_{r-1,k}\,,\ \beta_{r-1,k} - \beta_{r,k}\Big\rangle\,.
\end{aligned}$$

Since

$$\begin{aligned}
\Big\langle u_{r-1,k}\,,\ u_{r-1,k}(t_k^r - t_k^{r-1})\Big\rangle &= (t_k^r - t_k^{r-1})\,\langle u_{r-1,k}\,,\, u_{r-1,k}\rangle \\
&= (t_k^r - t_k^{r-1})\,\|u_{r-1,k}\|^2\,,
\end{aligned}$$

thus, for all positive integer number k and all $r \in \{1, 2, ..., m(k)-1\}$ we have

$$\begin{aligned}
&V\Big(x'(t_k^r) - \beta_{r,k}\Big) - V\Big(x'(t_k^{r-1}) - \beta_{r-1,k}\Big) \ \geq \\
&\qquad \geq \int_{t_k^{r-1}}^{t_k^r} \|x''(t)\|^2\,dt - \Big\langle \gamma_{r-1,k}\,,\ \int_{t_k^{r-1}}^{t_k^r} x''(t)\,dt \Big\rangle \\
&\qquad\quad + \Big\langle u_{r-1,k} - \gamma_{r-1,k}\,,\ \beta_{r-1,k} - \beta_{r,k}\Big\rangle\,.
\end{aligned} \tag{16}$$

Also, from (2)–(i) we have $t_k^{m(k)-1} \leq T < t_k^{m(k)}$, then from (15) when $r = m(k)$ one has,

$$\begin{aligned}
&V\Big(x_k{}'(T)\Big) - V\Big(y_{m(k)-1,k} - \beta_{m(k)-1,k}\Big) \ \geq \\
&\qquad \geq \Big\langle u_{m(k)-1,k} - \gamma_{m(k)-1,k}\,,\ x_k{}'(T) - y_{m(k)-1,k} + \beta_{m(k)-1,k}\Big\rangle \\
&\qquad = \Big\langle u_{m(k)-1,k} - \gamma_{m(k)-1,k}\,,\ x_k{}'(T) - x_k{}'(t_k^{m(k)-1}) + \beta_{m(k)-1,k}\Big\rangle \\
&\qquad = \Big\langle u_{m(k)-1,k}\,,\ \int_{t_k^{m(k)-1}}^{T} x_k{}''(t)\,dt \Big\rangle - \Big\langle \gamma_{m(k)-1,k}\,,\ \int_{t_k^{m(k)-1}}^{T} x_k{}''(t)\,dt \Big\rangle \\
&\qquad\quad + \Big\langle u_{m(k)-1,k} - \gamma_{m(k)-1,k}\,,\ \beta_{m(k)-1,k}\Big\rangle\,.
\end{aligned}$$

Then

$$\begin{aligned}
&V\Big(x_k{}'(T)\Big) - V\Big(x_k{}'(t_k^{m(k)-1}) - \beta_{m(k)-1,k}\Big) \ \geq \\
&\qquad \geq \int_{t_k^{m(k)-1}}^{T} \|x''(t)\|^2\,dt \ - \ \Big\langle \gamma_{m(k)-1,k}\,,\ \int_{t_k^{m(k)-1}}^{T} x_k{}''(t)\,dt \Big\rangle \\
&\qquad\quad + \Big\langle u_{m(k)-1,k} - \gamma_{m(k)-1,k}\,,\ \beta_{m(k)-1,k}\Big\rangle\,.
\end{aligned} \tag{17}$$

By adding the $m(k)-1$ inequalities from (16) and inequality from (17) we get

$$
\begin{aligned}
&V\Big(x_k{}'(T)\Big) - V\Big(y_\circ - \beta_{\circ,k}\Big) \;= \\
&\qquad = V\Big(x_k{}'(T)\Big) - V\Big(x'(t_k^{m(k)-1}) - \beta_{m(k)-1,k}\Big) \\
&\qquad\quad + V\Big(x_k{}'(t_k^{m(k)-1}) - \beta_{m(k)-1,k}\Big) - V\Big(x_k{}'(t_k^{m(k)-2}) - \beta_{m(k)-2,k}\Big) \\
&\qquad\quad + \cdots + \\
&\qquad\quad + V\Big(x_k{}'(t_k{}^1) - \beta_{1,k}\Big) - V(y_\circ - \beta_{\circ,k}) \\
&\qquad \geq \int_0^T \|x_k{}''(t)\|^2\, dt \,+\, \rho(k)\;,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\rho(k) \;=\; & -\sum_{r=1}^{m(k)-1} \Big\langle \gamma_{r-1,k}\,,\; \int_{t_k^{r-1}}^{t_k^r} x_k{}''(t)\,dt \Big\rangle \\
& + \sum_{r=1}^{m(k)-1} \Big\langle u_{r-1,k} - \gamma_{r-1,k}\,,\;\; \beta_{r-1,k} - \beta_{r,k} \Big\rangle \\
& - \Big\langle \gamma_{m(k)-1}\,,\; \int_{t_k^{m(k)-1}}^{T} x_k{}''(t)\,dt \Big\rangle \\
& + \Big\langle u_{m(k)-1,k} - \gamma_{m(k)-1,k}\,,\;\; \beta_{m(k)-1,k} \Big\rangle\;.
\end{aligned}
$$

We have

$$
\begin{aligned}
|\rho(k)| \;\leq\; & \sum_{r=1}^{m(k)-1} \|\gamma_{r-1,k}\| \int_{t_k^{r-1}}^{t_k^r} \|x_k{}''(t)\|\,dt \\
& + \sum_{r=1}^{m(k)-1} \|u_{r-1,k} - \gamma_{r-1,k}\|\; \|\beta_{r-1,k} - \beta_{r,k}\| \\
& + \|\gamma_{m(k)-1}\| \int_{t_k^{m(k)-1}}^{T} \|x_k{}''(t)\|\,dt \\
& + \|u_{m(k)-1,k} - \gamma_{m(k)-1,k}\|\; \|\beta_{m(k)-1,k}\| \\
\leq\; & \sum_{r=1}^{m(k)-1} \frac{1}{k}\,\|u_{r-1,k}\|\,(t_k^r - t_k^{r-1}) \;+\; \sum_{r=1}^{m(k)-1} \|v_{r-1,k}\|\,\frac{2}{k} \\
& + \frac{1}{k}\,\|u_{m(k)-1,k}\|\,(T - t_k^{m(k)-1}) \;+\; \|v_{r-1,k}\|\,\frac{1}{k} \\
\leq\; & \sum_{r=1}^{m(k)-1} \frac{M+1}{k^2} \;+\; \sum_{r=1}^{m(k)-1} \frac{2\,M}{k} + \frac{M+1}{k^2} + \frac{M}{k}\;,
\end{aligned}
$$

and this yields $\lim\limits_{k\to\infty}\rho(k)=0$. By (18) we obtain

$$\lim_{k\to\infty} V(x_k{}'(T)) - V(y_\circ) \;\geq\; \lim_{k\to\infty}\sup\int_0^T \|x_k{}''(t)\|^2\,dt\;.$$

Therefore, by (14) we have

$$\int_0^T \|x'(t)\|^2\,dt \;=\; V(x'(T)) - V(x'(0)) \;\geq\; \lim_{k\to\infty}\sup\int_0^T \|x_k{}''(t)\|^2\,dt\;. \tag{19}$$

Since $(x_k{}'')$ converges weakly to x'' in $L^2([0,T],H)$ then relation (19) implies that $(x_k{}'')$ converges strongly to x'' in $L^2([0,T],H)$. Consequently, there is a subsequence of $x_k{}''$, denoted again by $(x_k{}'')$ converges to x'' almost everywhere on $[0,T]$. From (12) we obtain,

$$\lim_{k\to\infty} d\Big(\Big(x_k(t), x_k{}'(t), x_k{}''(t)\Big),\ \operatorname{graph} F\Big) \;=\; 0\;. \tag{20}$$

By the assumptions on F and by Prop. 1.1.2 [3] the graph of F is closed. Then relation (20) yields that

$$x''(t)\in F(x(t),x'(t)) \qquad \text{a.e. on}\ \ [0,T]\;.$$

It remains to prove that $x(t)\in P(x(t))$, for all $t\in[0,T]$ and if $s>t$ then $x(s)\in P(x(t))$.

In order to do this for all $k\geq 1$ let $\delta_k\colon[0,T]\to[0,T]$ be a function defined by:

$$\delta_k(0)=0\,,\qquad \delta_k(t)=t_k^{r-1}$$

for all $t\in(t_k^{r-1},t_k^r]$ and all $r\in\{1,2,...,m(k)\}$. Since

$$t_k^r - t_k^{r-1} < \frac{1}{k} \qquad \text{we get} \qquad \lim_{k\to\infty} x_k(\theta_k(t)) \;=\; \lim_{k\to\infty} x_k(\delta_k(t)) \;=\; x(t)$$

for all $t\in[0,T]$. Let $t\in[0,T]$ be fixed. For every positive integer k there is $r\in\{1,2,...,m(k)\}$ such that $t\in(t_k^{r-1},t_k^r]$. We have

$$x_k(\theta_k(t)) = x_k(t_k^r) \;\in\; P(x_k(t_k^{r-1}))\;. = P(x_k(\delta_k(t)))$$

Since the graph of P is closed, we conclude that

$$x(t)\in P(x(t))\;.$$

Now let $t,s\in[0,T]$ be such that $s>t$. Then for k large enough we can find $r,q\in\{1,2,...,m(k)-1\}$ such that $r>q$, $s\in[t_k^{r-1},t_k^r]$ and $t\in[t_k^{q-1},t_k^q]$.

Assume that $r = q + j$. Clearly $1 \le j < m(k)$. We have

$$x_k(t_k^{r-1}) \in P(x_k(t_k^{r-2})) \ .$$

Since P is transitive,

$$P(x_k(t_k^{r-1})) \subset P(x_k(t_k^{r-2})) \ .$$

Similarly,

$$P(x_k(t_k^{r-2})) \subset P(x_k(t_k^{r-3})) \ .$$

We continue for j steps, hence we get

$$P(x_k(t_k^{r-1})) \subset P(x_k(t_k^{q})) \ .$$

But

$$x_k(t_k^{r}) \in P(x_k(t_k^{r-1})) \ .$$

We obtain

$$x_k(t_k^{r}) \in P(x_k(t_k^{q})) \ .$$

This means that

$$x_k(\theta_k(s)) \in P(x_k(\theta_k(t))) \ .$$

Since $\lim\limits_{k\to\infty} \theta_k(t) = t$, $\lim\limits_{k\to\infty} \theta_k(s) = s$ and the graph of P is closed, we get

$$x(s) \in P(x(t)) \ . \blacksquare$$

If we consider the particular case when $P(x) = K$ for all $x \in K$ we obtain the following Viability Theorem.

Theorem 3.3. *Let K be a closed subset of H, Ω be an open subset of H such that $Q = K \times \Omega$ be a locally compact subset of $H \times H$, F be an upper semicontinuous set-valued function from Q to the family of nonempty compact subsets of H.*

Assume that condition (H3) of Theorem 3.2 and the following condition are satisfied:

(**H4**) *For all $(x, y) \in Q$, $F(x, y) \cap T_K^{(2)}(x, y) \neq \emptyset$.*

Then for all $(x_\circ, y_\circ) \in Q$ there exists $T > 0$ and an absolutely continuous function $x\colon [0, T] \to H$ with absolutely continuous derivative such that:

$$x''(t) \in F(x(t), x'(t)) \quad \text{a.e. on } [0, T] \ ,$$

$$(x(t), x'(t)) \in Q \, , \quad \forall\, t \in [0, T] \ ,$$

$$x(0) = 0 \, , \quad x'(0) = y_\circ \ . \blacksquare$$

Remark. If we suppose that the dimension of H is finite, in Theorem 3.3, we obtain Theorem 2.1 of [20]. □

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