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# A NONLOCAL FRICTION PROBLEM FOR A CLASS OF NON-NEWTONIAN FLOWS

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**Abstract:** In this work we study the flow for a class of non-Newtonian fluids with a nonlocal friction condition obtained by the mollification of the normal stresses on part of the boundary. Considering a reformulated problem using an abstract boundary operator, we prove an existence result for the steady case. The mathematical framework of the paper is mainly constituted by the duality theory of convex analysis and an application of a fixed point theorem to multivalued mappings.

## 1 – Introduction and statement of the problem

The class of non-Newtonian fluids considered in this work includes the "power law" of Ostwald and de Waele, the Carreau, Prandtl-Eyring, Williamson, Cross, Ellis models (see [BAH]), the Ladyzenskaya model (see [L1,2]) and Bingham fluids (see [DL]).

Here we suppose that the classical adherence of the fluid to the boundary enclosing its flows does not hold everywhere and there exists a friction phenomenum in part of it. We extend the previous result for Bingham fluids already obtained in [Co].

The friction problem for fluids was proposed in [Se]. Serrin suggested to introduce a local friction law on a part of the boundary of the domain. However a local law does not give enough regularity to the normal stress tensor as it was observed first in elastostatic problems ([DL] and [D]).

Nonlocal friction laws arising in elasticity have been considered by various authors. References can be found in [DL], [KO], [M], [P] and in particular for

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elastodynamics problems (see [OM]). Roughly speaking, a nonlocal friction law proposes that impending motion at a point of contact between two deformable continuous media will occur when the shear stress at that point reaches a value proportional to a weighted measure of the normal stresses in a neighborhood of the point. The character of the effective local neighborhood and the manner in which neighborhood stresses contribute to the sliding condition depends upon features of the microstructure of the materials in contact.

Different simplifications were used to contourn the difficulty of lack of regularity of the normal stress tensor, for instance, it is assumed nonpositive by [C], [NJH] and [J], or even constant by [C1] and [EMS]. Here we extend the approach of [D] and [P] for a class of non-Newtonian flows.

As examples of the effect of friction on the boundary for some stationary flows, we refer the papers of [Mi] and [HP].

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ with a sufficiently smooth boundary $\Gamma$ which consists of the union of the closure of two open subsets, e.g., $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$. The flow problem is formulated by the equations of motion for an incompressible non-Newtonian fluid:

$$\nabla\cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{in } \Omega ;$$
$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot\sigma + \mathbf{f} \qquad \text{in } \Omega .$$

Here $\mathbf{u} = (u_i)_{i=1,\dots,n}$ denotes the velocity vector and $\mathbf{f}$ the body forces. The stress tensor $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$ satisfies the constitutive law (see [C2]):

$$\tau = \pi I + \sigma \in \partial\mathcal{F}(D(\mathbf{u})) ,$$

where $D(\mathbf{u}) = [D_{ij}\mathbf{u}] = (1/2)\,[\partial u_i/\partial x_j + \partial u_j/\partial x_i]$ is the symmetric part of $\nabla\mathbf{u}$, $\pi$ the pressure, $I$ the identity matrix and the functional $\mathcal{F}$ is the sum of the products of constant viscosities $\mu_l$ and arbitrary convex functions $F_l$ $(l = 1, ..., L)$:

$$\mathcal{F}(\tau) = \sum_{l=1}^{L} \mu_l\, F_l(|\tau|) ,$$

reminding that $2\,D_{II} = D\!:\!D = tr(D^2) = |D|^2$. This law means that the viscous part $\tau = (\tau_{ij})$ of the stress tensor belongs to the subdifferential of the functional $\mathcal{F}$ at the point given by the velocity of strain tensor $D(\mathbf{u})$. This class of fluids have been studied in particular in the papers [Co1] and [Co2] with classical boundary conditions.

We assume for sake of simplicity that the density $\rho \equiv 1$. The boundary condition is prescribed on the part of the boundary where the fluid adheres to the wall, i.e.,

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0 .$$

In the remaining part of the boundary we assume a friction condition on $\Gamma_1$ (see [DL]):

$$\mathbf{u}_N = 0 \qquad \text{and} \quad \begin{cases} |\sigma_T| < k\,|\sigma_N| \implies \mathbf{u}_T = \mathbf{0}\,, \\ |\sigma_T| = k\,|\sigma_N| \implies \exists\,\lambda \geq 0, \quad \mathbf{u}_T = -\lambda\,\sigma_T\,. \end{cases} \tag{1}$$

We denote by $k > 0$ the coefficient of friction,

$$\mathbf{u}_T = \mathbf{u} - \mathbf{u}_N\,\mathbf{n}\,, \quad \mathbf{u}_N = \mathbf{u}\cdot\mathbf{n} \quad \text{on } \Gamma_1 \tag{2}$$

the tangential and normal velocities, where $\mathbf{n}$ represents the unit outward normal to the boundary, and

$$\sigma_T = \sigma\cdot\mathbf{n} - \sigma_N\mathbf{n}\,, \quad \sigma_N = (\sigma\cdot\mathbf{n})\cdot\mathbf{n} \tag{3}$$

the tangential and normal stress tensors on the boundary, respectively.

**Remark 1.** Notice that (1) is equivalent to (see [DL] or [KO])

$$\sigma_T\,\mathbf{u}_T + k\,|\sigma_N|\,|\mathbf{u}_T| \;=\; 0 \quad \text{on a.e. } \Gamma_1\ ;$$

which implies that, for arbitrary smooth $\mathbf{v}$,

$$\sigma_T(\mathbf{v}_T - \mathbf{u}_T) + k\,|\sigma_N|\,\{|\mathbf{v}_T| - |\mathbf{u}_T|\} \;\geq\; 0 \quad \text{on a.e. } \Gamma_1\ . \ \square$$

Instead the classical pointwise Coulomb law we shall adopt a non local friction law introduced in [D]

$$\begin{cases} |\sigma_T| < k\,\Phi(\sigma_N) \implies \mathbf{u}_T = 0\,, \\ |\sigma_T| = k\,\Phi(\sigma_N) \implies \exists\,\lambda \geq 0\colon \ \mathbf{u}_T = -\lambda\,\sigma_T\,. \end{cases}$$

One possible approach to the boundary operator $\Phi$ would consist of defining it as the convolution of the given $\sigma_N$ with a $C_0^\infty$ mollifier $\rho$, i.e.,

$$\Phi(\sigma_N) = |\rho * \sigma_N|\ ,$$

considering

$$\rho * \sigma_N(x,t) = \int_{\Gamma_1} \rho(|x-s|)\, \sigma_N(s,t)\ ds\ ,$$

where $ds$ denotes the measure over the boundary $\Gamma_1$.

Next section, we introduce the weak formulation of above problem and we state the existence theorem. In section 3, we recall some preliminary results. In section 4, we solve an auxiliary problem, its dual and we prove the continuous dependence of the solutions. In section 5 we prove the main theorem.

## 2 – The weak formulation

Let be $\Omega \in C^{1,1}$. Concerning the variational formulation, we define the functional spaces, for $p, r > 1$,

$$V_p = \Big\{ \mathbf{v} \in [W^{1,p}(\Omega)]^n :\ \nabla\cdot \mathbf{v} = 0 \text{ in } \Omega,\ \ \mathbf{v} = 0 \text{ on } \Gamma_0,\ \ v_N = 0 \text{ on } \Gamma_1 \Big\}\ ;$$

$$H_r = \Big\{ \mathbf{v} \in [L^r(\Omega)]^n :\ \nabla\cdot \mathbf{v} = 0 \text{ in } \Omega,\ \ v_N = 0 \text{ on } \partial\Omega \Big\}\ ;$$

$$L^p_{sym}(\Omega) = \Big\{ \tau = (\tau_{ij}) :\ \tau_{ij} = \tau_{ji} \in L^p(\Omega) \Big\}\ ;$$

equipped with their natural norms, assuming always that $\text{meas}(\Gamma_0) > 0$, so that the Poincaré's inequality holds in $\Omega$. We denote by $\langle \cdot, \cdot \rangle$ every duality pairing.

We now state the variational formulation of the above problem.

**(Pb)** *Find* $\mathbf{u} \in V_p$ , $\sigma \in L^{p'}_{sym}(\Omega)$ *and* $\pi \in L^{p'}(\Omega)$ *satisfying*

$$\begin{cases} \tau = \pi I + \sigma \ \in\ \partial\mathcal{F}(D(\mathbf{u}))\ ; \\ \langle (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}-\mathbf{u} \rangle + \displaystyle\int_\Omega \mathcal{F}(D(\mathbf{v}))\, dx - \int_\Omega \mathcal{F}(D(\mathbf{u}))\, dx\ + \\ \qquad\qquad + \displaystyle\int_{\Gamma_1} k\, \Phi(\sigma_N(\mathbf{u}))\, \{|\mathbf{v}_T| - |\mathbf{u}_T|\}\ ds\ \geq\ \langle \mathbf{f}, \mathbf{v}-\mathbf{u} \rangle\,, \qquad \forall\, \mathbf{v} \in V_p\ . \end{cases}$$

Since the functional $\mathcal{F}$ is not differentiable, a variational inequality appears. The main step of the proof of existence of solutions is to introduce Lagrange multipliers with the purpose to replace the inequality by a system of equalities. Hence, $\tau$ appears as a Lagrange multiplyer associated with the structure of the problem. Then, the proof of the existence of solutions requires a fixed point theorem for a suitable multivalued mapping.

In order to state the main result, some standard assumptions are presented.

We assume that for each $l = 1, ..., L$:

(4) $\mu_l \in \mathbb{R}, \quad \mu_l > 0 \ ;$

(5) $F_l\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a convex function such that $F_l(0) = 0$ ,

(6) $\exists\, p_l > 1, \ \ \exists\, C_l > 0\colon \quad F_l(d) \le C_l(d^{p_l} + 1) \quad \forall\, d > 0 \ ;$

and exists $l_0 \in \{1, ..., L\}$, defining $p = p_{l_0} = \max\{p_l\colon\ l = 1, ..., L\}$

(7) $$\exists\, \alpha > 0\colon \qquad \alpha\, d^p \le F_{l_0}(d) \quad \forall\, d \ge 0 \ .$$

We define as a positive cone for $1 \le s \le +\infty$

$$L^s_+(\Gamma_1) \ = \Big\{\psi \in L^s(\Gamma_1)\colon\ \psi \ge 0\Big\} \ ,$$

and we assume

(8) $$\mathbf{f} \in [L^{p'}(\Omega)]^n \quad \text{and} \quad k \in L^\infty_+(\Gamma_1) \ .$$

**Theorem 1.** *If we assume (4)–(8), $F_{l_0}$ is strictly convex and*

$$\Phi\colon\ L^{p'}(\Gamma_1) \to L^{p'}_+(\Gamma_1) \ \ \textit{is a weakly continuous operator}, \ \ \textit{i.e.},$$

(9) $$\tau_\eta \rightharpoonup \tau \ \ \textit{in}\ L^{p'}(\Gamma_1) \quad \Longrightarrow \quad \Phi(\tau_\eta) \rightharpoonup \Phi(\tau) \ \ \textit{in}\ L^{p'}(\Gamma_1) \ ,$$

*then for $p > 3n/(n+2)$ the problem (Pb) has at least one solution.*

## 3 – Preliminary results

We recall some results that will be used later. First, a fixed point theorem for multivalued mappings (see [BC], pages 218–220).

**Theorem 2.** *Let $E$ be a locally convex Hausdorff topological vector space and let $K$ be a non-empty convex compact set in $E$. If*

$$\Psi\colon\ K \longrightarrow \Big\{R \in \mathcal{P}(K)\colon\ R \ne \emptyset,\ R \ \textit{closed convex}\ \Big\}$$

*is an upper semi-continuous mapping then $\Psi$ has at least one fixed point, i.e., $e \in \Psi(e)$ for some $e \in K$.*

**Remark 2.** *Notice that* $\Psi$ *is upper semi-continuous iff* $G_{KK}(\Psi) := \{(x,y) \in K\times K\colon\ y\in\Psi(x)\}$ *is closed in* $K\times K$ *with the product topology (see [BC], page 413).* $\square$

Next we summarize some existence results from the duality theory of convex optimization in the following proposition (see [ET], pages 50–52).

**Proposition 1.** *Let* $U$ *and* $Z$ *be locally convex Hausdorff topological vector spaces and* $\Upsilon\colon U\times Z \to \mathbb{R}$ *be a convex functional. If there exists* $u_0 \in U$ *such that* $\varsigma \mapsto \Upsilon(u_0,\varsigma)$ *is finite and continuous at* $0$ $(\in Z)$*, and* $\Upsilon(\mathbf{u},0)$ *is finite where* $\mathbf{u}$ *is a minimizer in* $U$ *of the initial problem* $(\varsigma = 0)$*:*

$$(\Pi_\varsigma) \qquad \inf_{\mathbf{v}\in U} \Upsilon(\mathbf{v},\varsigma)\ ;$$

*then there exists, at least, a Lagrange multiplyer* $\varsigma^* \in Z'$ *solution of the dual problem*

$$(\Pi^*) \qquad \sup_{\varsigma^*\in Z'} \left[-\Upsilon^*(0,\varsigma^*)\right]\ ,$$

*where the conjugate mapping* $\Upsilon^*\colon U'\times Z' \to \mathbb{R}$ *is defined by*

$$\Upsilon^*(\mathbf{v}^*,\varsigma^*) \ = \sup_{(\mathbf{v},\varsigma)\in D(\Upsilon)} \Big[\langle(\mathbf{v}^*,\varsigma^*),(\mathbf{v},\varsigma)\rangle - \Upsilon(\mathbf{v},\varsigma)\Big]\ .$$

*Moreover,* $\mathbf{u}\in U$ *and* $\varsigma^* \in Z'$ *satisfy the relation*

$$\Upsilon(\mathbf{u},0) + \Upsilon^*(0,\varsigma^*) \ = \ 0\ . \tag{10}$$

We recall now some properties of the convective term.

**Lemma 1.** *The anti-symmetrical trilinear form:*

$$b(\mathbf{w},\mathbf{u},\mathbf{v}) \ := \ \langle(\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle \ = \int_\Omega \mathbf{w}\otimes\mathbf{v} : (\nabla\mathbf{u})^T\,dx$$

*is well defined for* $\mathbf{u},\mathbf{v}\in V_p$ *with* $p>1$*, and for* $\mathbf{w}\in H_r$ *with*

$$r \geq \frac{p\,n}{n\,p+p-2\,n} \quad \text{if } p<n, \qquad \text{or} \qquad r\geq p' \quad \text{if } p\geq n\ ; \tag{11}$$

*if* $\mathbf{w}_m \to \mathbf{w}$ *in* $L^r(\Omega)$ *and* $\mathbf{u}_m \rightharpoonup \mathbf{u}$ *weakly in* $V_p$ *then*

$$b(\mathbf{w}_m,\mathbf{u}_m,\mathbf{v}) \ \longrightarrow \ b(\mathbf{w},\mathbf{u},\mathbf{v})\ .$$

Let us introduce the existence of the pressure which proof can be found in [G, pages 164 and 180].

**Proposition 2.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ satisfying the cone condition and $\mathbf{g} \in [W^{-1,p'}(\Omega)]^n$. If*

$$\langle \mathbf{g}, \mathbf{v} \rangle = 0 \qquad \forall\, \mathbf{v} \in \Big\{ \mathbf{v} \in [W_0^{1,p}(\Omega)]^n \colon\ \nabla\cdot\mathbf{v} = 0 \ \text{ in } \Omega \Big\}$$

*then*

$$\exists^1\, \pi \in L_0^{p'}(\Omega) \;=\; \Big\{ \pi \in L^{p'}(\Omega) \colon\ \int_\Omega \pi\, dx = 0 \Big\}$$

*such that*

$$\langle \mathbf{g}, \mathbf{v} \rangle \;=\; \int_\Omega \pi\, \nabla\cdot\mathbf{v}\; dx \qquad \forall\, \mathbf{v} \in [W_0^{1,p}(\Omega)]^n \ ;$$

*and the following estimate holds*

$$\exists\, c > 0 \colon \quad \|\pi\|_{p',\Omega} \;\le\; c \sup_{\mathbf{v}\in[W_0^{1,p}(\Omega)]^n} \frac{|\langle \nabla\pi, \mathbf{v}\rangle|}{\|D(\mathbf{v})\|_{p,\Omega}} \;=\; c\, \|\nabla\pi\|_{[W^{-1,p'}(\Omega)]^n} \ .$$

Finally we state a "decomposed" Green's formula (see, for instance, [KO, page 90] or [S]).

**Proposition 3.** *Let be $\gamma_i$ the restriction to $\Gamma_i$ $(i = 0,1)$ of the trace operator and*

$$V = \Big\{ \mathbf{v} \in [W^{1,p}(\Omega)]^n \colon\ \nabla\cdot\mathbf{v} = 0 \ \text{ in } \Omega,\ \gamma_0\mathbf{v} = 0 \Big\} \ ;$$

$$W = \Big\{ \gamma_1\mathbf{v} \colon\ \mathbf{v} \in V \Big\} \ ;$$

$$W_p = \Big\{ \gamma_1\mathbf{v} \colon\ \mathbf{v} \in V_p \Big\} \ .$$

*Then there exist uniquely determined linear continuous mappings $\varpi$ from $V_\sigma := \{\sigma \in L^{p'}_{sym}(\Omega) \colon\ \nabla\cdot\sigma \in V'\}$ into $W'$ and $\varpi_T$ from $\{\sigma \in L^{p'}_{sym}(\Omega) \,:\ \nabla\cdot\sigma \in (V_p)'\}$ into $(W_p)'$ such that*

**i)** $\varpi(\sigma) = \sigma|_{\Gamma_1}\cdot\mathbf{n}$ *and* $\varpi_T(\sigma) = \sigma|_{\Gamma_1}\cdot\mathbf{n} - [(\sigma|_{\Gamma_1}\cdot\mathbf{n})\cdot\mathbf{n}]\mathbf{n}$ *if* $\sigma \in [C^1(\bar\Omega)]^{n\times n}$*;*

**ii)** *the Green's formula is satisfied*

$$\langle \sigma, D(\mathbf{v})\rangle + \langle \nabla\cdot\sigma, \mathbf{v}\rangle \;=\; \langle \varpi\,\sigma, \gamma_1\mathbf{v}\rangle_{W'\times W} \;=\; \langle \varpi_T\,\sigma, \mathbf{v}_T\rangle_{W_p'\times W_p} \ ,$$
$$\forall\, \sigma \in L^{p'}_{sym}(\Omega), \quad \forall\, \mathbf{v} \in V_p \ .$$

For simplicity, we use the notation (2)–(3).

## 4 – An auxiliary problem

Let us introduce the auxiliary problem.

$(\mathbf{Pb})_{\mathbf{w},\psi}$: *Given* $\mathbf{w} \in H_r$ *with (11) and* $\psi \in L_+^{p'}(\Gamma_1)$, *find* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \psi)$ *such that, for all* $\mathbf{v} \in V_p$,

$$(12) \qquad b(\mathbf{w}, \mathbf{u}, \mathbf{v}-\mathbf{u}) + \int_\Omega \{\mathcal{F}(D(\mathbf{v})) - \mathcal{F}(D(\mathbf{u}))\}\, dx + \int_{\Gamma_1} k\, \psi \{|\mathbf{v}_T| - |\mathbf{u}_T|\}\, ds \geq \geq \langle\, \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle\ .$$

The existence and uniqueness of solution is proved in the following proposition.

**Proposition 4.** *Under the previous conditions (4)–(8), for all* $p > 1$, *with* $F_{l_0}$ *being a strictly convex function, then there exists an unique solution* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \psi) \in V_p$ *of the problem* $(Pb)_{\mathbf{w},\psi}$ *such that*

$$(13) \qquad \|\mathbf{u}\|_{V_p}^p \leq \frac{\|\mathbf{f}\|_{p',\Omega}^{p'}}{(\mu_{l_0}\, \alpha)^{p'}}\ .$$

**Proof:** Taking into account that the functional

$$\mathbf{v} \in V_p \ \longmapsto \int_\Omega \mathcal{F}(D(\mathbf{v}))\, dx + \int_{\Gamma_1} k\, \psi\, |\mathbf{v}_T|\, ds$$

is strictly convex, continuous and coercive on $V_p$, the existence of an unique solution to (12) is an consequence of a classical existence theorem (see [ET, pages 39 and 44], for instance).

Taking $\mathbf{v} = \mathbf{0}$ in $(Pb)_{\mathbf{w},\tau}$, the estimate (13) easily follows. ∎

The following proposition establishes that the dual problem of the auxiliary problem $(Pb)_{\mathbf{w},\tau}$ has at least a solution.

**Proposition 5.** *For each solution* $\mathbf{u}$ *of* $(Pb)_{\mathbf{w},\psi}$, *there exists a Lagrange multiplyer* $\varsigma^* = (\varsigma_l^*)_{l=0,1,\dots,L} \in [L^{p'}(\Gamma_1)]^n \times L_{sym}^{p_1'}(\Omega) \times \dots \times L_{sym}^{p_L'}(\Omega)$ *and a pressure field* $\pi \in L^{p'}(\Omega)$ *such that the stress tensor is given by*

$$\sigma = -\pi I - \sum_{l=1}^{L} \varsigma_l^*\ ,$$

*satisfying*

(14) $\quad \langle \varsigma_0^* - \sigma_T, \mathbf{v}_T\rangle = 0 \quad$ *for all* $\mathbf{v} \in V_p$ ;

(15) $\quad \displaystyle\int_{\Gamma_1} k\,\psi\,|\mathbf{u}_T|\,ds = \langle -\varsigma_0^*, \mathbf{u}_T\rangle \quad$ *and* $\quad |\varsigma_0^*| \leq k\,\psi \quad$ *on* $\Gamma_1$ ;

(16) $\quad \displaystyle\int_\Omega \mu_l\,F_l\Big(|D(\mathbf{u})|\Big)\,dx + \int_\Omega \mu_l\,F_l^*\left(\left|\frac{\varsigma_l^*}{\mu_l}\right|\right)dx \;=\; \langle -\varsigma_l^*, D(\mathbf{u})\rangle\,, \qquad l = 1,...,L$ ;

(17) $\quad \displaystyle(\mathbf{w}\cdot\nabla)\mathbf{u} + \sum_{l=1}^{L} \nabla\cdot(\varsigma_l^*) \;=\; \mathbf{f} - \nabla\pi \quad$ *in* $\Omega$ .

*Moreover, for each* $l = 1,...,L$ *we have the following estimate*

(18) $\quad \displaystyle\|\varsigma_l^*\|_{p_l',\Omega}^{p_l'} \;\leq\; \frac{2^{p_l}}{p_l-1}\,(\mu_l\,C_l\,p_l)^{p_l'}\,\|D(\mathbf{u})\|_{p_l,\Omega}^{p_l} + \frac{2}{p_l-1}\,(\mu_l C_l p_l)^{p_l'}\,|\Omega|$ .

*Conversely, if* $\mathbf{u} \in V_p$ *and* $\varsigma^* \in [L^{p'}(\Gamma_1)]^n \times \prod_{l=1}^L L_{sym}^{p_l'}(\Omega)$ *satisfy (14), (15), (16) and (17) then* $\mathbf{u}$ *is the solution of* $(Pb)_{\mathbf{w},\psi}$.

**Proof:** The converse is straightforward. The existence of, at least, a Lagrange multiplyer such that satisfies (14)–(17) follows from Proposition 1 if we set $U := V_p$, $Z := [L^p(\Gamma_1)]^n \times L_{sym}^{p_1}(\Omega) \times ... \times L_{sym}^{p_L}(\Omega)$ and

$$\Upsilon(\mathbf{v},\varsigma) \;=\; J(\mathbf{v}) + G_0(\mathbf{v}_T - \varsigma_0) + \sum_{l=1}^{L} G_l\Big(D(\mathbf{v}) - \varsigma_l\Big)\;,$$

where

$$J\colon\;\; \mathbf{v}\in U \longmapsto \langle(\mathbf{w}\cdot\nabla)\mathbf{u} - \mathbf{f}, \mathbf{v}\rangle$$

(19) $$G\colon\;\; \varsigma \in Z \;\longmapsto\; G(\varsigma) = \sum_{l=0}^{L} G_l(\varsigma_l) := \int_{\Gamma_1} k\,\psi\,|\varsigma_0|\,ds + \sum_{l=1}^{L}\int_\Omega \mu_l\,F_l(|\varsigma_l|)\,dx\;.$$

Indeed, if we define

$$\Lambda(\mathbf{v}) = (\mathbf{v}_T, D(\mathbf{v}), ..., D(\mathbf{v}))$$

then there exists $\varsigma^* = (\varsigma_l^*)_{l=0,1,...,L} \in [L^{p'}(\Gamma_1)]^n \times L_{sym}^{p_1'}(\Omega) \times ... \times L_{sym}^{p_L'}(\Omega)$ and (10) is decomposed into the conditions:

$$\begin{cases} G_0(\gamma_1\mathbf{u}) + G_0^*(-\varsigma_0^*) + \langle\varsigma_0^*, \gamma_1\mathbf{u}\rangle \;=\; 0\;, \\ G_l(D(\mathbf{u})) + G_l^*(-\varsigma_l^*) + \langle\varsigma_l^*, D(\mathbf{u})\rangle \;=\; 0 \qquad l = 1,...,L\;, \\ J(\mathbf{u}) + J^*(\Lambda^*\varsigma^*) + \langle\Lambda^*\varsigma^*, \mathbf{u}\rangle \;=\; 0\;. \end{cases}$$

For each $l = 1, ..., L$ the conjugate function $G_l^*: Y_{p_l'} \to \mathbb{R}$ is expressed by

$$G_l^*(\varsigma) = \int_\Omega [\mu_l F_l(|\varsigma|)]^* dx = \int_\Omega \mu_l F_l^* \left( \left| \frac{\varsigma}{\mu_l} \right| \right) dx \ ;$$

while for $l = 0$, the computation of the conjugate function gives

$$G_0^*(v) = \begin{cases} 0 & \text{if } |v| \le k\,\psi \\ +\infty & \text{if } |v| > k\,\psi \ . \end{cases}$$

Consequently, the relations (16) and (15) arises.

We observe that

$$J^*(\Lambda^*\varsigma^*) = \begin{cases} 0 & \text{if (17) and (14) hold} \\ +\infty & \text{otherwise} \end{cases}$$

because

$$\begin{aligned} J^*(\Lambda^*\varsigma^*) &= \sup_{\mathbf{v}\in V_p} \Big[ \langle \Lambda^*\varsigma^*, \mathbf{v}\rangle - J(\mathbf{v}) \Big] = \sup_{\mathbf{v}\in V_p} \Big[ \langle \varsigma^*, \Lambda\mathbf{v}\rangle - J(\mathbf{v}) \Big] \\ &\ge \sup_{\mathbf{v}\in \mathcal{V}} \Big[ \langle \varsigma^*, \Lambda\mathbf{v}\rangle - \langle (\mathbf{w}\cdot\nabla)\mathbf{u} - \mathbf{f}, \mathbf{v}\rangle \Big] \end{aligned} \tag{20}$$

which permits to deduce (17).

Defining $\sigma = -\pi I - \sum_{l=1}^L \varsigma_l^* \in L^{p'}_{sym}(\Omega)$ and introducing it in (20), we obtain

$$J^*(\Lambda^*\varsigma^*) = \sup_{\mathbf{v}\in V_p} \left[ \langle \varsigma_0^*, \mathbf{v}_T \rangle + \left\langle \sum_{l=1}^L \varsigma_l^*, D(\mathbf{v}) \right\rangle - \langle \nabla\cdot\sigma, \mathbf{v}\rangle \right] .$$

Since $\nabla\cdot\sigma \in (V_p)'$, (14) is a consequence of the Green's formula (cf. Proposition 3).

Finally, for each $l = 1, ..., L$ the estimate (18) follows from introducing in (16) the conjugate inequality of (6)

$$\frac{(\mu_l\, C_l\, p_l)^{1-p_l'}}{p_l'} \int_\Omega |\varsigma_l^*|^{p_l'} dx - \mu_l\, C_l\, |\Omega| \le \int_\Omega \mu_l F_l^* \left( \left| \frac{\varsigma_l^*}{\mu_l} \right| \right) dx \le \|\varsigma_l^*\|_{p_l',\Omega} \, \|D(\mathbf{u})\|_{p_l,\Omega}$$

and applying the Young inequality. ∎

**Remark 3.** Notice that the set of solutions is convex if $\mathbf{u}$ is unique. □

Some estimates are presented in the following proposition.

**Proposition 6.** *Under the same assumptions as before,*

$$\|\pi\|_{p',\Omega} \;\le\; \|(\mathbf{w}\cdot\nabla)\mathbf{u}\|_{p',\Omega} + \Big\|\sum_{l=1}^{L} \varsigma_l^*\Big\|_{p',\Omega} + \|\mathbf{f}\|_{p',\Omega}\;, \tag{21}$$

*and consequently there exists a constant* $C = C(|\Omega|, p_l, \mu_{l_0}\alpha, \|\mathbf{f}\|_{p',\Omega}, \mu_l C_l p_l)$ *such that*

$$\|\sigma\|_{p',\Omega} \;\le\; \|\mathbf{w}\|_{s,\Omega} \left\{\frac{\|\mathbf{f}\|_{p',\Omega}}{\mu_{l_0}\alpha}\right\}^{1/(p-1)} + C\;. \tag{22}$$

**Proof:** It is consequence of Propositions 5 and 2. ∎

Finally, we state the following continuous dependence result.

**Proposition 7.** *Let* $\{\mathbf{w}_m\}$ *and* $\{\psi_m\}$ *be sequences in* $H_r$ *and in* $L_+^{p'}(\Gamma_1)$, *respectively, such that* $\mathbf{w}_m \to \mathbf{w}$ *in* $L^r(\Omega)$ *and* $\psi_m \rightharpoonup \psi$ *in* $L^{p'}(\Gamma_1)$. *If* $\sigma_m$ *are stress tensors associated to the solutions* $\mathbf{u}_m$ *of* $(Pb)_{\mathbf{w}_m,\psi_m}$, *for every* $m \in \mathbb{N}$, *as defined in Proposition 5, then*

- **i)** *there exists* $\mathbf{u}$, *the solution of* $(Pb)_{\mathbf{w},\psi}$, *such that* $\mathbf{u}_m \rightharpoonup \mathbf{u}$ *in* $V_p$;
- **ii)** *there exists a subsequence* $\sigma_m$ *such that*

$$\sigma_m \rightharpoonup \sigma \;=\; -\pi I - \sum_{l=1}^{L} \varsigma_l^* \quad \text{in } \; L_{sym}^{p'}(\Omega) \tag{23}$$

*where* $\varsigma^* = (\varsigma_l^*)_{l=0,1,\dots,L}$ *is a Lagrange multiplyer of the solution of* $(Pb)_{\mathbf{w},\psi}$ *and* $\pi$ *is uniquely determined in the space* $L_0^{p'}(\Omega)$.

**Proof:**

**i)** From the estimate (13), we can extract a subsequence of $\mathbf{u}_m$, also denoted $\mathbf{u}_m$, weakly convergent to $\mathbf{u}$ in $V_p$. Passing to the limit $(Pb)_{\mathbf{w}_m,\psi_m}$, using Lemma 1, the sequential weak lower semicontinuity of the functional $\mathcal{F}$, the compact imbedding $V_p \subset\subset [L^p(\Gamma_1)]^n$ and Fatou's lemma, we conclude that $\mathbf{u}$ is solution of $(Pb)_{\mathbf{w},\psi}$. By uniqueness, whole initial sequence converges.

**ii)** From the estimates (15) and (18) we can extract a subsequence of $\varsigma_m^*$, denoted $\varsigma_{m0}^*$ and $\varsigma_{ml}^*$ $(l = 1, \dots, L)$ respectively, weakly convergent to $\varsigma$ in $[L^{p'}(\Gamma_1)]^n \times L_{sym}^{p_1'}(\Omega) \times \dots \times L_{sym}^{p_L'}(\Omega)$ satisfying

$$\exists\, \pi_m \in L_0^{p'}(\Omega)\colon \;\; \langle \varsigma_{m0}^*, \mathbf{v}_T\rangle + \Big\langle \varpi_T\Big(\pi_m I + \sum_{l=1}^{L} (\varsigma_{ml}^*)\Big), \mathbf{v}_T\Big\rangle = 0 \quad \forall\, \mathbf{v} \in V_p\;; \tag{24}$$

$$(25)\qquad \int_{\Gamma_1} k\,\psi_m\,|\mathbf{u}_{mT}|\,ds = \langle -\varsigma^*_{m0}, \mathbf{u}_{mT}\rangle \quad \text{and} \quad |\varsigma^*_{m0}| \le k\,\psi_m \ \text{ on } \Gamma_1\ ;$$

$$(26)\qquad \int_\Omega \mu_l\, F_l\Big(|D(\mathbf{u}_m)|\Big)\,dx + \int_\Omega \mu_l\, F_l^*\left(\left|\frac{\varsigma^*_{ml}}{\mu_l}\right|\right)dx \;=\; \langle -\varsigma^*_{ml}, D(\mathbf{u}_m)\rangle\ ;$$

$$(27)\qquad \left\langle (\mathbf{w}_m\cdot\nabla)\mathbf{u}_m + \nabla\cdot\left(\pi_m I + \sum_{l=1}^{L}\varsigma^*_{ml}\right),\, \mathbf{v}\right\rangle \;=\; \langle \mathbf{f}, \mathbf{v}\rangle\,, \qquad \forall\,\mathbf{v}\in[\mathcal{D}(\Omega)]^n\ .$$

From (21), we can extract a subsequence such that $\pi_m \rightharpoonup \pi$ in $L^{p'}_0(\Omega)$. Therefore, we pass to the limit (24), since $\varpi_T$ is linear continuous. To the first expression of (25) we use the compact imbedding $V_p \subset\subset [L^p(\Gamma_1)]^n$ while to the second one we take into account the sequential weak lower semicontinuity property. Using Green's formula we also pass easily to the limit (27).

Finally, the term $\langle -\varsigma^*_{ml}, D(\mathbf{u}_m)\rangle$ of (26) can not pass directly to the limit because we only have weak convergences.

Summing the relations from each component of Lagrange multiplyer and using Green's formula, for all $\mathbf{v}\in V_p$

$$\begin{aligned} b(\mathbf{w}_m,\mathbf{u}_m,\mathbf{v}-\mathbf{u}_m) \;=\; & \langle \mathbf{f}, \mathbf{v}-\mathbf{u}_m\rangle + \langle \varsigma^*_{m0}, \mathbf{v}_T\rangle + \sum_{l=1}^{L}\langle \varsigma^*_{ml}, D(\mathbf{v})\rangle \\ & + \int_{\Gamma_1} k\,\psi_m\,|\mathbf{u}_{mT}|\,ds + \int_\Omega \mathcal{F}(D(\mathbf{u}_m))\,dx \\ & + \sum_{l=1}^{L}\int_\Omega \mu_l\,F_l^*\left(\left|\frac{\varsigma^*_{ml}}{\mu_l}\right|\right)dx\ . \end{aligned}$$

From the definition of conjugate function, it follows that the last term is also weakly sequential lower semicontinuous and convex. Then, we are able to pass to the limit as before, obtaining

$$(28)\qquad \begin{aligned} b(\mathbf{w},\mathbf{u},\mathbf{v}-\mathbf{u}) \;\ge\; & \langle \mathbf{f}, \mathbf{v}-\mathbf{u}\rangle + \langle \varsigma_0, \mathbf{v}_T\rangle + \sum_{l=1}^{L}\langle \varsigma_l, D(\mathbf{v})\rangle \\ & + \int_{\Gamma_1} k\,\psi\,|\mathbf{u}_T|\,ds + \int_\Omega \mathcal{F}(D(\mathbf{u}))\,dx \\ & + \sum_{l=1}^{L}\int_\Omega \mu_l\,F_l^*\left(\left|\frac{\varsigma_l}{\mu_l}\right|\right)dx\,, \qquad \forall\,\mathbf{v}\in V_p\ . \end{aligned}$$

If we set $\mathbf{v}=\mathbf{u}$, applying (15) we deduce

$$\int_\Omega \mathcal{F}(D(\mathbf{u}))\,dx + \sum_{l=1}^{L}\int_\Omega \mu_l\,F_l^*\left(\left|\frac{\varsigma_l}{\mu_l}\right|\right)dx \;\le\; -\sum_{l=1}^{L}\langle \varsigma_l, D(\mathbf{u})\rangle\ .$$

Since the opposite inequality always holds, this is a sufficient condition to obtain (see [ET], page 21 and 26)

$$-\sum_{l=1}^{L} \varsigma_l \;\in\; \partial \sum_{l=1}^{L} \int_\Omega \mu_l\, F_l(D(\mathbf{u}))\, dx \;=\; \sum_{l=1}^{L} \partial \int_\Omega \mu_l\, F_l(D(\mathbf{u}))\, dx \;.$$

Thus, we conclude that there exists $\varsigma^* = (\varsigma_l^*)_{l=0,1,\dots,L}$ satisfying $\sum_{l=1}^{L} \varsigma_l^* = \sum_{l=1}^{L} \varsigma_l$ and (14)–(17). ∎

## 5 – Proof of Theorem 1

**Proof:** Let us consider $V_p$ and $L^{p'}(\Gamma_1)$ endowed with the weak topologies, becoming locally convex Hausdorff topological vector spaces. The idea of the proof is to apply the fixed point Theorem 2 to a multivalued mapping defined on a compact convex set. To this aim we are going to define a ball which is a compact convex subset of $V_p \times L^{p'}(\Gamma_1)$ for the weak topologies:

$$\begin{aligned} K \;:=\; &\left\{\mathbf{w} \in V_p\colon\; \|\mathbf{w}\|_{V_p} \le \left\{\frac{\|\mathbf{f}\|_{p',\Omega}}{\mu_{l_0}\alpha}\right\}^{1/(p-1)}\right\} \times \\ &\times \left\{\tau \in L^{p'}(\Gamma_1)\colon\; \|\tau\|_{p',\Gamma_1} \le R\left[\left\{\frac{\|\mathbf{f}\|_{p',\Omega}}{\mu_{l_0}\alpha}\right\}^{2/(p-1)} + C\right]\right\}\;, \end{aligned}$$

where $R$ is a positive constant due to the continuity of trace operator and $C$ given by (22). We define by $\Psi$ the following multivalued mapping

$$\Psi\colon\;\; (\mathbf{w},\tau) \;\in\; V_p \times L^{p'}(\Gamma_1) \longmapsto \{\mathbf{u}\} \times S \;.$$

Since $p > 3n/(n+2)$, we may choose $r$ satisfying (11) such that $V_p \subset\subset H_r$ and $\mathbf{u} = \mathbf{u}(\mathbf{w}, \Phi(\tau))$ is the solution of the auxiliary problem (12). From Proposition 5, the corresponding stress tensor $\sigma$ satisfies

$$\sigma := -\pi I - \sum_{l=1}^{L} \varsigma_l^* \;\in\; L^{p'}_{sym}(\Omega) \quad\text{ and }\quad \nabla\cdot\sigma = (\mathbf{w}\cdot\nabla)\mathbf{u} - \mathbf{f} \;\in\; [L^{p'}(\Omega)]^n \;.$$

Using the notations (3) and recalling Proposition 3, we define

$$S := \left\{\sigma_N = \varpi\Big(-\pi I - \sum_{l=1}^{L} \varsigma_l^*\Big)\cdot\mathbf{n}\right\} \;\subset\; L^{p'}(\Gamma_1) \;.$$

We pretend to apply Theorem 2 to the function $\Psi$. In fact $\Psi$ is a well defined multivalued mapping from $K$ to $2^K_{\blacksquare ck} := \{R \in \mathcal{P}(K) \colon R \neq \emptyset,\ R \text{ closed convex}\}$ because, for each $(\mathbf{w},\tau) \in K$, $\Psi(\mathbf{w},\tau)$ is a non empty convex set. In addition $\Psi(\mathbf{w},\tau)$ is closed and $\Psi$ is an upper semicontinuous mapping if the set $G_{KK}(\Psi)$ will be closed in $K \times K$ (see Remark 2).

Thus, it remains to prove that $G_{KK}(\Psi) = \overline{G_{KK}(\Psi)}$.

Taking $((\mathbf{w},\tau),(\mathbf{u},\zeta))$ in $(V_p \times L^{p'}(\Gamma_1))^2$ such that there exist sequences $(\mathbf{w}_m,\tau_m)$ from $K$ and $(\mathbf{u}_m,\zeta_m)$ from $\Psi((\mathbf{w}_m,\tau_m))$ verifying

$$\mathbf{w}_m \rightharpoonup \mathbf{w}\,,\ \ \mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{in } V_p \qquad \text{and} \qquad \tau_m \rightharpoonup \tau\,,\ \ \zeta_m \rightharpoonup \zeta \quad \text{in } L^{p'}(\Gamma_1)\ ,$$

we deduce $(\mathbf{w},\tau) \in K$ by weak convergence property.

To prove that $\mathbf{u}$ is the required solution, we apply Proposition 7 considering the compact imbedding $\mathbf{w}_m \to \mathbf{w}$ in $H_r$ and the assumption (9) on $\Phi$. So, by uniqueness $\mathbf{u} = \mathbf{u}(\mathbf{w},\Phi(\tau))$. Next, by definition of $\Psi$ there exist Lagrange multipliers $\varsigma^*_m$ and pressures $\pi_m$ such that

$$\zeta_m \;=\; \varpi\Big(-\pi_m I - \sum_{l=1}^{L} \varsigma^*_{ml}\Big)$$

and we will prove that the weak limit $\zeta$ takes also this form.

Indeed, applying Proposition 7 we obtain

$$\exists\, \varsigma^*, \pi: \quad \sigma_m \rightharpoonup \sigma = -\pi I - \sum_{l=1}^{L} \varsigma^*_l \quad \text{in } L^{p'}_{sym}(\Omega)\ .$$

Since $\varpi$ is linear continuous, the result $\zeta = \varpi(\sigma) \in S$ follows from the uniqueness of weak limit.

Finally, Theorem 2 garantees at least a fixed point of $\Psi$, that yields one solution to the problem. ∎

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