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STABILITY AND CONTINUOUS DEPENDENCE OF SOLUTIONS OF ONE-PHASE STEFAN PROBLEMS FOR SEMILINEAR PARABOLIC EQUATIONS

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Abstract: We consider a one-phase Stefan problem for the heat equation with a superlinear reaction term and we prove the stability of fastly decaying global solutions. Also, we establish a result of continuous dependence of local solutions up to the maximum existence time, needed for the stability proof.

1 – Introduction and results

Consider the following reaction-diffusion problem with free boundary:

$$\text{(SP)} \qquad \begin{cases} u_t - u_{xx} = u^p, \quad 0<t<T, \quad 0<x<s(t), \\ u(0,x) = u_0(x) \geq 0, \quad 0<x<s_0, \quad s(0)=s_0>0, \\ u(t,s(t)) = u_x(t,0) = 0, \quad 0<t<T, \\ s'(t) = -u_x(t,s(t)), \quad 0<t<T\ , \end{cases}$$

where $p>1$ is a fixed real number. Problem (SP) can be viewed as a simple model of a chemically reactive and heat-diffusive liquid surrounded by ice. Here $u \geq 0$ represents the temperature of the liquid phase, and the ice is assumed to be at temperature 0.

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We say that (u_0, s_0) are (*admissible*) *initial data* if

$$s_0 > 0, \quad u_0 \in C^1([0, s_0]), \quad u_0 \geq 0 \quad \text{and} \quad (u_0)_x(0) = u_0(s_0) = 0 \ .$$

Under these assumptions, which will be in force through out this paper, it is well known that there exists a unique, maximal in time, classical solution (u, s) of (SP), which satisfies $u \geq 0$ and $s' \geq 0$ (see [6, 1]). The maximal existence time is denoted by $T \in (0, \infty]$ and we say that (u, s) is a global solution if $T = \infty$.

In what follows, the function u_0 (resp., $u(t, .)$) is extended by 0 for $x > s_0$ (resp. $x > s(t)$), and $|.|_\infty$ denotes the supremum over $(0, \infty)$. Also we will use the couple $(\overline{u}, \overline{s})$ to denote another solution of (SP), associated to initial data $(\overline{u}_0, \overline{s}_0)$, with maximal existence time $\overline{T}$.

Nonglobal solutions to (SP) were studied in [9, 1], where the shape of some blowing-up solutions was investigated. A sufficient blowup condition of energy type was obtained in [8]. Global solutions were studied in [2, 3, 8, 7]. In [8, 7], it was shown that all global solutions decay uniformly to 0 as $t \to \infty$ and satisfy uniform a priori estimates for $t \geq 0$. Moreover, introducing the notions of *fast* and *slow global solutions*, the following classification for the asymptotic behavior of global solutions has been obtained in [8].

Theorem A. *Let u be a global solution of* (SP). *Then it holds $\lim_{t\to\infty} |u(t)|_\infty = 0$. Moreover, if we let $s_\infty = \lim_{t\to\infty} s(t) \leq \infty$, then one of the following two possibilities occurs:*

(i) *u is a* **fast** *solution i.e., $s_\infty < \infty$ and there exist real numbers C, $\alpha > 0$ (depending on u) such that*

$$|u(t)|_\infty \leq C\, e^{-\alpha t}, \quad t \geq 0 \ ;$$

(ii) *u is a* **slow** *solution i.e., $s_\infty = \infty$ and one has the estimates*

$$s(t) = O(t^{2/3}), \quad t \to \infty \qquad \textit{and} \qquad \liminf_{t\to\infty} s^{2/(p-1)}(t)\, |u(t)|_\infty > 0$$

hence, in particular,

$$\liminf_{t\to\infty} t^{4/(3(p-1))}\, |u(t)|_\infty > 0 \ .$$

Concerning the existence of global fast and slow solutions, the following result was proved.

Theorem B.

(i) (see [8]) *There exists* $K = K(p) > 0$ *such that if*

$$|u_0|_\infty < K \min(1, s_0^{-2/(p-1)}) \ ,$$

then u *is a global fast solution.*

(ii) (see [7]) *Let* $\phi \in C^1([0, s_0])$ *satisfy* $\phi \geq 0$, $\phi \not\equiv 0$, *with* $\phi_x(0) = \phi(s_0) = 0$. *There exists* $\lambda^* > 0$ *such that the solution of* (SP) *with initial data* $u_0 = \lambda\phi$ *is a global fast solution for* $0 < \lambda < \lambda^*$ *and a global slow solution for* $\lambda = \lambda^*$.

The fact that suitably small data yield global fast solutions was proved earlier in [2] in the case $p > 2$. In [3], still for $p > 2$, the following stability property for global fast solutions was obtained.

Theorem C. *Assume* $p > 2$ *and let* (u, s) *be a global fast solution of (SP). For some* $q = q(p) > 1$ *and for all* $A > 0$, *there exists* $\eta = \eta(u, s, p, A) > 0$ *such that*

$$|\overline{u}_0|_\infty < A \quad \text{and} \quad |\overline{u}_0 - u_0|_{L^q(0,\infty)} + |\overline{s}_0 - s_0| < \eta \tag{1.1}$$

implies that $(\overline{u}, \overline{s})$ *is a global fast solution.*

The goal of the present paper is twofold:

(i) First, we want to show that a stability property of global fast solutions is actually true for all $p > 1$.

(ii) Second, we would like to provide a precise result on continuous dependence of local solutions of (SP), up to the maximum existence time T, i.e. on each interval $[0, T_1]$ with $T_1 < T$. Besides its own interest, this is one of the main ingredients of our stability proof. Of course, results on continuous dependence of solutions of problems of type (SP) were proved in the past by several authors (see [6, 11]), but their formulations do not seem suitable to our needs (see Remark 2.1 (c)).

2 – Main results

Theorem 2.1 (**Stability of global fast solutions**). *Assume* $p > 1$ *and let* (u, s) *be a global fast solution of (SP). There exists* $\eta = \eta(u, s, p) > 0$ *such that*

$$|\overline{u}_0 - u_0|_\infty + |\overline{s}_0 - s_0| < \eta \tag{2.1}$$

implies that $(\overline{u}, \overline{s})$ *is a global fast solution.*

Theorem 2.2 (**Continuous dependence up to** T). *Assume* $p > 1$ *and let* (u, s) *be a maximal solution of (SP), with maximal existence time* $T \in (0, \infty]$. *For all* $0 < T_1 < T$ *and all* $\varepsilon > 0$, *there exists* $\eta = \eta(u, s, p, T_1, \varepsilon) > 0$ *such that*

$$|\overline{u}_0 - u_0|_\infty + |\overline{s}_0 - s_0| < \eta$$

implies

$$\overline{T} > T_1 \quad \text{and} \quad \sup_{t \in [0,T_1]} |\overline{u}(t,.) - u(t,.)|_\infty + |\overline{s}(t) - s(t)| < \varepsilon \ . \tag{2.2}$$

Remarks 2.1.

(**a**) The result of Theorem 2.2 remains valid if the nonlinearity u^p is replaced by any function $f(u)$ with $f : [0, \infty) \to [0, \infty)$ locally Lipschitz.

(**b**) The stability result Theorem C from [3] is actually proved for weak solutions of (SP) and the smallness condition (1.1) involves a weaker norm than our condition (2.1). Of course the main improvement in Theorem 2.1 is to assume $p > 1$ instead of $p > 2$.

(**c**) Results on continuous dependence of solutions of problems of type (SP) were proved in the past by several authors, but their formulations do not seem suitable for the proof of Theorem 2.1. For instance, in [6, Theorem 2], continuous dependence is stated only for small time and it is assumed that $s_0 = \overline{s}_0$. In [11, pp. 130–134] this is proved for all times but it is assumed a priori that $\overline{T} > T_1$, and since $|u - \overline{u}|$ is estimated only for $x \le \min(s(t), \overline{s}(t))$, it is not clear if this assumption can be relaxed. Moreover, in both [6] and [11], the closeness of u_0 and $\overline{u}_0$ is measured in C^1 norm while we wish to use only L^∞ norm. The paper [10] treats a linear heat equation with nonlinear free-boundary conditions arising from chemical applications. Continuous dependence is also proved there only for small time and with respect to the C^1 norm. On the other hand, the works [12, 5] (see also [4]) treat the classical Stefan problem for the linear heat equation $u_t = u_{xx}$, for which all solutions exist globally. The continuous dependence results there are global in time and involve the L^∞ norm. Our method is different from [6, 10, 11] and related to that in [12, 5]. □

3 – Proofs

The proof of Theorem C from [3] relies on rather involved energy arguments. The proof of Theorem 2.1 is simpler. It is a consequence of Theorem B (i) from [8] (which was based on the construction of a suitable supersolution) and on Theorem 2.2. Let us first give the proof of Theorem 2.1, assuming Theorem 2.2 is proved.

Proof of Theorem 2.1: Since $s_\infty < \infty$ and $\lim_{t\to\infty} |u(t)|_\infty = 0$ by assumption, it follows that

$$|u(t_0)|_\infty < K \min(1, s(t_0)^{-2/(p-1)})$$

for some large t_0. By the continuous dependence property of Theorem 2.2, we deduce that for $\eta = \eta(u, s, p, t_0) > 0$ sufficiently small, (2.1) implies $\overline{T} > t_0$ and

$$|\overline{u}(t_0)|_\infty < K \min(1, \overline{s}(t_0)^{-2/(p-1)}) \ .$$

But in view of Theorem B (i), this implies that $(\overline{u}, \overline{s})$ is a global fast solution. Theorem 2.1 is proved. ∎

Iw view of the proof of Theorem 2.2, we prepare two lemmas. The first one is a special case of Theorem 2.2, for which we can make use of the comparison principle. In what follows, we say that (u_0, s_0) and $(\overline{u}_0, \overline{s}_0)$ are *ordered* if $s_0 \le \overline{s}_0$ and $u_0 \le \overline{u}_0$ or if $s_0 \ge \overline{s}_0$ and $u_0 \ge \overline{u}_0$.

Lemma 3.1. *Let (u, s) be a maximal solution of (SP), with maximal existence time $T \in (0, \infty]$. For all $0 < T_1 < T$ and all $\varepsilon, A > 0$, there exists $\eta = \eta(u, s, p, T_1, \varepsilon, A) > 0$ such that, if*

$$|\overline{u}_0|_{C^1([0,\overline{s}_0])} < A, \qquad |\overline{u}_0 - u_0|_\infty + |\overline{s}_0 - s_0| < \eta$$

and

$$(u_0, s_0) \text{ and } (\overline{u}_0, \overline{s}_0) \text{ are ordered },$$

then (2.2) holds.

Proof: For all $t \in [0, \min(T, \overline{T}))$, we define

$$\sigma(t) = \min(s(t), \overline{s}(t)) \,, \qquad \delta(t) = \sup_{\tau\in[0,t]} |\overline{s}(\tau) - s(\tau)| \ ,$$

$$w = \overline{u} - u \quad \text{ and } \quad \mu(t) = |w(t)|_\infty \ .$$

Take M, $L > 0$ such that $|u(t)|_\infty \le M$ and $s(t) \le L$ for $t \in [0, T_1]$. Assume

$$|\overline{u}_0|_{C^1([0,\overline{s}_0])} < A \quad \text{and} \quad |\overline{u}_0 - u_0|_\infty + |\overline{s}_0 - s_0| < \eta < \tfrac{1}{2}\min(1, s_0) \ ,$$

where η will be chosen later. In the rest of the proof we denote by C any positive constant depending only on u, s, p, A and T_1 (but not on η).

Now suppose there is a first $t_0 \in (0, \overline{T})$ with $t_0 \le T_1$, such that $\mu(t_0) + \delta(t_0) = 1$. In particular we have

$$|\overline{u}(t)|_\infty \le N := M + 1 \quad \text{and} \quad \overline{s}(t) \le L + 1, \quad 0 \le t \le t_0 \ .$$

It then follows from [1] (see the proof of Lemma 3.3) that there exists $M' = M'(M, L, |u_0|_{C^1([0,s_0])}, A, p, T_1) > 0$ such that

$$\sup_{[0,s(t)]} |u_x(t,x)|, \ \sup_{[0,\overline{s}(t)]} |\overline{u}_x(t,x)| \le M' \ , \qquad 0 \le t \le t_0 \ .$$

This implies in particular

$$|w(t,x)| \le M'\delta(t) \ , \qquad 0 \le t \le t_0, \quad x \ge \sigma(t) \ . \tag{3.1}$$

On the other hand w satisfies

$$\begin{cases} w_t - w_{xx} = a(t,x)w \ , & 0 < t < t_0, \quad 0 < x < \sigma(t), \\ w_x(t,0) = 0 \ , & 0 < t < t_0, \\ |w(t,\sigma(t))| \le M'\delta(t) \ , & 0 < t < t_0 \ , \end{cases}$$

where $|a(t,x)| \le p\,N^{p-1}$. Since $\delta(t)$ is nondecreasing, it follows from the maximum principle and (3.1) that

$$\mu(t) \le M'\delta(t) + |w(0)|_\infty\, e^{pN^{p-1}T_1} \le C\eta + C\delta(t) \ , \qquad 0 \le t \le t_0 \ . \tag{3.2}$$

Now we use the assumption that the initial data are ordered, say, $\overline{s}_0 \ge s_0$ and $\overline{u}_0 \ge u_0$. By the comparison principle (see, e.g., [1]), it follows that $\overline{s} \ge s$ and $\overline{u} \ge u$. On the other hand, by integrating $(\mathrm{SP})_1$, one obtains

$$\begin{aligned} \overline{s}(t) - s(t) + \int_0^{\overline{s}(t)} \overline{u}(t) - \int_0^{s(t)} u(t) \ &= \\ = \ \overline{s}_0 - s_0 + \int_0^{\overline{s}_0} \overline{u}_0 - \int_0^{s_0} u_0 &+ \int_0^t\!\!\int_0^{\overline{s}(\tau)} \overline{u}^p - \int_0^t\!\!\int_0^{s(\tau)} u^p \ . \end{aligned}$$

Therefore,

$$0 \ \le \ \overline{s}(t) - s(t) \le (s_0 + A + 1)\eta + pN^{p-1}L\int_0^t \mu(\tau)\,d\tau + N^p\int_0^t \delta(\tau)\,d\tau, \quad 0 \le t \le t_0 \ .$$

Combining this with (3.2), we get $\delta(t) \le C\eta + C\int_0^t \delta(\tau)\,d\tau$. By Gronwall's Lemma, it follows that $\delta(t) \le C\eta$, hence

$$\mu(t) + \delta(t) \le C\eta, \quad 0 \le t \le t_0 \ . \tag{3.3}$$

In particular, if η is chosen sufficiently small (depending on $M, L, |u_0|_{C^1([0,\overline{s}_0])}$, A, p, T_1), then necessarily $t_0 \ge T_1$. Since nonglobal solutions must satisfy $\limsup_{t\to\overline{T}} |\overline{u}(t)|_\infty = \infty$ (see [1, Proposition 3.1]), we deduce that $\overline{T} > T_1$ and the conclusion follows from (3.3). ∎

The next approximation lemma enables one to reduce the general case to Lemma 3.1 (and to remove the dependence of η on $|\overline{u}_0|_{C^1}$).

Lemma 3.2. *For all admissible initial data (u_0, s_0) and all $\eta \in (0, s_0/2)$, there exist admissible initial data $(u_0^\pm, s_0^\pm)$ with the following properties:*

$$s_0^- \le s_0 - \eta, \qquad u_0^- \le \max(u_0 - \eta, 0) \ , \tag{3.4}$$

$$s_0^+ \ge s_0 + \eta, \quad u_0^+ \ge u_0 + \eta \quad \textit{for } x \le s_0 + \eta \tag{3.5}$$

and

$$|u_0^\pm - u_0|_\infty + |s_0^\pm - s_0| \le C\eta\,, \quad |u_0^\pm|_{C^1([0,s_0^\pm])} \le \max(|u_0|_{C^1([0,s_0])}, 1) + C\eta \ , \tag{3.6}$$

where $C = C(u_0, s_0) > 0$.

Proof: Extend u_0 by 0 for $x > s_0$ and symmetrically for $x < 0$, and define the function $z_0(x) = u_0(x) + 2\eta$ if $|x| \le s_0 + 2\eta$, $z_0(x) = (s_0 + 4\eta - x)_+$ if $|x| > s_0 + 2\eta$, where $t_+ = \max(t, 0)$. One then puts $u_0^+ = z_0 * \rho_n$ and $u_0^- = (u_0 - B\eta)_+ * \rho_n$, where $B = 1 + 2k$, $k = |u_0|_{C^1([0,s_0])}$ and ρ_n is a standard mollifier.

Observe that $|y| \le 2\eta$ implies $u_0(x-y) \le u_0(x) + 2k\eta$, hence $(u_0(x-y) - B\eta)_+ \le (u_0(x) - \eta)_+$. Therefore, for $n \ge n_0(\eta)$ and all $x \ge 0$, we get $u_0^-(x) \le (u_0(x) - \eta)_+$. Similarly, we have $(u_0(y) - B\eta)_+ = 0$ for $y \ge s_0 - 2\eta$, so that $u_0^-(x) = 0$ for $x \ge s_0 - \eta$ and $n \ge n_1(\eta)$. Therefore we may take $s_0^- = s_0 - \eta$ and (3.4) is proved.

On the other hand, noting that $u_0 + 2\eta\,\mathbf{1}_{\{|x| \le s_0 + 2\eta\}} \le z_0 \le u_0 + 2\eta\,\mathbf{1}_{\{|x| \le s_0 + 4\eta\}}$ and taking $n \ge n_2(\eta)$, (3.5) follows easily with $s_0^+ = s_0 + 5\eta$. Finally, (3.6) is a consequence of the above and of usual properties of convolution. ∎

Proof of Theorem 2.2: Fix (u_0, s_0) and $\eta \in (0, s_0/2)$, and denote by $(u^\pm, s^\pm)$ the solutions corresponding to the initial data $(u_0^\pm, s_0^\pm)$ given by Lemma

3.2, with maximal existence times $T^{\pm}$. For any admissible data satisfying $|\overline{u}_0 - u_0|_\infty + |\overline{s}_0 - s_0| < \eta$, it follows from (3.4) and (3.5) that $u_0^- \le \overline{u}_0 \le u_0^+$ and $s_0^- \le \overline{s}_0 \le s_0^+$. By the comparison principle, we then have

$$u^- \le \overline{u} \le u^+ \quad \text{and} \quad s^- \le \overline{s} \le s^+ , \qquad 0 \le t < \min(\overline{T}, T^{\pm}) \ . \tag{3.7}$$

For any $0 < T_1 < T$, we deduce from (3.6) and Lemma 3.1 that

$$T^{\pm} > T_1 \quad \text{and} \quad \sup_{t\in[0,T_1]} |u^{\pm}(t,.) - u(t,.)|_\infty + |s^{\pm}(t) - s(t)| < \varepsilon \ ,$$

whenever $\eta = \eta(u, s, p, T_1, \varepsilon) > 0$ small enough. The conclusion (2.2) then follows from (3.7) and the fact that nonglobal solutions cannot remain uniformly bounded. ∎

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