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ON THE EXTREMAL BEHAVIOR OF SUB-SAMPLED SOLUTIONS OF STOCHASTIC DIFFERENCE EQUATIONS

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Abstract: Let $\{X_k\}$ be a process satisfying the stochastic difference equation

$$X_k = A_k X_{k-1} + B_k \,, \qquad k = 1, 2, \dots \,,$$

where $\{A_k, B_k\}$ are i.i.d. $\mathbb{R}^2$-valued random pairs. Let $Y_k = X_{Mk}$ be the sub-sampled series corresponding to a fixed systematic sampling interval $M > 1$. In this paper, we look at the extremal properties of $\{Y_k\}$. Motivation comes from the comparison of schemes for monitoring financial and environmental processes. The results are applied to the class of bilinear and ARCH processes.

1 – Introduction

Stochastic difference equations (SDE) play a crucial role in fields such as finance, economics, and insurance mathematics. Interest in these equations arose from the well-known fact that many non-linear processes, including ARCH, GARCH and bilinear processes, can be embedded in SDE. This implies that the extremal behavior of these processes can be investigated via the study of stochastic difference equations and their extremal behavior.

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Let $\{X_k\}$ be a process satisfying the stochastic difference equation

$$X_k = A_k X_{k-1} + B_k\,, \qquad k = 1, 2, \ldots\,, \tag{1}$$

where $\{A_k, B_k\}$ are i.i.d. $\mathbb{R}^2$-valued random pairs with some given joint distribution and X_0 is independent of this random pair, with some given starting distribution. Kesten [9], Vervaat [16] and Goldie [5] have studied the existence of stationary solutions for equation (1). They show essentially that under certain conditions, the process $\{X_k\}$ will have a stationary solution whose distribution converges to the distribution of

$$R = \sum_{s=1}^{\infty} \prod_{r=1}^{s-1} A_r\, B_s\ . \tag{2}$$

Although it is difficult to get an explicit solution for R in (2), it is possible to say something about its tail behavior. Kesten [9], Goldie [5], Grey [7], and Goldie and Grübel [6] have studied how the tail behavior of the distribution of R is determined by the joint distribution of $\{A_1, B_1\}$.

The extremal properties of $\{X_k\}$ were first studied by de Haan et al. [8] and then by Perfekt [10]. de Haan et al. [8] proved the compound Poisson process result, when neither A_k nor B_k are heavy tailed but A_k can take values outside the interval $[-1, 1]$. As example of application, de Haan and co-workers obtained the extremal behavior of the ARCH(1) process. Perfekt [10] extended their results to Markov processes and his results include stochastic difference equations given in (1) with possibly negative A_k and B_k as a special case. More recently, Turkman and Amaral Turkman [15] have derived the extremal properties of the first order bilinear process.

An important feature when dealing with time series is to assess the impact of different sampling frequencies on the extremes values of the process. For example, if the assets of a company are monitored daily, it would be important to know how much larger peak values are to be expected if the sampling was done on an hourly basis. Similar questions occur in other economical and environmental studies; Robinson and Tawn [12] give examples on the latter. Although this problem has important practical implications, it has not been addressed sufficiently.

From equation (1) we define the sub-sampled series

$$Y_k = X_{Mk}\,, \qquad k = 1, 2, \ldots\,, \tag{3}$$

corresponding to a fixed systematic sampling interval $M > 1$. Our purpose is to understand how the extremes of $\{Y_k\}$ should behave when X_k satisfies (1).

As example of application, results on the extremal behavior of the first order bilinear process and the ARCH(1) process are presented.

The rest of the paper is organized as follows: in Section 2, conditions for the existence of stationary solutions of the sub-sampled $\{Y_k\}$ process are obtained. Section 3 deals with the tail of the stationary distribution of Y_k when the random pair $\{A_1, B_1\}$ have different tail behavior. In Section 4, the extremal behavior of $\{Y_k\}$ is obtained. Finally, in Section 5 some concluding remarks are given.

2 – Conditions for the existence of stationary solutions of sub-sampled series

We first discuss conditions for the existence of stationary solutions of sub-sampled series embedded in SDE. Since $\{X_k\}$ satisfies the first order SDE in (1), it is natural to ask if the sub-sampled process $\{Y_k\}$ can also be embedded in a SDE.

Proposition 2.1. *Let $\{X_k\}$ be a process satisfying (1), then the sub-sampled $\{Y_k\}$ process satisfies the SDE*

$$Y_k = A_k^y Y_{k-1} + B_k^y\,, \qquad k = 1, 2, \dots\,, \tag{4}$$

where $\{A_k^y, B_k^y\}$ are i.i.d. random pairs with

$$A_k^y \;=\; \prod_{i=0}^{M-1} A_{Mk-i} \tag{5}$$

and

$$B_k^y \;=\; \sum_{j=1}^{M} B_{Mk-j+1} \prod_{i=1}^{j-1} A_{Mk-i+1}\;. \tag{6}$$

We use the convention that $\prod_{i=1}^{0} = 1$.

Proof: Note that for a fixed value of $M > 1$

$$\begin{aligned}
X_{Mk} &= A_{Mk}\,X_{Mk-1} + B_{Mk}\\
&= A_{Mk}(A_{Mk-1}X_{Mk-2} + B_{Mk-1}) + B_{Mk}\\
&= A_{Mk}\,A_{Mk-1}X_{Mk-2} + A_{Mk}B_{Mk-1} + B_{Mk}\\
&\;\;\vdots
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&= \prod_{i=0}^{M-1} A_{Mk-i}\, X_{M(k-1)} + \sum_{j=1}^{M} B_{Mk-j+1} \prod_{i=1}^{j-1} A_{Mk-i+1}\\
&= A_k^y\, X_{M(k-1)} + B_k^y \ .
\end{aligned}
$$

Thus relation (4) follows since by definition $X_{M(k-1)} = Y_{k-1}$. To verify that $\{A_k^y, B_k^y\}$ form a sequence of independent random pairs, note that $\{A_k^y, B_k^y\}$ and $\{A_{k+1}^y, B_{k+1}^y\}$ have no common terms for each $k \geq 1$. ∎

In what follows, we investigate the existence of a stationary solution for $\{Y_k\}$. Lemma 2.1 below summarizes the main result of this section.

Lemma 2.1. *Let $\{Y_k\}$ be the sub-sampled process corresponding to a fixed systematic interval $M > 1$ satisfiying the SDE in (4), where $\{A_k^y, B_k^y\}$ are i.i.d. $\mathbb{R}^2$-valued random pairs defined as in (5) and (6). Define*

$$
R^y = \sum_{s=1}^{\infty} \prod_{r=1}^{s-1} A_r^y\, B_s^y \ . \tag{7}
$$

Then

1. *If $E(\log|A_i|) \in (-\infty, 0)$ then the sum in (7) converges almost surely and (4) has a strictly stationary solution Y with distribution equivalent to the distribution of R^y in (7) if and only if*

$$
E\left(\log^+ |B_1^y|\right) < \infty \ . \tag{8}
$$

2. *If $E(\log|A_i|) = -\infty$ then the sum in (7) converges almost surely and (4) has a strictly stationary solution Y with distribution equivalent to the distribution of R^y in (7) either if (8) holds or $A_i = 0$ with positive probability.* ∎

The proof is a straight forward extention of Theorem 1.6 of Vervaat [16] and the details will be omitted.

3 – Tail behavior

Since it is very difficult to obtain an explicit solution for Y, we concentrate on characterazing its tail behavior. Since the tail behavior of Y will depend on the joint distribution of A_1 and B_1, we consider first the case when B_1 is heavy tailed.

3.1. When B_1 is heavy tailed and A_1 has comparatively lighter tails

Throughout this section we assume that $\{A_k, B_k\}$ are i.i.d. $\mathbb{R}_+^2$-valued random pairs. The first question we want to answer is: assuming that

$$P[B_1 > x] = x^{-\alpha} L(x)\,, \qquad \alpha > 0\,, \tag{9}$$

where L is a function slowly varying at infinity, can we derive the tail behavior of B_1^y? The answer is yes. Next result shows that the distributions associated with B_1 and B_1^y are tail equivalent.

Lemma 3.1. *Let $\{A_k, B_k\}$ be i.i.d. $\mathbb{R}_+^2$-valued random pairs such that $E(A_1)^\beta < \infty$ for some $\beta > \alpha > 0$, then*

$$\lim_{x\to\infty} \frac{P[B_1^y > x]}{P[B_1 > x]} = \frac{1-(EA_1^\alpha)^M}{1-EA_1^\alpha}\,. \tag{10}$$

Proof: We give an outline of the proof, details can be found in Scotto and Turkman [14]. Note that B_1^y can be rewritten as

$$B_1^y = \sum_{j=1}^{M} W_j\,,$$

where $W_j = B_{M-j+1} \prod_{i=1}^{M-1} A_{M-i+1}$. For a fixed value of $1 \le j \le M$,

$$\begin{aligned} \lim_{x\to\infty} \frac{P[W_j > x]}{P[W_1 > x]} &= \lim_{x\to\infty} \frac{P\Big[B_{M-j+1} \prod_{i=1}^{j-1} A_{M-i+1} > x\Big]}{P[B_1 > x]} \\ &= (EA_1^\alpha)^{j-1}, \end{aligned} \tag{11}$$

which follows from the Breiman's result quoted in Davis and Resnick [3], page 1197. In addition, for $1 \le j_1 < j_2 \le M$, we need to prove that as $x \to \infty$,

$$\frac{P\Big[W_{j_1} > x,\, W_{j_2} > x\Big]}{P[W_1 > x]} \to 0\,. \tag{12}$$

In doing so, define

$$U_{j_1} = \prod_{i=1}^{j_1-1} A_{M-i+1}\,, \qquad U_{j_1,j_2} = B_{M-j_2+1} \prod_{i=j_1}^{j_2-1} A_{M-i+1}\,.$$

Thus

$$P\Big[W_{j_1} > x,\, W_{j_2} > x\Big] \;=\; E\Big\{P\Big[\min\{B_{M-j_1+1}, U_{j_1,j_2}\} > x\, U_{j_1}^{-1}\Big]\Big\}\ ,$$

where the expectation is taken over U_{j_1}. Since the random variables B_{M-j_1+1} and U_{j_1,j_2} are independent for fixed values of $j_1 \neq j_2$ and $M > 1$, it follows from (11) that as $x \to \infty$,

$$P\Big[\min\{B_{M-j_1+1}, U_{j_1,j_2}\} > x\, U_{j_1}^{-1}\Big] \;\sim\; x^{-2\alpha}\, U_{j_1}^{2\alpha} (EA_1^\alpha)^{j_2-j_1-1}$$

and

$$E\Big\{P\Big[\min\{B_{M-j_1+1}, U_{j_1,j_2}\} > x\, U_{j_1}^{-1}\Big]\Big\} \;\sim\; x^{-2\alpha} E\, U_{j_1}^{2\alpha} (EA_1^\alpha)^{j_2-j_1-1}\ .$$

Thus

$$\frac{P\Big[W_{j_1} > x,\, W_{j_2} > x\Big]}{P[W_1 > x]} \;\sim\; x^{-\alpha} \times \text{constant} \;\to\; 0\,, \qquad x \to \infty\ .$$

Finally, the result follows as an application of Lemma 2.1 of Davis and Resnick [3]. ∎

Hence the tail behavior of Y follows as an application of Theorem 1 in Grey [7].

Lemma 3.2. *If $\{A_k, B_k\}$ are such that $E\,\log^+ B_1 < \infty$, $P[A_1 > 0 = 1]$, and for some $\beta > \alpha > 0$, $EA_1^\alpha < 1$ and $EA_1^\beta < \infty$, then*

1. $E\,\log^+ \sum_{j=1}^M B_{M-j+1} \prod_{i=0}^{j-1} A_{M-i+1} < \infty$.
2. *$\prod_{i=0}^{M-1} A_i$ takes non-negative values with probability one.*
3. *For some $\beta > \alpha > 0$, $E(\prod_{i=0}^{M-1} A_i)^\alpha < 1$ and $E(\prod_{i=0}^{M-1} A_i)^\beta < \infty$, for any integer $M \geq 1$.*

Moreover, the following two statements are equivalent:

$$\lim_{x\to\infty} \frac{P[B_1^y > x]}{P[B_1 > x]} = \frac{1 - (EA_1^\alpha)^M}{1 - EA_1^\alpha} \tag{13}$$

and

$$\lim_{x\to\infty} \frac{P[Y > x]}{P[B_1 > x]} = \frac{1}{1 - EA_1^\alpha}\ . \tag{14}$$

Proof: Conditions 1, 2 and 3 follow from Lemma 2 of Grey [7]. (14) follows from (13) as an application of Lemma 3.1, Lemma 2.1 in Davis and Resnick [3], and the arguments outlined in Resnick [11], page 228. Conversely, (13) follows from (14) as an application of the *if* part of Lemma 2 in Grey [7], page 173. ∎

3.2. When neither A_1 nor B_1 are heavy tailed but A_1 can take values outside the interval $[-1,1]$

In this case, the clusters of large values of the sequence $\{\prod_{k=1}^{n} A_k^y\}$ dominate the distribution of Y_k, in contrast to the case when B_1 is heavy tailed. We now extend Theorem 1.1 in de Haan et al. [8] to characterize the tail behavior of Y_k.

Lemma 3.3. *If for some* $\kappa > 0$, $E|A_1|^\kappa = 1$, $E|A_1|^\kappa \log^+|A_1| < \infty$, *and* $0 < E|B_1|^\kappa < \infty$, *then* $E\left|\prod_{i=0}^{M-1} A_i\right|^\kappa = 1$, $E\left|\prod_{i=0}^{M-1} A_i\right|^\kappa \log^+\left|\prod_{i=0}^{M-1} A_i\right| < \infty$,

$$0 < E\left|\sum_{j=1}^{M} B_{M-j+1} \prod_{i=0}^{j-1} A_{M-i+1}\right|^\kappa < \infty ,$$

and Y_k *has a strictly stationary solution* Y *with distribution equivalent to the distribution of* (7). *Moreover, as* $x \to \infty$

$$P[Y > x] \sim c_+ x^{-\kappa} , \quad P[Y < -x] \sim c_- x^{-\kappa} , \tag{15}$$

such that at least one of the contants is strictly positive. Further, if $P[A_1^y < 0] > 0$ *then* $c_+ = c_- > 0$. ∎

The proof follows from straight forward extention of the argument given in de Haan et al. [8]. For details, see Scotto and Turkman [14].

Remark. The exact values of c_+ and c_- are given in Theorem 4.1 in Goldie [5]. Unfortunately, these values are in general not very useful as they depend on the unknown distribution of Y. □

4 – Extremal behavior

In this section we present the main results regarding the extremal behavior of the sub-sampled process $\{Y_k\}$.

4.1. When B_1 is heavy tailed

We will assume again throughout this section that A_1 takes non-negative values with probability one. Note that if we define $\{\hat{Y}_k\}$ as the associated independent process of $\{Y_k\}$, i.e., $\hat{Y}_1, \hat{Y}_2, ...$ are i.i.d. random variables with the stationary distribution of $\{Y_k\}$ then from Lemma 3.2 and classical extreme value theory, we obtain

$$(16) \qquad \lim_{n\to\infty} P\left[\max_{1\le k\le [n/M]} \hat{Y}_k \le a_n x\right] = \exp\left(-\frac{1}{M}\,\frac{1}{1-EA_1^\alpha}\,x^{-\alpha}\right), \qquad x\ge 0\,,$$

where a_n is the $1-n^{-1}$ quantile of $\hat{Y}_1$, i.e.

$$a_n = \inf\left\{x\colon\; P[\hat{Y}_1 > x] \le \frac{1}{n}\right\}.$$

Hence the maximum of the associated independent process $\{\hat{Y}_k\}$ belongs to the domain of attraction of the Fréchet distribution. In the dependent case the limit distribution is still Fréchet but will depend on the extremal index θ_M, which under general conditions has an informal interpretation as the reciprocal of the limiting expected cluster size. In order to describe the clustering of extremes in more detail, we consider the time-normalized point process N_n of exceedances of an appropriately high chosen u_n given by

$$(17) \qquad N_n(\cdot) = \sum_{k=1}^{\infty} \epsilon_{\left(\frac{Mk}{n}\right)}(\cdot)\; 1_{(Y_k>u_n)}\,.$$

We show that this point process converges to a compound Poisson process N, whose events are the clusters of consecutive large values of $\{Y_k\}$. We derive the intensity and the distribution of the cluster centers.

Theorem 4.1. *For a fixed value of $M>1$,*

1. *Y_k has an extremal index θ_M given by*

$$\theta_M = \int_1^\infty P\left[\max_{1\le r\le\infty} \prod_{s=1}^{rM} A_s \le y^{-1}\right] \alpha\, y^{-\alpha-1}\, dy\,,$$

and

$$\lim_{n\to\infty} P\left[\max_{1\le k\le [n/M]} Y_k \le a_n\, x\right] = \exp\left(-\frac{\theta_M}{M}\,\frac{1}{1-EA_1^\alpha}\,x^{-\alpha}\right).$$

2. *N_n converges to a compound Poisson process with intensity $\frac{\theta_M}{M}\frac{1}{1-EA_1^\alpha}\,x^{-\alpha}$, and compounding probabilities $\pi_l = (\xi_l - \xi_{l+1})/\theta_M$, where*

$$\xi_l \;=\; \int_1^\infty P\left[\#\left\{r \ge 1\colon \prod_{s=1}^{rM} A_s > y^{-1}\right\} \;=\; l-1\right]\alpha\, y^{-\alpha-1}\; dy \;.$$

Proof: The proof of the above result is an application of Theorem 4.1 of Rootzén [13] and follows closely the proof of Theorem 2 given in Turkman and Amaral Turkman [15]; see also de Haan et al. [8]. For the first part of the theorem, we need to show that the $D(u_n)$ condition holds for $u_n = n^{1/\alpha}x$, $x>0$ and that

$$\lim_{\gamma\to 0}\limsup_{n\to\infty}\left|P\left[\max_{1\le r\le[\frac{n}{M}\gamma]} Y_k \le a_n \mid\; Y_0 > a_n\right] - \theta_M\right| \;=\; 0\;. \tag{18}$$

The mixing condition $D(u_n)$ is verified following the arguments given in de Haan et al. [8]. We now concentrate on verifying (18). Following de Haan et al. [8], set $Y_k^+ = \prod_{r=1}^{[n/M]} A_r^y\, Y_0$ and $\Delta_k = Y_k - Y_k^+$. Let $M_{[\frac{n}{M}\gamma]} = \max_{1\le r\le[\frac{n}{M}\gamma]} Y_r$, for any $\gamma>0$. Then

$$P\left[M_{[\frac{n}{M}\gamma]} > a_n \mid Y_0 > a_n\right] \;\ge\; P\left[\max_{1\le r\le[\frac{n}{M}\gamma]} Y_r^+ \;-\; \max_{1\le r\le[\frac{n}{M}\gamma]} \Delta_r > a_n \mid\; Y_0 > a_n\right].$$

Define $M^+ = \max_{1\le r\le[\frac{n}{M}\gamma]} Y_r^+$ and $M_\Delta = \max_{1\le r\le[\frac{n}{M}\gamma]} \Delta_r$. Then, for any $\delta>0$

$$\left\{M^+ - M_\Delta > a_n\right\} \;\supseteq\; \left\{M^+ > (1+\delta)\,a_n\right\} - \left\{M^+ > (1+\delta)\,a_n \cap M_\Delta > \delta\, a_n\right\}\,.$$

Hence

$$\begin{aligned} P\left[M_{[\frac{n}{M}\gamma]} > a_n \mid Y_0 > a_n\right] \;&\ge \\ &\ge\; P\Big[M^+ > (1+\delta)\,a_n \mid Y_0 > a_n\Big] - P\Big[M_\Delta > \delta\, a_n \mid Y_0 > a_n\Big]\,. \end{aligned} \tag{19}$$

Now, $\Delta_0 = 0 \le Y_0 = Y$, $\Delta_r \le Y$ and

$$P\Big[M_\Delta > \delta\, a_n \mid Y_0 > a_n\Big] \;\le\; \left[\frac{n}{M}\gamma\right] P[Y > \delta\, a_n] \;\to\; 0\;,$$

as $\gamma\to 0$. Similarly from de Haan et al. [8]

$$\begin{aligned} &P\Big[M^+ > (1+\delta)\,a_n \mid Y_0 > a_n\Big] \;= \\ &\quad= \int_1^\infty P\left[\max_{1\le r\le[\frac{n}{M}\gamma]} \prod_{s=1}^{r} A_s^y\, Y_0 \;>(1+\delta)\,a_n \mid\; Y_0 > a_n\, y\right] \frac{P[a_n Y_0 \in dy]}{P[a_n Y_0 > 1]} \end{aligned}$$

and since $P[Y_0 > a_n\, y]/P[Y_0 > a_n] \to y^{-\alpha}$ uniformly for $y \geq 1$, then from (19) we find

$$\begin{aligned}
\lim_{\gamma\to 0} \liminf_{n\to\infty} P\left[M_{[\frac{n}{M}\gamma]} > a_n \mid Y_0 > a_n\right] \;\geq \\
&\geq \lim_{\gamma\to 0} \int_1^\infty P\left[\max_{1\leq r\leq [\frac{n}{M}\gamma]} \prod_{s=1}^{r} A_s^y > y^{-1}(1+\delta)\right] \alpha\, y^{-\alpha-1}\, dy \\
&\quad - \lim_{\gamma\to 0} \gamma\, \delta^{-\alpha}\, \frac{1}{M}\, \frac{1}{1-EA_1^\alpha} \\
&\to \int_1^\infty P\left[\max_{1\leq r\leq [\frac{n}{M}\gamma]} \prod_{s=1}^{r} A_s^y > y^{-1}\right] \alpha\, y^{-\alpha-1}\, dy \\
&= 1-\theta_M \;,
\end{aligned}$$

as $\delta \to 0$. The upper bound is obtained by similar arguments and takes the form

$$\begin{aligned}
\lim_{\gamma\to 0} \limsup_{n\to\infty} P\left[M_{[\frac{n}{M}\gamma]} > a_n \mid Y_0 > a_n\right] \;\geq \\
&\leq (1-\delta)^{-\alpha} \int_{(1-\delta)^{-1}}^\infty P\left[\max_{1\leq r\leq [\frac{n}{M}\gamma]} \prod_{s=1}^{r} A_s^y > y^{-1}\right] \alpha\, y^{-\alpha-1}\, dy \\
&\to \int_1^\infty P\left[\max_{1\leq r\leq [\frac{n}{M}\gamma]} \prod_{s=1}^{r} A_s^y > y^{-1}\right] \alpha\, y^{-\alpha-1}\, dy \\
&= 1-\theta_M \;,
\end{aligned}$$

as $\delta \to 0$. This, and the fact that $\prod_{s=1}^{r} A_s^y = \prod_{s=1}^{rM} A_s$, shows (18) and hence the first part of the theorem. The second part of the theorem follows by introducing some straightforward changes in the arguments given in the first part of the theorem; see Rootzén [13] for further details. ∎

4.2. When neither A_1 nor B_1 are heavy tailed but A_1 can take values outside the interval $[-1,1]$

By means of the same machinery developed in Section 4.1 we establish the extremal properties of the sub-sampled process $\{Y_k\}$.

Theorem 4.2. *Assume that $P[A_1^y < 0] > 0$. Then, for a fixed value of $M > 1$*

1. *Y_k has an extremal index θ_M given by*

$$\theta_M = \int_1^\infty P\left[\max_{1\leq r\leq\infty} \prod_{s=1}^{rM} A_s \leq y^{-1}\right] \kappa\, y^{-\kappa-1}\, dy \;,$$

and

$$\lim_{n\to\infty} P\left[\max_{1\le k\le [n/M]} Y_k \le a_n\, x\right] = \exp\left(-\frac{c_+\,\theta_M}{M}\, x^{-\kappa}\right).$$

2. *N_n converges to a compound Poisson process with intensity $\frac{c_+\,\theta_M}{M}\, x^{-\kappa}$ and compounding probabilities $\pi_l = (\xi_l - \xi_{l+1})/\theta_M$, where*

$$\xi_l = \int_1^\infty P\left[\#\left\{r \ge 1 \colon \prod_{s=1}^{rM} A_s > y^{-1}\right\} = l-1\right] \kappa\, y^{-\kappa-1}\, dy$$

Proof: The proof is very similar to the proof of Theorem 4.1 and will not be given. (See Scotto and Turkman, [14] for details). However, note that when $M=1$, our results are consistent with those obtained by de Haan et al. [8]. ∎

5 – Examples

In order to illustrate the results given above, we study the tail and extremal behavior of a first order bilinear process with heavy-tailed innovations and an ARCH(1) process with light tailed innovations.

5.1. Sub-sampled bilinear processes

Assume that X_k satisfies the recursive equation

$$X_k = b\, X_{k-1} Z_{k-1} + Z_k\,, \qquad k = 1, 2, \ldots\,, \tag{20}$$

where $\{Z_k\}$ are i.i.d. non-negative random variables with common distribution F whose tail satisfies $P[Z_1 > x] = x^{-\alpha} L(x)$, $\alpha > 0$, where L is a function slowly varying at infinity, and b is a positive constant. First note that the representation given in (20) is not Markovian and the random pair $\{A_k, B_k\} = \{b\, Z_{k-1}, Z_k\}$ forms an 1-dependent, identically distributed pair. However, by setting $S_k = b\, Z_k X_k$ we see that

$$S_k = A^s_k\, S_{k-1} + B^s_k\,,$$

where $\{A^s_k, B^s_k\} = \{b Z_k, b Z^2_k\}$ forms an i.i.d. random sequence. Note that X_k can be expressed in the form

$$X_k = S_{k-1} + Z_k\,, \tag{21}$$

where S_k has a Markovian representation. Hence, Y_k can be written as

$$Y_k = V_{k-1} + Z_k^y \,, \tag{22}$$

where $Z_k^y = Z_{Mk}$. From (4), (5) and (6),

$$V_k = A_k^y\, V_{k-1} + B_k^y \,, \tag{23}$$

with $\{A_k^y, B_k^y\} = \left\{\prod_{i=0}^{M-1} bZ_{Mk-i},\ \sum_{j=1}^{M} b^j \left(\prod_{i=1}^{j-1} Z_{Mk-i+1}\right) Z^2_{Mk-j+1}\right\}$. Note that $\{A_k^y, B_k^y\}$ forms an i.i.d. random sequence. Theorem 3.2 given above can be used to describe the tail behavior of V_k. First note that since $P[Z_1^2 > x]$ is regularly varying with index $-\alpha/2$, it follows from Lemma 3.1 that

$$\lim_{x\to\infty} \frac{P[B_1^y > x]}{P[Z_1^2 > x]} \;=\; b^{\alpha/2}\, \frac{1-(b^{\alpha/2} E Z_1^{\alpha/2})^M}{1-b^{\alpha/2} E Z_1^{\alpha/2}} \;. \tag{24}$$

The tail behavior of V_k now follows from (24) and Lemma 3.2:

$$\lim_{x\to\infty} \frac{P[V_1 > x]}{P[Z_1^2 > x]} \;=\; \frac{b^{\alpha/2}}{1-b^{\alpha/2} E Z_1^{\alpha/2}} \;.$$

It is worth noting that the reason in considering the tail behavior of V_k rather than Y_k itself is due to the fact that the contribution of the term Z_{Mk} on the extremal behavior of Y_k is negligible; see Turkman and Amaral Turkman [15] for further details. Since Z_1^2 is regularly varying with index $-\alpha/2$, by Theorem 4.1 V_k has extremal index θ_M given by

$$\theta_M \;=\; \int_1^\infty P\left[\max_{1\le r\le\infty} \prod_{s=1}^{rM} b\, Z_s \le y^{-1}\right] \frac{\alpha}{2}\, y^{-\alpha/2-1}\; dy$$

and

$$\lim_{n\to\infty} P\left[\max_{1\le k\le[n/M]} V_k \le a_n^2\, x\right] \;=\; \exp\left(-\frac{\theta_M}{M}\, \frac{b^{\alpha/2}}{1-b^{\alpha/2} E Z_1^{\alpha/2}}\; x^{-\alpha/2}\right).$$

In addition, by defining N_n^v as the time-normalized exceedance point process of V_k, it follows from Theorem 4.1 that N_n^v converges to a compound Poisson process with intensity $\frac{\theta_M}{M}\, \frac{b^{\alpha/2}}{1-b^{\alpha/2} E Z_1^{\alpha/2}}\; x^{-\alpha/2}$, and compounding probabilities $\pi_l = (\xi_l - \xi_{l+1})/\theta_M$, where

$$\xi_l \;=\; \int_1^\infty P\left[\#\left\{r\ge 1\colon\ \prod_{s=1}^{rM} b\, Z_s > y^{-1}\right\} = l-1\right] \frac{\alpha}{2}\, y^{-\alpha/2-1}\; dy \;.$$

Finally, in order to prove that the time-normalized point process N_n converge to the same limit, it is suffices to show that (Resnick [11], page 232)

$$N_n^y(f) - N_n^v(f) \to 0 ,$$

in probability, where f is any positive, continuous and bounded function defined on $[0,\infty)$. From the definition of the vague metric (Resnick [11], page 148) it is suffice to check that

$$\sum_{k=1}^{[n/M]} f\left(\frac{Mk}{n}\right) 1_{(Y_k > a_n^2 x)} - \sum_{k=1}^{[n/M]} f\left(\frac{Mk}{n}\right) 1_{(V_k > a_n^2 x)} \to 0 , \tag{25}$$

in probability. The rest of the proof follows by the arguments given in Turkman and Amaral Turkman [15] with some minor changes. We skip the details.

5.2. Sub-sampled ARCH processes

The most widely used models of dynamic conditional variance on financial time series have been the ARCH models, first introduced by Engle [4]. This class was extended by Bollerslev [2] who suggested an alternative and more flexible dependence structure for describing log-returns (i.e. daily logarithmic differences of financial returns), the generalized ARCH or GARCH models. We consider the ARCH(1) model defined as

$$X_k = Z_k \sqrt{\sigma_k} , \qquad k = 1, 2, \dots ,$$

where $\{Z_k\}$ is a sequence of i.i.d. random variables with zero-mean and unit variance and σ_k a time-varying, positive and measurable function of the $k-1$ information set, satisfying the recurrence equation $\sigma_k = a_0 + a_1 X_{k-1}^2$ with $a_0 > 0$ and $0 < a_1 < 1$. For deriving probabilistic properties of the ARCH(1) process we will make extensive use of the fact that the squared process $\{X_k^2\}$ satisfies the SDE,

$$X_k^2 = A_k X_{k-1}^2 + B_k , \qquad k = 1, 2, \dots , \tag{26}$$

where $\{A_k, B_k\} = \{a_1 Z_k^2, a_0 Z_k^2\}$. The tail behavior of Y_k is discussed below.

5.2.1. When neither A_1 nor B_1 are heavy tailed but A_1 can take values outside the interval $[-1,1]$

In view of (26), Y_k takes the form

$$Y_k = Z_k^y \sqrt{\sigma_k^y} \;, \tag{27}$$

with $Z_k^y = Z_{Mk}$ and $\sigma_k^y = \sigma_{Mk}$, which implies that Y_k^2 has a representation as in (4) in the form

$$Y_k^2 = A_k^y \, Y_{k-1}^2 + B_k^y \;,$$

with $\{A_k^y, B_k^y\} = \Big\{a_1^M \prod_{i=0}^{M-1} Z_{Mk-i}^2, \, \sum_{j=1}^M a_0 \, Z_{Mk-j+1}^2 \, \prod_{i=1}^{j-1} a_1 Z_{Mk-i+1}^2\Big\}$. By Theorem 3.3, $\kappa = \kappa(a_1)$ is the unique solution of the equation $E(a_1 Z_1^2)^\kappa = 1$, and

$$P[Y^2 > x] \;\sim\; c_+ \, x^{-\kappa} \;,$$

as $x \to \infty$. Finally, by Theorem 4.2 the sub-sampled squared ARCH process $\{Y_k^2\}$ has extremal index given by,

$$\theta_M \;=\; \int_1^\infty P\left[\max_{1 \le r \le \infty} \prod_{s=1}^{rM} a_1 Z_s^2 \le y^{-1}\right] \kappa \, y^{-\kappa-1} \; dy \;,$$

and

$$\lim_{n\to\infty} P\left[\max_{1 \le k \le [n/M]} Y_k^2 \le a_n \, x\right] \;=\; \exp\left(-\frac{c_+ \, \theta_M}{M} \, x^{-\kappa}\right) .$$

Moreover, N_n converges to a compound Poisson process with intensity $\frac{c_+ \theta_M}{M} \, x^{-\kappa}$ and compounding probabilities π_l where

$$\xi_l \;=\; \int_1^\infty P\left[\#\left\{r \ge 1\colon \prod_{s=1}^{rM} a_1 Z_s^2 > y^{-1}\right\} = l-1\right] \kappa \, y^{-\kappa-1} \; dy \;.$$

6 – Notes and comments

In this paper we have obtained limit results for sub-sampled processes generated by stationary solutions of 1-dimensional SDE. It would be interesting to extend those results in a multivariate setting. The reason is that many non-linear models, can be studied in the context of multivariate SDEs. Let us consider a few of them.

- GARCH(1,1). Assume that, for an i.i.d. sequence $\{Z_k\}$, the process $\{X_k\}$ satisfies the equations

$$\begin{aligned} X_k &= \sqrt{\sigma_k}\, Z_k \ , \\ \sigma_k &= a_0 + a_1 X_{k-1}^2 + b_1\, \sigma_{k-1} \ , \end{aligned}$$

where a_0, a_1 and b_1 are fixed constants. The GARCH(1,1) process can be rewritten as a two-dimensional SDE in the form (1) as follows:

$$\begin{pmatrix} X_{k+1}^2 \\ \sigma_{k+1} \end{pmatrix} = \mathbf{A}_{k+1} \begin{pmatrix} X_k^2 \\ \sigma_k \end{pmatrix} + \mathbf{B}_{k+1}\,, \qquad k \geq 1 \ ,$$

where

$$\mathbf{A}_{k+1} = \begin{pmatrix} a_1 Z_k^2 & b_1 Z_k^2 \\ a_1 & b_1 \end{pmatrix}\,, \qquad \mathbf{B}_{k+1} = \begin{pmatrix} a_0 Z_k^2 \\ a_0 \end{pmatrix}.$$

- GARCH(p,q) process. Assume $\{X_k\}$ is a solution to the GARCH equations

$$\begin{aligned} X_k &= \sigma_k\, Z_k \ , \\ \sigma_k^2 &= a_0 + a_1\, X_{k-1}^2 + \cdots + a_p\, X_{n-p} + b_1\, \sigma_{k-1}^2 + \cdots + b_q\, \sigma_{k-q}^2 \ . \end{aligned}$$

Following Basrak [1], one possibility for $\{X_k\}$ to be embedded in a SDE is the following: define $\mathbf{X}_k = (X_k^2, ..., X_{k-p+1}^2, \sigma_k^2, ..., \sigma_{k-q+1})'$, and notice that this $(p+q)$-dimensional process satisfies the SDE

$$\mathbf{X}_k = \mathbf{A}_k\, \mathbf{X}_{k-1} + \mathbf{B}_k\,, \qquad k \geq 1 \ , \tag{28}$$

with

$$\mathbf{A}_{k+1} = \begin{pmatrix} a_1 Z_k^2 & \cdots & a_{p-1} Z_k^2 & a_p Z_k^2 & b_1 Z_k^2 & \cdots & b_{q-1} Z_k^2 & b_q Z_k^2 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & \cdots & a_{p-1} & a_p & b_1 & \cdots & b_{q-1} & b_q \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{B}_k = (a_0\, Z_k^2,\, 0,\, ...,\, 0,\, a_0,\, 0,\, ...,\, 0)' \ .$$

The study of the extremal properties of sub-sampled processes associated with those processes remains as a topic of future research.

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