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OPTIMAL CONTROL AND "STRANGE TERM" FOR A STOKES PROBLEM IN PERFORATED DOMAINS

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Abstract: We study a problem of optimal control for Stokes equations in perforated domains with Dirichlet conditions on the boundary of holes. We consider different sizes of holes.

1 – Introduction

The aim of this paper is to study an optimal control problem for Stokes equations in perforated domains with Dirichlet conditions on the boundary of holes.

Let Ω be a bounded connected open set in $\mathbb{R}^n$ (n_2) with Lipschitz boundary $\partial\Omega$. Let ε be a sequence of positive real numbers which tends to zero. We cover the set Ω with a regular mesh of size 2ε, each cell is a cube P_i^ε, $i = 1, ..., N(\varepsilon)$, similar to $[-\varepsilon, \varepsilon]^n$. We make a hole T_i^ε at the center of each cube P_i^ε, included in Ω. We define the holes as follows: each hole T_i^ε is equal to $a_\varepsilon T$ where T is a given closed set independent of ε, and a_ε is the size of the hole $(0 < a_\varepsilon < \varepsilon)$. Then the perforated domain Ω_ε is defined by $\Omega_\varepsilon = \Omega \backslash \bigcup T_i^\varepsilon$. There are different possible sizes of the holes which can be considered ("critical", smaller and larger holes). So we define a ratio σ_ε between the current size of the holes and the critical one:

$$(1.1) \qquad \sigma_\varepsilon = (\varepsilon^n / a_\varepsilon^{n-2})^{1/2} \quad \text{for } n \geq 3 \,, \qquad \sigma_\varepsilon = \varepsilon \Big(\log(a_\varepsilon/\varepsilon)\Big)^{1/2} \quad \text{for } n = 2 \,.$$

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If the limit of σ_ε as ε tends to zero, is positive and finite then the size of the holes is called critical. If the $\lim\limits_{\varepsilon\to 0}\sigma_\varepsilon = +\infty$, the size of holes is smaller and if $\lim\limits_{\varepsilon\to 0}\sigma_\varepsilon = 0$, the holes are larger (cf. Cioranescu and Murat [2] and Allaire [1]).

Throughout all the sequel, we use the convention of summation over repeated indices.

We denote by $\tilde{}$ the extension by zero onto the holes.

Let $B = (b_{ij})$ be a symmetric matrix such that

$$\alpha_m\, \xi_i\, \xi_i \leq b_{ij}(x)\, \xi_i\, \xi_j \leq \alpha_M\, \xi_i\, \xi_i \quad \text{a.e. in } \Omega \text{ and } b_{ij} \in L^\infty(\Omega)\ , \tag{1.2}$$

where α_m and α_M are constants such that $\alpha_M > \alpha_m > 0$.

For $\varepsilon > 0$ fixed, we define the optimal control problem as follows.

Let $\mathcal{U}^\varepsilon_{ad} \subset L^2(\Omega_\varepsilon)^n$ be a closed convex set. Let $f \in L^2(\Omega)^n$ be a given function and let $N > 0$ be a given constant. For $\theta_\varepsilon \in \mathcal{U}^\varepsilon_{ad}$, we define the state equation of the Stokes problem by

$$\left\{\begin{aligned} \nabla p_\varepsilon - \Delta u_\varepsilon &= f + \theta_\varepsilon && \text{in } \Omega_\varepsilon,\\ \operatorname{div} u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon,\\ u_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon\ . \end{aligned}\right. \tag{1.3}$$

where u_ε, p_ε are respectively the velocity, the pressure of the fluid and θ_ε is the control.

The cost functional is then given by

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\; dx + \frac{N}{2}\int_{\Omega_\varepsilon} \theta_\varepsilon^2\; dx\ . \tag{1.4}$$

The second integral corresponds to the cost of the control whereas the first one corresponds to the energy of the fluid. The matrix B is used in order to generalize the usual energy (we obtain this energy when the matrix B is equal to identity).

The optimal control $\theta^\star_\varepsilon$ is the function in $\mathcal{U}^\varepsilon_{ad}$ which minimizes $J_\varepsilon(\theta_\varepsilon)$ for $\theta_\varepsilon \in \mathcal{U}^\varepsilon_{ad}$, i.e.

$$\theta^\star_\varepsilon \in \mathcal{U}^\varepsilon_{ad} \quad \text{and} \quad J_\varepsilon(\theta^\star_\varepsilon) = \min_{\theta_\varepsilon \in \mathcal{U}^\varepsilon_{ad}} J_\varepsilon(\theta_\varepsilon)\ . \tag{1.5}$$

This problem admits a unique optimal solution $\theta^\star_\varepsilon$ (see Lions [6]).

The problem (1.3)–(1.5) can be reduced to a system of equations by introducing the adjoint state $(v_\varepsilon, p'_\varepsilon)$ of $(u_\varepsilon, p_\varepsilon)$. Thus we get

$$\left\{\begin{aligned} \nabla p'_\varepsilon + \Delta v_\varepsilon &= \operatorname{div}(B\,\nabla u_\varepsilon) && \text{in } \Omega_\varepsilon,\\ \operatorname{div} v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon,\\ v_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon\ . \end{aligned}\right. \tag{1.6}$$

where $(v, p'_\varepsilon) \in (H^1(\Omega)^n \times L^2_0(\Omega))$.

The optimal control $\theta_\varepsilon^\star$ is characterized by the variational inequality

$$(1.7)\qquad \theta_\varepsilon^\star \in \mathcal{U}_{ad}^\varepsilon \quad \text{and} \quad \int_{\Omega_\varepsilon} (v_\varepsilon + N\,\theta_\varepsilon^\star)\,(\theta_\varepsilon - \theta_\varepsilon^\star)\; dx \;\geq\; 0 \quad \forall\,\theta_\varepsilon \in \mathcal{U}_{ad}^\varepsilon\,.$$

Our aim is to study the limiting behaviour of the optimal control $\theta_\varepsilon^\star$ as $\varepsilon \to 0$.

In fact, it can be shown that (up to a subsequence) $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)^n$. Our objective is to characterize θ_0^* as the optimal control of a similar problem set in the non-perforated domain Ω.

The type of optimal control problem which we consider, was studied by Kesavan and Vanninathan [5], Kesavan and Saint Jean Paulin [3] in non-perforated domains and by Kesavan and Saint Jean Paulin [4] in perforated domains. They studied in [4] the Laplace problem with Neumann conditions on the boundary. Also Rajesh [7] considered the optimal control problem for the Dirichlet problem in perforated domains and he obtained a "strange term" in the limit.

This paper is organized as follows. In Section 2, we recall some hypotheses (H1)–(H6) in perforated domains concerning the holes (see Allaire [1]) and the main results of the homogenization of Stokes equations. In Section 3, we consider the critical case and we homogenize the adjoint problem and establish convergence results of energies which appear in the cost functional. In Section 4, we obtain the limiting optimal control problem. In Section 5, we study the optimal problem for smaller sizes of holes (for which $\lim\limits_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$).

Notation. Throughout this paper, C denotes various real positive constants independent of ε. The duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and between $(H_0^1(\Omega))^n$ and $(H^{-1}(\Omega))^n$, are each denoted by $\langle\ ,\ \rangle$. □

We denote by $(e_k)_{1\leq k\leq n}$ the canonical basis of $\mathbb{R}^n$.

Definition 1.1. We define the set $L_0^2(\Omega)$ by

$$(1.8)\qquad L_0^2(\Omega) \;=\; \left\{ f \in L^2(\Omega) \mid \int_\Omega f(x)\,dx = 0 \right\}\ . \ \square$$

2 – Hypotheses on the perforations and preliminary results

We make on the holes the same assumptions as Allaire [1], so there exist functions $(\omega_k^\varepsilon, r_k^\varepsilon, \mu_k)$ and a linear mapping R_ε such that

(H1) $\omega_k^\varepsilon \in H^1(\Omega)^n,\ r_k^\varepsilon \in L^2(\Omega)$,

(H2) $\operatorname{div}\omega_k^\varepsilon = 0$ in Ω and $\omega_k^\varepsilon = 0$ in T_i^ε,

(H3) $\omega_k^\varepsilon \rightharpoonup e_k$ weakly in $H^1(\Omega)^n$ and $r_k^\varepsilon \rightharpoonup 0$ weakly in $L_0^2(\Omega)$,

(H4) $\mu_k \in W^{-1,\infty}(\Omega)^n$,

(H5) $\forall\, v_\varepsilon$ and $\forall\, v$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)^n$, $v_\varepsilon = 0$ in T_i^ε and $\forall\,\phi \in \mathcal{D}(\Omega)$,

$$\left\langle \nabla r_k^\varepsilon - \Delta\omega_k^\varepsilon\,,\ \phi\, v_\varepsilon \right\rangle \to \langle \mu_k, \phi v\rangle\ ,$$

$$\text{(H6)}\qquad \begin{cases} R_\varepsilon \in \mathcal{L}(H_0^1(\Omega)^n, H_0^1(\Omega_\varepsilon)^n), \\ \text{If } u \in H_0^1(\Omega_\varepsilon)^n \text{ then } R_\varepsilon\,\widetilde{u} = u \text{ in } \Omega_\varepsilon, \\ \text{If } \operatorname{div} u = 0 \text{ in } \Omega \text{ then } \operatorname{div}(R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ ||R_\varepsilon u||_{H_0^1(\Omega_\varepsilon)^n} \le c\,||u||_{H_0^1(\Omega)^n}\ . \end{cases}$$

Example 2.1. The assumptions (H1)–(H6) are satisfied in the particular case where each hole T_i^ε is a ball of radius a_ε where $a_\varepsilon = C_0\,\varepsilon^{n/n-2}$ for $n \ge 3$ and $a_\varepsilon = e^{-C_0/\varepsilon^2}$ for $n = 2$ with $C_0 > 0$ and in a such geometry we can compute explicitly the functions ω_k^ε, r_k^ε and μ_k which satisfy (H1)- -(H6) (see [1]). In this case, the diameter of the holes is such that $a_\varepsilon << \varepsilon$.

Note also that, the case where the diameter of the holes a_ε is of the same order as ε corresponds to the classical homogenization. ▫

Assumptions (H1)–(H6) hold throughout the paper.

We define the matrix $M \in (W^{-1,\infty}(\Omega))^{n\times n}$ by (see [1])

$$M e_k = \mu_k\ . \tag{2.1}$$

This matrix is symmetric and under the above assumptions, we have the following result which is due to Allaire [1].

The extension $\widetilde{u_\varepsilon}$ of the velocity u_ε and the extension $P^\varepsilon p_\varepsilon$ of the pressure p_ε (defined by Allaire [1]) satisfy

Theorem 2.2 (Allaire [1]). *Depending on the size of the holes, there are three different limit flow regimes for the solution of (1.3):*

(i) *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$ *then* $(\widetilde{u_\varepsilon}, P^\varepsilon p_\varepsilon)$ *converges strongly to* (u,p) *in* $H_0^1(\Omega)^n \times L_0^2(\Omega)$, *where* (u,p) *is the unique solution of the Stokes problem*

$$\left\{\begin{aligned} \nabla p - \Delta u &= f+\theta \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.2}$$

(ii) *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0$ *then there exist a measure* μ^k *and a matrix* M *such that* $M e_k = \mu^k$ *such that* $(\widetilde{u_\varepsilon}, P^\varepsilon p_\varepsilon)$ *converges weakly to* (u,p) *in* $H_0^1(\Omega)^n \times L_0^2(\Omega)$, *where* (u,p) *is the unique solution of the Brinkman-type law*

$$\left\{\begin{aligned} \nabla p - \Delta u + Mu &= f+\theta \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.3}$$

Remark 2.3. Under hypotheses similar to (H1)–(H6) (with a scaling depending of σ_ε), if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$ then there exist a matrix M_0 such that $(\widetilde{u_\varepsilon}/\sigma_\varepsilon^2, P^\varepsilon p_\varepsilon)$ converges strongly to (u,p) in $L^2(\Omega)^n \times L_0^2(\Omega)$, where (u,p) is the unique solution of Darcy's law

$$\left\{\begin{aligned} u = M_0^{-1}(f - \nabla p + \theta) &\quad \text{in } \Omega,\\ \operatorname{div} u = 0 &\quad \text{in } \Omega,\\ u\,.\,n = 0 &\quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.4}$$

with n the exterior normal vector to Ω (see Allaire [1] for more details concerning these hypotheses and the matrix M_0). □

3 – Homogenization and convergence of some energies

In this section and in Section 4, we assume that

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0\,. \tag{3.1}$$

Following the approach of Kesavan and Saint Jean Paulin [4], we introduce the adjoint state variable and pass to the limit in the resulting system.

Assuming (3.1), there exists a sequence $(\omega_k^\varepsilon, r_k^\varepsilon)$ satisfying (H1)–(H6). We show that there exists n distributions μ_B^k $(k = 1, ..., n)$ and a matrix M_B

defined below by (3.8) such that, given any $f \in L^2(\Omega)^n$, if $(u_\varepsilon, p_\varepsilon)$ solves the Stokes problem (1.3), then (up to a subsequence), we have the following convergence of energies.

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon \nabla u_\varepsilon\,dx \;\to\; \int_\Omega B\,\nabla u\,\nabla u\;dx \;+\; \langle M_B\,u,\,u\rangle \tag{3.2}$$

$$B\,\nabla u_\varepsilon \nabla u_\varepsilon\,dx \;\to\; B\,\nabla u\,\nabla u \;+\; {}^t(M_B\,u)\,u \quad \text{in } \mathcal{D}'(\Omega)\;, \tag{3.3}$$

where (u,p) solves the problem (2.3).

This type of results was shown by Rajesh [7] for the Dirichlet problem for the Laplace operator.

We introduce some auxiliary test functions which are used to homogenize the adjoint problem (1.6).

Lemma 3.1. *Assume (3.1) and let $(\psi_k^\varepsilon, s_k^\varepsilon) \in H_0^1(\Omega_\varepsilon)^n \times L_0^2(\Omega_\varepsilon)$ be the solution of the auxiliary system*

$$\begin{cases} \nabla s_k^\varepsilon + \Delta\psi_k^\varepsilon \;=\; -\operatorname{div}({}^tB\,\nabla\omega_k^\varepsilon) & \text{in } \Omega_\varepsilon,\\ \qquad \operatorname{div}\psi_k^\varepsilon \;=\; 0 & \text{in } \Omega_\varepsilon,\\ \qquad\quad \psi_k^\varepsilon \;=\; 0 & \text{on } \partial\Omega_\varepsilon\;. \end{cases} \tag{3.4}$$

Then there exist ψ_k and s_k such that (for a subsequence)

$$\widetilde{\psi_k^\varepsilon} \rightharpoonup \psi_k \quad \text{weakly in } H_0^1(\Omega)^n\;, \tag{3.5}$$

$$P_\varepsilon(s_k^\varepsilon) \rightharpoonup s_k \quad \text{weakly in } L_0^2(\Omega)\;. \tag{3.6}$$

Proof: Multiplying the first equation of (3.4) by ψ_k^ε, integrating by parts and taking into account the boundedness of ω_k^ε in $H^1(\Omega)^n$, we have the announced result. ∎

Definition 3.2. Let us define the distributions $\mu_B^k \in \mathcal{D}'(\Omega)$, $k = 1,...,n$ by

$$\mu_B^k \;=\; -M\,\psi_k + (\nabla s_k + \Delta\psi_k)\;, \tag{3.7}$$

and the matrix $M_B \in (W^{-1,\infty})^{n\times n}$ by

$$M_B\,e_k \;=\; \mu_B^k\;. \;\square \tag{3.8}$$

Proposition 3.3. *Let $f \in L^2(\Omega)$. Define M by (2.1) and M_B by (3.8). Assume that (3.1) holds and that θ_ε is such that $\widetilde{\theta_\varepsilon}$ is bounded in $L^2(\Omega)^n$. Let $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ in $(H^1_0(\Omega_\varepsilon)^n \times L^2_0(\Omega_\varepsilon))^2$ be the solution of the system*

$$\left\{\begin{aligned} \nabla p_\varepsilon - \Delta u_\varepsilon &= f + \theta_\varepsilon && \text{in } \Omega_\varepsilon, \\ \nabla p'_\varepsilon + \Delta v_\varepsilon &= \operatorname{div}(B\,\nabla u_\varepsilon) && \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = \operatorname{div} v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ u_\varepsilon = v_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon\ . \end{aligned}\right. \tag{3.9}$$

Then, up to subsequences

$$\left\{\begin{array}{ll} \widetilde{\theta_\varepsilon} \rightharpoonup \theta & \text{weakly in } L^2(\Omega)^n, \\ \widetilde{u_\varepsilon} \rightharpoonup u & \text{weakly in } H^1_0(\Omega)^n, \\ \widetilde{v_\varepsilon} \rightharpoonup v & \text{weakly in } H^1_0(\Omega)^n \end{array}\right. \tag{3.10}$$

and

$$\left\{\begin{array}{ll} P^\varepsilon p_\varepsilon \rightharpoonup p & \text{weakly in } L^2_0(\Omega), \\ P^\varepsilon p'_\varepsilon \rightharpoonup p' & \text{weakly in } L^2_0(\Omega)\ , \end{array}\right. \tag{3.11}$$

where the limits (u,p) and (v,p') are solution of the Brinkman type system

$$\left\{\begin{aligned} \nabla p - \Delta u + Mu &= f + \theta && \text{in } \Omega, \\ \nabla p' + \Delta v - Mv &= \operatorname{div}(B\,\nabla u) - {}^tM_B\, u && \text{in } \Omega, \\ \operatorname{div} u = \operatorname{div} v &= 0 && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega\ . \end{aligned}\right. \tag{3.12}$$

Proof:

Step 1: *A priori estimates*

Since $\widetilde{\theta_\varepsilon}$ is bounded in $L^2(\Omega)$, it is clear that $\widetilde{u_\varepsilon}$ and $\widetilde{v_\varepsilon}$ are uniformly bounded in $H^1_0(\Omega)^n$ and, also $\{P^\varepsilon p_\varepsilon\}$ and $\{P^\varepsilon p'_\varepsilon\}$ are uniformly bounded in $L^2_0(\Omega)$.

Hence we can extract a subsequence (again indexed by ε for convenience) such that (3.10) and (3.11) holds.

The homogenization of the state equation (1.3) is known (see Theorem 2.1 (ii)).

Step 2: *Energy method*

To pass to the limit in the second equation in (3.9), we use the test functions $(\omega_k^\varepsilon, r_k^\varepsilon)$ defined in (H1)–(H6) and the auxiliary functions $(\psi_k^\varepsilon, s_k^\varepsilon)$ defined by (3.4).

Let $\phi \in \mathcal{D}(\Omega)$. Multiplying the second equation in (3.9) by $\phi\omega_k^\varepsilon$ and integrating by parts and using assumption (H2), we get

$$\begin{aligned}\int_{\Omega_\varepsilon} p'_\varepsilon\, \nabla\phi\; \omega_k^\varepsilon\; dx \;=\; &-\int_{\Omega_\varepsilon} z_\varepsilon\, \nabla\phi\; \omega_k^\varepsilon\; dx \;-\int_{\Omega_\varepsilon} \nabla v_\varepsilon\, \nabla\omega_k^\varepsilon\; \phi\; dx \\ &+\int_{\Omega_\varepsilon} B\, \nabla u_\varepsilon\, \nabla\omega_k^\varepsilon\; \phi\; dx\;, \end{aligned} \tag{3.14}$$

where

$$z_\varepsilon = \nabla v_\varepsilon - B\, \nabla u_\varepsilon\;. \tag{3.15}$$

Similarly, multiplying the first equation in (3.9) by $\phi\,\psi_k^\varepsilon$, integrating by parts and taking into account the definition of ψ_k^ε (see equation (3.4)), we obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon} (f+\theta_\varepsilon)\, \phi\; \psi_k^\varepsilon\; dx \;+\int_{\Omega_\varepsilon} p_\varepsilon\, \nabla\phi\; \psi_k^\varepsilon\; dx \;= \\ &\quad= \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\phi\; \psi_k^\varepsilon\; dx \;-\int_{\Omega_\varepsilon} u_\varepsilon\, \nabla\phi\; s_k^\varepsilon\; dx \;-\int_{\Omega_\varepsilon} u_\varepsilon\, \nabla\phi\, \nabla\psi_k^\varepsilon\; dx \\ &\qquad -\int_{\Omega_\varepsilon} ({}^tB\, \nabla\omega_k^\varepsilon)\, \nabla u_\varepsilon\; \phi\; dx \;-\int_{\Omega_\varepsilon} ({}^tB\, \nabla\omega_k^\varepsilon)\, u_\varepsilon\, \nabla\phi\; dx\;. \end{aligned} \tag{3.16}$$

Adding (3.14) and (3.16) and transforming all the integrals over Ω_ε into integrals over Ω, we get

$$\begin{aligned}&\int_\Omega (f+\widetilde{\theta}_\varepsilon)\, \phi\; \widetilde{\psi_k^\varepsilon}\; dx \;+\int_\Omega P^\varepsilon\, p_\varepsilon\, \nabla\phi\; \widetilde{\psi_k^\varepsilon}\; dx \;+\int_\Omega P^\varepsilon\, p'_\varepsilon\, \nabla\phi\; \omega_k^\varepsilon\; dx \;= \\ &\quad= -\int_\Omega \widetilde{z_\varepsilon}\, \nabla\phi\; \omega_k^\varepsilon\; dx \;-\int_\Omega \nabla\widetilde{v_\varepsilon}\, \nabla\omega_k^\varepsilon\; \phi\; dx \;+\int_\Omega \nabla\widetilde{u_\varepsilon}\, \nabla\phi\; \widetilde{\psi_k^\varepsilon}\; dx \\ &\qquad -\int_\Omega b_k^\varepsilon\, \widetilde{u_\varepsilon}\, \nabla\phi\; dx \;-\int_\Omega \widetilde{u_\varepsilon}\, \nabla\phi\; P^\varepsilon\, s_k^\varepsilon\; dx\;, \end{aligned} \tag{3.17}$$

where

$$b_k^\varepsilon \;=\; {}^tB\, \nabla\omega_k^\varepsilon + \nabla\widetilde{\psi_k^\varepsilon}\;. \tag{3.18}$$

Since $\operatorname{div} v_\varepsilon = 0$ in Ω_ε, we get

$$\int_\Omega r_k^\varepsilon\, \phi\, \operatorname{div}\widetilde{v_\varepsilon}\; dx \;=\; 0\;. \tag{3.19}$$

Adding (3.17) and (3.19) and integrating by parts, we get

$$\tag{3.20}\begin{aligned}\int_\Omega (f+\widetilde{\theta}_\varepsilon)\,\phi\,\widetilde{\psi_k^\varepsilon}\,dx &+ \int_\Omega P^\varepsilon p_\varepsilon\,\nabla\phi\,\widetilde{\psi_k^\varepsilon}\,dx + \int_\Omega P^\varepsilon p'_\varepsilon\,\nabla\phi\,\omega_k^\varepsilon\,dx = \\ &= -\int_\Omega \widetilde{z_\varepsilon}\,\nabla\phi\,\omega_k^\varepsilon\,dx + \left\langle \Delta\omega_k^\varepsilon - \nabla r_k^\varepsilon,\,\phi\,\widetilde{v_\varepsilon}\right\rangle + \int_\Omega \widetilde{v_\varepsilon}\,\nabla\omega_k^\varepsilon\,\nabla\phi\,dx \\ &\quad - \int_\Omega r_k^\varepsilon\,\nabla\phi\,\widetilde{v_\varepsilon}\,dx + \int_\Omega \nabla\widetilde{u_\varepsilon}\,\nabla\phi\,\widetilde{\psi_k^\varepsilon}\,dx \\ &\quad - \int_\Omega b_k^\varepsilon\,\widetilde{u_\varepsilon}\,\nabla\phi\,dx - \int_\Omega \widetilde{u_\varepsilon}\,\nabla\phi\,P^\varepsilon s_k^\varepsilon\,dx\,.\end{aligned}$$

Step 3: *Passing to the limit*

We now pass to the limit in (3.20) as ε tends to 0. In order to do so, we need some preliminary results.

Using (H3), we have

$$\tag{3.21} \nabla\omega_k^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)^{n\times n}\,.$$

By the definition (3.18) and using the convergences (3.5) and (3.21), we can extract a subsequence such that

$$\tag{3.22} b_k^\varepsilon \rightharpoonup \nabla\psi_k \quad \text{weakly in } L^2(\Omega)^{n\times n}\,.$$

Also by the definition (3.15) and using the convergence (3.10), we get (up to subsequences)

$$\tag{3.23} \widetilde{z_\varepsilon} \rightharpoonup z = \nabla v - B\,\nabla u \quad \text{weakly in } L^2(\Omega)^{n\times n}\,.$$

Now passing to the limit in (3.20), taking into account the convergences in (H3), (H5), (3.5), (3.6), (3.10), (3.11) and (3.21)–(3.23), we get, (up to subsequences)

$$\tag{3.24}\begin{aligned}\int_\Omega (f+\theta)\,\phi\,\psi_k\,dx &+ \int_\Omega p\,\nabla\phi\,\psi_k\,dx + \int_\Omega p'\,\nabla\phi\,e_k\,dx = \\ &= -\int_\Omega z\,\nabla\phi\,e_k\,dx - \langle\mu_k,\,\phi v\rangle + \int_\Omega \nabla u\,\nabla\phi\,\psi_k\,dx \\ &\quad - \int_\Omega \nabla\psi_k\,u\,\nabla\phi\,dx - \int_\Omega u\,\nabla\phi\,s_k\,dx\,.\end{aligned}$$

Therefore, integrating by parts the right-hand side of (3.24) and using Theorem 2.1 (ii), we have

$$\tag{3.25}\begin{aligned}\int_\Omega Mu\,\phi\,\psi_k\,dx &- \int_\Omega \nabla p'\,\phi\,e_k\,dx = \\ &= \int_\Omega (\operatorname{div} z)\,\phi\,e_k\,dx - \langle\mu_k,\phi v\rangle + \int_\Omega \Delta\psi_k\,u\,\phi\,dx + \int_\Omega \nabla s_k\,u\,\phi\,dx\,.\end{aligned}$$

Since the above relation holds for all $\phi \in \mathcal{D}(\Omega)$ and since M is symmetric, we have

$$\nabla p' + \operatorname{div} z - Mv \;=\; -\,{}^tM_B\,u \tag{3.26}$$

i.e. (u,p) and (v,p') satisfy (3.12).

Since M is symmetric and positive definite, the solutions (u,p) and (v,p') of (3.12) are unique, and therefore, it follows that the whole sequences $(u_\varepsilon, P^\varepsilon p_\varepsilon)$ and $(v_\varepsilon, P^\varepsilon p'_\varepsilon)$ converge. This completes the proof of the proposition. ∎

Now, we treat the convergence of the energies $\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx$. This type of convergence have been studied by Rajesh [7] for the Dirichlet problem. He has shown in [7] that "a strange term" for the energy appears in the limit using ideas of [2]. Similarly, we show a same type of result i.e. a strange term in the limiting energy for Stokes problem following ideas of [1] and [7].

Theorem 3.4. *Let $f \in L^2(\Omega)^n$ and $(u_\varepsilon, p_\varepsilon)$ be the solution of the Stokes problem (1.3). Let M_B given by (3.8). Then*

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;\to\; \int_\Omega B\,\nabla u\,\nabla u\;dx \;+\; \langle M_B\,u,\,u\rangle \tag{3.27}$$

and

$$B\,\nabla\widetilde{u_\varepsilon}\;\nabla\widetilde{u_\varepsilon} \;\to\; B\,\nabla u\;\nabla u \;+\; {}^t(M_B\,u)\,u \quad \text{in } \mathcal{D}'(\Omega)\;. \tag{3.28}$$

Proof: Using the fact that $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ are solution of (3.9), we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;&=\; -\int_{\Omega_\varepsilon} (\nabla v_\varepsilon - B\,\nabla u_\varepsilon)\,\nabla u_\varepsilon\;dx \;+\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; -\int_{\Omega_\varepsilon} \nabla p'_\varepsilon\,u_\varepsilon\;dx \;+\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; \int_{\Omega_\varepsilon} v_\varepsilon(f+\theta_\varepsilon)\;dx \;=\; \int_\Omega \widetilde{v_\varepsilon}(f+\widetilde{\theta_\varepsilon})\;dx\;.
\end{aligned} \tag{3.29}$$

Therefore, integrating by parts and using the homogenization results of Proposition 3.3 and the fact that M is symmetric, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx &= \int_\Omega v(f+\theta)\;dx \\
&= \int_{\Omega_\varepsilon} v(\nabla p - \Delta u + Mu) \\
&= -\int_\Omega \Delta v\;u\;dx + \int_\Omega Mv\;u\;dx \\
&= \langle -\Delta v + Mv,\,u\rangle \\
&= \Big\langle \nabla p' - \operatorname{div}(B\,\nabla u) + {}^t M_B\,u,\,u\Big\rangle \\
&= \int_\Omega B\,\nabla u\,\nabla u\;dx + \langle M_B\,u,\,u\rangle\ ,
\end{aligned} \tag{3.30}$$

which proves (3.27).

Let $\phi\in\mathcal{D}(\Omega)$. Set z_ε defined by (3.15), integrating by parts and using the problem (3.9), we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx &= \int_{\Omega_\varepsilon}\nabla v_\varepsilon\nabla u_\varepsilon\;\phi\;dx - \int_{\Omega_\varepsilon} z_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx \\
&= \int_{\Omega_\varepsilon} v_\varepsilon(f+\theta-\nabla p_\varepsilon)\,\phi\;dx - \int_{\Omega_\varepsilon} v_\varepsilon\,\nabla u_\varepsilon\,\nabla\phi\;dx \\
&\quad - \int_{\Omega_\varepsilon}\nabla p'_\varepsilon\,u_\varepsilon\,\phi\;dx + \int_{\Omega_\varepsilon} z_\varepsilon\,u_\varepsilon\,\nabla\phi\;dx\ .
\end{aligned} \tag{3.31}$$

Using the same arguments as in the proof of Proposition 3.3 and using system (3.12), we derive

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx = \\
&= \int_\Omega v(f+\theta-\nabla p)\,\phi\;dx - \int_\Omega v\,\nabla u\,\nabla\phi\;dx - \int_\Omega \nabla p'\,u\,\phi\;dx + \int_\Omega z\,u\,\nabla\phi\;dx \\
&= \langle Mu,\,v\,\phi\rangle + \int_\Omega \nabla u\,\nabla v\;\phi\;dx - \int_\Omega z\,\nabla u\;\phi\;dx + \langle {}^t M_B\,u - Mv,\,\phi u\rangle\ .
\end{aligned} \tag{3.32}$$

Therefore, using the fact that M is symmetric, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx &= \int_\Omega (\nabla v - z)\,\nabla u\;\phi\;dx + \langle {}^t M_B\,u,\,\phi u\rangle \\
&= \int_\Omega B\,\nabla u\,\nabla u\;\phi\;dx + \langle {}^t(M_B\,u)\,u,\,\phi\rangle\ .
\end{aligned} \tag{3.33}$$

This holds for all $\phi\in\mathcal{D}(\Omega)$. This proves (3.28) and completes the proof. ∎

Now we give some properties concerning the functions $(\mu_B^k)_{1\le k\le n}$.

Theorem 3.5. *Let μ_B^k be as defined in (3.7). Then*

$$\mu_B^k\, e i \;=\; \lim_{\varepsilon\to 0} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon \quad \textit{in}\;\; \mathcal{D}'(\Omega)\ . \tag{3.34}$$

Proof: Let $\phi\in\mathcal{D}(\Omega)$. Using the problem (3.4), the expression (3.19) and integrating by parts, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon\;\phi\;dx \;&=\; \int_{\Omega_\varepsilon}(\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\,\nabla\omega_i^\varepsilon\;\phi\;dx \;-\int_{\Omega_\varepsilon}\nabla\psi_k^\varepsilon\,\nabla\omega_i^\varepsilon\;\phi\;dx\\
&=\;-\int_{\Omega_\varepsilon} s_\varepsilon^k\,\omega_i^\varepsilon\,\nabla\phi\;dx\;-\int_{\Omega_\varepsilon}(\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\;\omega_i^\varepsilon\,\nabla\phi\;dx\\
&\qquad+\int_{\Omega_\varepsilon}\psi_k^\varepsilon\,\Delta\omega_i^\varepsilon\;\phi\;dx\;+\int_{\Omega_\varepsilon}\psi_k^\varepsilon\,\nabla\omega_i^\varepsilon\,\nabla\phi\;dx\\
&=\;-\int_{\Omega_\varepsilon} s_k^\varepsilon\,\omega_i^\varepsilon\,\nabla\phi\;dx\;-\int_{\Omega_\varepsilon}(\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\;\omega_i^\varepsilon\,\nabla\phi\;dx\\
&\qquad-\Big\langle\nabla r_i^\varepsilon - \Delta\omega_i^\varepsilon,\;\widetilde{\psi_k^\varepsilon}\,\phi\Big\rangle\;-\int_{\Omega_\varepsilon} r_i^\varepsilon\,\psi_k^\varepsilon\,\nabla\phi\;dx\ .
\end{aligned}\tag{3.35}$$

Passing to the limit (using the convergences (H3), (H5), (3.5) and (3.6)), we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon\;\phi\;dx \;&=\;-\int_\Omega s_k\;e_i\,\nabla\phi\;dx\;-\int_\Omega\nabla\psi_k\;e_i\,\nabla\phi\;dx\;-\;\langle\mu_i,\,\psi_k\,\phi\rangle\\
&=\;\int_\Omega\nabla s_k\;e_i\,\phi\;dx\;+\int_\Omega\Delta\psi_k\,e_i\,\phi\;dx\;-\;\langle\mu_i,\,\psi_k\,\phi\rangle\\
&=\;\Big\langle\nabla s_k + \Delta\psi_k - M\psi_k,\;\phi\,e_i\Big\rangle\\
&=\;\langle\mu_B^k,\,\phi\,e_i\rangle\\
&=\;\langle\mu_B^k\,e_i,\,\phi\rangle\ .
\end{aligned}\tag{3.36}$$

This proves (3.34). ∎

Corollary 3.6. *If B is symmetric positive definite, then μ_B^k is a positive measure and M_B is symmetric.*

Proof: This is a consequence of Theorem 3.5. ∎

In the next section, we return to the control problem we started with.

4 – Optimal control

We denote by χ_ε the characteristic function of Ω_ε. We now consider the optimal control problem (1.3)–(1.5) where the convex set $\mathcal{U}^\varepsilon_{ad} \subset L^2(\Omega_\varepsilon)$ is one of the following ones (see [3] and [4]).

$$\mathcal{U}^\varepsilon_{ad} = L^2(\Omega_\varepsilon)^n \ , \tag{4.1}$$

$$\mathcal{U}^\varepsilon_{ad} = \Big\{\theta \in L^2(\Omega_\varepsilon)^n \ | \ \widetilde{\theta} \geq \chi_\varepsilon\, \psi \ \text{ a.e. in } \Omega\Big\} \ , \tag{4.2}$$

$$\mathcal{U}^\varepsilon_{ad} = \Big\{\theta \in L^2(\Omega_\varepsilon)^n \ | \ \chi_\varepsilon\, \psi_1 \leq \widetilde{\theta} \leq \chi_\varepsilon\, \psi_2 \ \text{ a.e. in } \Omega\Big\} \ , \tag{4.3}$$

where ψ, ψ_1 and ψ_2 are given functions in $L^2(\Omega)^n$.

Now, since $\theta^\star_\varepsilon$ is optimal we have

$$\frac{N}{2}\int_{\Omega_\varepsilon} (\theta^\star_\varepsilon)^2\, dx \ \leq \ J_\varepsilon(\theta^\star_\varepsilon) \ \leq \ J_\varepsilon(\Theta_\varepsilon) \qquad \forall\, \Theta_\varepsilon \in \mathcal{U}^\varepsilon_{ad} \ . \tag{4.4}$$

This relation holds in particular with the following choice of Θ_ε

$$\Theta_\varepsilon \ = \begin{cases} \chi_\varepsilon & \text{in the case of (4.1)}, \\ \chi_\varepsilon \psi & \text{in the case of (4.2)}, \\ \chi_\varepsilon \psi_2 & \text{in the case of (4.3)} \ . \end{cases} \tag{4.5}$$

In each of the three cases above, we have

Lemma 4.1. *The optimal control satisfies (up to a subsequence)*

$$\widetilde{\theta^\star_\varepsilon} \rightharpoonup \theta^\star_0 \quad \textit{weakly in } \ L^2(\Omega)^n \ . \tag{4.6}$$

Proof: Using (4.5), we have that $J_\varepsilon(\Theta_\varepsilon)$ is bounded in $L^2(\Omega_\varepsilon)^n$, so we derive from (4.4) the announced result. ∎

Lemma 4.2. *The characteristic function χ_ε of Ω_ε satisfies*

$$\chi_\varepsilon \rightharpoonup 1 \quad \textit{weakly} \star \textit{ in } \ L^\infty(\Omega) \ . \tag{4.7}$$

Proof: We have, up to a subsequence

$$\chi_\varepsilon \rightharpoonup \chi_0 \quad \text{weakly} \star \text{ in } \ L^\infty(\Omega) \ .$$

Since $\chi_\varepsilon\, \omega^\varepsilon_k = \omega^\varepsilon_k$, thus passing to the limit and by uniqueness, we obtain $\chi_0 = 1$. ∎

We proceed to characterize the limiting optimal control problem. We define the set $\mathcal{U}_{ad} \subset L^2(\Omega)$ as

$$\mathcal{U}_{ad} = L^2(\Omega)^n , \tag{4.8}$$

$$\mathcal{U}_{ad} = \left\{\theta \in L^2(\Omega)^n \mid \theta \geq \psi \ \text{ a.e. in } \Omega\right\} , \tag{4.9}$$

$$\mathcal{U}_{ad} = \left\{\theta \in L^2(\Omega)^n \mid \psi_1 \leq \theta \leq \psi_2 \ \text{ a.e. in } \Omega\right\} , \tag{4.10}$$

corresponding to the cases (4.1), (4.2) and (4.3) respectivly. We have the following convergence result of optimal control.

Theorem 4.3. *Let M_B given by (3.8). For $\theta \in \mathcal{U}_{ad}$, let $(u,p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$ be the solution of (2.2). Let J_0 be the cost functional defined by*

$$J_0(\theta) = \frac{1}{2}\int_\Omega B\,\nabla u\,\nabla u\ dx + \frac{1}{2}\,\langle M_B\, u,\, u\rangle + \frac{N}{2}\int_\Omega \theta^2\, dx\ . \tag{4.11}$$

Then $\theta_0^\star$ satisfy the condition of optimality

$$\theta_0^\star \in \mathcal{U}_{ad} \quad \text{and} \quad J_0(\theta_0^\star) = \min_{\theta\in\mathcal{U}_{ad}} J_0(\theta)\ . \tag{4.12}$$

Further we have the convergence of the minimal costs, i.e.

$$\lim_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^\star) = J_0(\theta_0^\star)\ . \tag{4.13}$$

Proof:

Step 1: It is clear from the definition of $\mathcal{U}_{ad}$, that if $\theta \in \mathcal{U}_{ad}$ then $\chi_\varepsilon\theta \in \mathcal{U}_{ad}^\varepsilon$. Further, since $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)^n$ and $\mathcal{U}_{ad}$ is a closed convex set, we have $\theta_0^\star \in \mathcal{U}_{ad}$.

Step 2: Let $(u_\varepsilon^\star, p_\varepsilon^\star)$ be the solution of the state equation (1.3) corresponding to $\theta_\varepsilon = \theta_\varepsilon^\star$. Using the convergence (4.6) of Lemma 4.1, we get

$$\begin{cases} \widetilde{u_\varepsilon^\star} \rightharpoonup u^\star & \text{weakly in } H_0^1(\Omega)^n, \\ P^\varepsilon p_\varepsilon^\star \rightharpoonup p^\star & \text{weakly in } L_0^2(\Omega)\ , \end{cases} \tag{4.14}$$

where $(u^\star, p^\star)$ is solution of (2.2) with $\theta = \theta^\star$ in the right-hand side.

Step 3: Let $(w_\varepsilon, q_\varepsilon) \in H_0^1(\Omega_\varepsilon)^n \times L_0^2(\Omega_\varepsilon)$ be the solution of the state equation (1.3) with the control $\chi_\varepsilon\theta,\ \theta \in \mathcal{U}_{ad}$, that is

$$\begin{cases} \nabla q_\varepsilon - \Delta w_\varepsilon = f + \chi_\varepsilon\,\theta & \text{in } \Omega_\varepsilon, \\ \operatorname{div} w_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon\ . \end{cases} \tag{4.15}$$

Since $\chi_\varepsilon \theta \rightharpoonup \theta$ weakly in $L^2(\Omega)^n$, it follows that $\widetilde{w_\varepsilon} \rightharpoonup w$ weakly in $H_0^1(\Omega)^n$ and $P^\varepsilon(q_\varepsilon) \rightharpoonup q$ weakly in $L_0^2(\Omega)$ where (w,q) satisfy the following Brinkmann-type problem

$$\text{(4.16)} \qquad \begin{cases} \nabla q - \Delta w + Mw = f + \theta & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega\ , \end{cases}$$

(see Proposition 3.3). Further, using Theorem 3.4 for θ fixed, we have

$$\text{(4.17)} \qquad \int_{\Omega_\varepsilon} B\, \nabla w_\varepsilon\, \nabla w_\varepsilon\ dx\ \to \int_\Omega B\, \nabla w\, \nabla w\ dx\ +\ \langle M_B\, w,\, w\rangle\ .$$

Thus

$$\text{(4.18)} \qquad J_\varepsilon(\chi_\varepsilon\theta) \to J_0(\theta)\ .$$

Once again, using Theorem 3.4 but now for $\theta_\varepsilon = \theta_\varepsilon^\star$, we get

$$\text{(4.19)} \qquad \int_{\Omega_\varepsilon} B\, \nabla u_\varepsilon^\star\, \nabla u_\varepsilon^\star\ dx\ \to \int_\Omega B\, \nabla u^\star\, \nabla u^\star\ dx\ +\ \langle M_B\, u^\star,\, u^\star\rangle\ .$$

Step 4: Passing to the limit in the inequality

$$\text{(4.20)} \qquad J_\varepsilon(\chi_\varepsilon\theta) \geq J_\varepsilon(\theta_\varepsilon^\star)\ .$$

and using Lemma 4.2, we get

$$\text{(4.21)} \quad J_0(\theta)\ \geq\ \frac{1}{2}\int_\Omega B\, \nabla u^\star\, \nabla u^\star\ dx\ +\ \limsup_{\varepsilon\to 0}\ \frac{N}{2}\int_{\Omega_\varepsilon} (\theta_\varepsilon^\star)^2\ dx\ +\ \frac{1}{2}\,\langle M_B\, u^\star,\, u^\star\rangle\ .$$

Thus taking $\theta = \theta^\star$ in the above inequality, we have

$$\text{(4.22)} \qquad \limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\theta_\varepsilon^\star)^2\ dx\ \leq \int_\Omega (\theta^\star)^2\ dx\ .$$

Moreover since $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)$, we get

$$\text{(4.23)} \qquad \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\theta_\varepsilon^\star)^2\ dx\ \geq \int_\Omega (\theta^\star)^2\ dx\ .$$

Thus using (4.22) and (4.23), we derive

$$\text{(4.24)} \qquad \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\theta_\varepsilon^\star)^2\ dx\ = \int_\Omega (\theta^\star)^2\ dx\ .$$

We deduce (4.13). Now (4.12) follows from (4.21) and (4.24). ∎

5 – Case of smaller holes

We now assume that the size of the holes is smaller than the critical size, i.e.

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty \,, \tag{5.1}$$

in other words,

$$a_\varepsilon << \varepsilon^{n/n-2} \text{ for } n \geq 3\,, \quad a_\varepsilon = \exp^{-1/C_\varepsilon} \quad \text{and} \quad C_\varepsilon << \varepsilon^2 \text{ for } n=2 \,. \tag{5.2}$$

Since the size of the holes satisfies (5.1), the hypothese (H3) is replaced by (see [1])

$$\omega_k^\varepsilon \to e_k \text{ strongly in } H^1(\Omega)^n \quad \text{and} \quad r_k^\varepsilon \to 0 \text{ strongly in } L_0^2(\Omega) \,. \tag{5.3}$$

Remark 5.1. Hypothese (5.3) is stronger than Hypothese (H3). □

We have the following result

Proposition 5.2. *Let the size of the holes satisfy (5.1). Assume that (H1), (H2), (H4)–(H6) and (5.3) hold. Let $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ be the unique solution of (3.9). Then up to subsequences*

$$\begin{cases} \widetilde{\theta_\varepsilon} \rightharpoonup \theta & \textit{weakly in } L^2(\Omega)^n, \\ \widetilde{u_\varepsilon} \to u & \textit{strongly in } H_0^1(\Omega)^n, \\ \widetilde{v_\varepsilon} \to v & \textit{strongly in } H_0^1(\Omega)^n \end{cases} \tag{5.4}$$

and

$$\begin{cases} P^\varepsilon(p_\varepsilon) \to p & \textit{strongly in } L_0^2(\Omega), \\ P^\varepsilon(p'_\varepsilon) \to p' & \textit{strongly in } L_0^2(\Omega) \,, \end{cases} \tag{5.5}$$

where (u,p) and (v,p') are solution of the Stokes problem

$$\begin{cases} \nabla p - \Delta u = f + \theta & \textit{in } \Omega, \\ \nabla p' + \Delta v = \operatorname{div}(B\,\nabla u) & \textit{in } \Omega, \\ \operatorname{div} u = \operatorname{div} v = 0 & \textit{in } \Omega, \\ u = v = 0 & \textit{on } \partial\Omega \,. \end{cases} \tag{5.6}$$

Proof: To prove this result, we use the same arguments as in the proof of Proposition 3.3.

Step 1: The fact that $M = 0$ was established by Allaire [1] and also the following convergence result

$$(5.7) \qquad \begin{cases} \widetilde{u_\varepsilon} \to u & \text{strongly in } H_0^1(\Omega)^n, \\ P^\varepsilon(p_\varepsilon) \to p & \text{strongly in } L_0^2(\Omega)\ . \end{cases}$$

Following Allaire [1], we show now that μ_B^k, defined by (3.7), is equal to zero. Since Hypotheses (H1), (H2), (H4)–(H6) and (5.3) are satisfied, all the results of Proposition 3.3 hold. But from (5.3) and Theorem 3.6, we deduce that $\mu_B^k = 0$ and hence $M_B = 0$. This proves that (u, p) and (v, p') satisfy the Stokes equations (5.6).

We complete the proof by showing the strong convergence of $\widetilde{v_\varepsilon}$ in $H_0^1(\Omega)^n$ and of $P^\varepsilon(p'_\varepsilon)$ in $L_0^2(\Omega)$.

Step 2: Using the convergences (3.10) and (3.11) of $\widetilde{v_\varepsilon}$ and $P^\varepsilon(p'_\varepsilon)$ respectively and defining v by (5.6), we have the following convergence (using classical arguments)

$$(5.8) \qquad \begin{cases} \widetilde{v_\varepsilon} \to v & \text{strongly in } H_0^1(\Omega)^n, \\ P^\varepsilon(p'_\varepsilon) \to p' & \text{strongly in } L_0^2(\Omega)\ . \end{cases}$$

This ends the proof. ∎

We now give a convergence result of the optimal control. Let $\mathcal{U}_{ad}^\varepsilon \subset L^2(\Omega_\varepsilon)^n$ given by (4.1)–(4.3) and $\mathcal{U}_{ad} \subset L^2(\Omega)^n$ by (4.8)–(4.10). We have the following result

Theorem 5.4. *Let $\theta_\varepsilon^\star$ be the optimal control for the Stokes problem (1.3) and let the cost functional be given by (1.4). Then*

$$(5.9) \qquad \widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star \quad \textit{weakly in } L^2(\Omega)^n$$

and $\theta_0^\star$ is the optimal control for the problem

$$(5.10) \qquad \begin{cases} \nabla p - \Delta u = f + \theta & \textit{in } \Omega, \\ \operatorname{div} u = 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega\ , \end{cases}$$

with the following cost functional

$$(5.11) \qquad J_0(\theta) = \frac{1}{2}\int_\Omega B\,\nabla u\,\nabla u\,dx + \frac{N}{2}\int_\Omega \theta^2\,dx\ .$$

Proof: By Lemma 4.1, we have (5.9). Now using Theorem 4.3 with the matrices M and M_B equal to zero, we have immediately the results. This completes the proof. ∎

Remark 5.5. In the case where $\sigma_\varepsilon \to 0$ (i.e. when the holes are larger), we have that $\widetilde{u_\varepsilon} \to 0$ strongly in $H^1(\Omega)^n$, hence $\nabla\widetilde{u_\varepsilon} \to 0$ strongly in $L^2(\Omega)^{n\times n}$. Then it is obvious that we have the following convergence of energy

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;=\; \int_\Omega B\,\nabla\widetilde{u_\varepsilon}\,\nabla\widetilde{u_\varepsilon}\;dx \;\to\; 0\;.$$

Unfortunately, we could not succeed to conclude concerning the optimal control problem in this case. □

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