PORTUGALIAE MATHEMATICA
Vol. 59 Fasc. 1 – 2002
Nova Série

THE INVARIANT SUBRINGS OF DEMEYER–KANZAKI GALOIS EXTENSIONS

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Abstract: Let B be a ring with 1, G a finite automorphism group of B, C the center of B, B^G the set of elements in B fixed under each element in G. When B is a DeMeyer–Kanzaki Galois extension of B^G with Galois group G, it was shown that a separable subring S of B over B^G is equal to B^K for some subgroup K of G if and only if $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$ where $J_g^{(S)} = \{s - g(s) \mid s \in S\}$. Moreover, the invariant subrings of C over C^G (i.e., $S = C^K$ for some subgroup K of G) and of $B * G$ over $(B * G)^{\bar{G}}$ are characterized in terms of the faithful B-module $BJ_g^{(S)}$ and the faithful C^G-module $C^G J_g^{(S)}$ respectively for $g \in G$.

1 – Introduction

Throughout this paper, B will represent a ring with 1, G a finite automorphism group of B, C the center of B, B^G the set of elements in B fixed under each element in G, $B * G$ a skew group ring over B in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$, and $\bar{G}$ the inner automorphism group of $B * G$ induced by G, that is, $\bar{g}(f) = gfg^{-1}$ for each $f \in B * G$ and $g \in G$. We note that $\bar{G}$ restricted to B is G.

Following the notations and facts in [5], B is called a Galois extension of B^G with Galois group G if there exist elements $\{c_i, d_i \text{ in } B,\ i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^m c_i\, g(d_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{c_i, d_i\}$ is

Received: September 8, 2000; *Revised*: November 8, 2000.
AMS Subject Classification: 16S35; 16W20.
Keywords: Galois extensions; center Galois extensions; DeMeyer–Kanzaki Galois extensions; separable extensions; H-separable extensions; Azumaya algebras.

called a G-Galois system for B. B is called a center Galois extension of B^G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$. Let A be a subring of a ring B with the same identity 1. $V_B(A)$ denotes the commutator subring of A in B. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A, and an Azumaya algebra is a separable extension of its center. B is called a DeMeyer–Kanzaki Galois extension with Galois group G if B is an Azumaya C-algebra and a center Galois extension with Galois group G. A ring F is called a H-separable extension of B if $F \otimes_B F$ is isomorphic to a direct summand of a finite direct sum of F as a F-bimodule. S is called a D-S-separable extension of A in B if S is a separable extension of A in B and a direct summand of a finite direct sum of B as a bimodule over S ([3]). We denote $\{s - g(s) \mid s \in S\}$ by $J_g^{(S)}$ and the A-module generated by $J_g^{(S)}$ by $AJ_g^{(S)}$ for $g \in G$.

The fundamental theorem for Galois extensions of a field or a commutative ring with no idempotents but 0 and 1 states that there exists a one-to-one correspondence between the set of subgroups of the Galois group G and the set of separable subrings of the Galois extension ([1], Chapter 3). In general, there exists no such a correspondence for Galois extensions of rings although there are some kind of correspondeces between certain sets of separable extensions of rings ([2]). For a Galois extension B it is easy to see that the map from the set of subgroups of G to the set of separable extensions of B^G in B given by $K \to B^K$ is one-to-one but not necessarily onto. So it is interesting to know what kind of separable subrings of B is invariant under a subgroup K of G. The purpose of the present paper is to characterize for a DeMeyer–Kanzaki Galois extension B the invariant separable subrings S of B over B^G, of C over C^G, and of $B * G$ over $(B * G)^{\bar{G}}$ respectively.

2 – Main results

In this section, we first characterize for a DeMeyer–Kanzaki Galois extension B the invariant separable subrings S of B over B^G, and then characterize for a center Galois extension B the invariant separable subrings S of C over C^G and of $B * G$ over $(B * G)^{\bar{G}}$ respectively. Consequently, results are derived for a DeMeyer–Kanzaki Galois extension B of B^G. We first give three lemmas.

Lemma 1. *Let B be a ring. Then, $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$ if and only if $V_{B*G}(C) = B$ where $J_g^{(C)} = \{c - g(c) \mid s \in C\}$ and $V_{B*G}(C)$ is the commutator subring of C in $B * G$.*

Proof: ($\Rightarrow$) Clearly, $B \subset V_{B*G}(C)$. Let $\sum_{g\in G} b_g g$ in $V_{B*G}(C)$ for some $b_g \in B$. Then $c(\sum_{g\in G} b_g g) = (\sum_{g\in G} b_g g)c$ for each c in C, so $cb_g = b_g g(c)$, that is, $b_g(c - g(c)) = 0$ for each $g \in G$ and $c \in C$. Since $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$, $b_g = 0$ for each $g \neq 1$. But then $\sum_{g\in G} b_g g = b_1 \in B$. Hence $V_{B*G}(C) \subseteq B$, and so $V_{B*G}(C) = B$.

($\Leftarrow$) By the above argument, we have that $V_{B*G}(C) = \{\sum_{g\in G} b_g g \mid b_g J_g^{(C)} = \{0\}$ for each $g \in G\}$. Thus, $V_{B*G}(C) = B$ implies that $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$. ∎

Lemma 2. *Let B be a ring such that $B = B^G C$, S a subring of B over B^G, and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$ if and only if $V_{B*G}(S) = C * K$.*

Proof: By hypothesis, $B = B^G C$. So $V_B(B^G) = V_B(B^G C) = V_B(B) = C$. Hence $V_{B*G}(B^G) = V_B(B^G)*G = C*G$. But $B^G \subset S$, so $V_{B*G}(S) \subset V_{B*G}(B^G) = C * G$. Thus, $V_{B*G}(S) = V_{C*G}(S)$. By a direct computation, $V_{C*G}(S) = C * K \oplus \sum_{g\notin K} I_g g$ where $I_g = \{c \in C \mid c(s - g(s)) = 0$ for each $s \in S\} = \mathrm{Ann}_C(J_g^{(S)})$, the annihilator of the C-module $CJ_g^{(S)}$. Therefore, $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$ if and only if $V_{B*G}(S) = C * K$. ∎

Lemma 3. *Assume that B is a ring such that $B = B^G C$ and $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$. Let S be a subring of B over B^G and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = B^K$ and $C * K$ satisfies the double centralizer property in $B * G$ if and only if $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$ and S satisfies the double centralizer property in $B * G$.*

Proof: ($\Leftarrow$) Since $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$, $V_{B*G}(S) = C*K$ by Lemma 2. Hence $V_{B*G}(V_{B*G}(S)) = V_{B*G}(C * K) = (V_{B*G}(C))^{\bar{K}} = B^K$ by Lemma 1 (for $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$). But $V_{B*G}(V_{B*G}(S)) = S$ by hypothesis, so $S = B^K$, and $V_{B*G}(V_{B*G}(C * K)) = V_{B*G}(S) = C * K$.

($\Rightarrow$) By hypothesis, $BJ_g^{(C)}$ is a faithful B-module for each $g \neq 1$, so $V_{B*G}(C) = B$ by Lemma 1. Hence $V_{B*G}(C * K) = (V_{B*G}(C))^{\bar{K}} = B^K = S$ by hypothesis. Thus $V_{B*G}(S) = V_{B*G}(V_{B*G}(C * K)) = C * K$. Therefore, $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$ by Lemma 2. Moreover, $V_{B*G}(V_{B*G}(S)) = V_{B*G}(C * K) = S$. This completes the proof. ∎

We now show a characterization for an invariant separable subring S of B over B^G for a DeMeyer–Kanzaki Galois extension.

Theorem 4. *If B is a DeMeyer–Kanzaki Galois extension of B^G with Galois group G, S a separable subring of B over B^G, and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = B^K$ if and only if $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$.*

Proof: Since B is a DeMeyer–Kanzaki Galois extension of B^G with Galois group G, B is an Azumaya C-algebra and B is a center Galois extension of B^G. Hence, by Theorem 3.2 and Lemma 3.1 in [5], $V_{B*G}(B) = C$, so $V_{B*G}(B*G) = (V_{B*G}(B))^{\bar{G}} = C^G$, that is, C^G is the center of $B*G$. Since B is a center Galois extension of B^G again, $B*G$ is H-separable over B and C is separable over C^G. Hence, $B*G$ is separable over C^G by the transitivity of separable extensions. Thus, $B*G$ is an Azumaya C^G-algebra. Since S is a separable extension over B^G which is separable over C^G, S is a separable C^G-subalgebra of the Azumaya algebra $B*G$ by the transitivity of separabe extensions. Hence S satisfies the double centralizer property in $B*G$ ([1], Theorem 4.3, page 57). On the other hand, by definition of the DeMeyer–Kanzaki Galois extension, C is a commutative Galois extension of C^G with Galois group G, so for any subgroup K of G, C is a Galois extension of C^K with Galois group K with the same Galois system. Hence $C*K$ is an Azumaya C^K-algebra and C^K is separable over C^G, and so $C*K$ is separable over C^G by the transitivity of separabe extensions. Thus, $C*K$ also satisfies the double centralizer property in $B*G$ for $B*G$ is an Azumaya C^G-algebra. Moreover, since B is a center Galois extension of B^G with Galois group G, by Theorem 3.2 in [5], $B = B^G C$ and $BJ_g^{(C)} = B$, which is a faithful B-module, for each $g \neq 1$ in G. Therefore, Theorem 4 holds by Lemma 3. ∎

To characterize for a center Galois extension B the invariant separable subrings S of C over C^G and of $B*G$ over $(B*G)^{\bar{G}}$ respectively, Theorem 1 in [3] plays an important role. For convenient, we state it here as a proposition.

Proposition 5. ([3], Theorem 1) *Let A be a H-separable extension of E. Then if A is left or right E-finitely generated projective, there exists a one-to-one correspondence $V\colon S \to V_A(S)$ such that V^2 is an identity between the set of D-S-separable extensions of E in A and the set of $Z(A)$-separable subalgebras of $V_A(E)$ where $Z(A)$ is the center of A.* ∎

Theorem 6. *Let B be a center Galois extension of B^G, S a separable extension of C^G in C and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $BJ_g^{(S)}$ is a faithful B-module for each $g \notin K$.*

Proof: ($\Leftarrow$) By a direct computation, we have $V_{B*G}(S) = B*K \oplus \sum_{g\notin K} I_g\, g$ where $I_g = \mathrm{Ann}_B(BJ_g^{(S)})$. But, $BJ_g^{(S)}$ is a faithful B-module for each $g \notin K$, so $I_g = \{0\}$ for each $g \notin K$; and so $V_{B*G}(S) = B*K$. Hence, $V_{B*G}(V_{B*G}(S)) = V_{B*G}(B*K) = (V_{B*G}(B))^{\bar{K}} = C^K$. Next, we prove that S satisfies the double centralizer property in $B*G$; and so $S = V_{B*G}(V_{B*G}(S)) = C^K$. In fact, since B is a center Galois extension of B^G, $B = BJ_g^{(C)}$ for each $g \neq 1$ in G ([5], Theorem 3.2). Hence, $B*G$ is H-separable over B and B-finitely generated projective ([5], Lemma 3.1-(3)). Moreover, by Lemma 3.1-(4) in [5], $V_{B*G}(B) = C$. Therefore, S is a separable C^G-subalgebra of $V_{B*G}(B)(= C)$. Thus, $V_{B*G}(V_{B*G}(S)) = S$ by Proposition 5.

($\Rightarrow$) By the above argument, $V_{B*G}(B) = C$ and $V_{B*G}(S) = B*K\oplus\sum_{g\notin K} I_g\, g$. Hence, to show that $BJ_g^{(S)}$ is a faithful B-module for each $g \notin K$, that is, $I_g = \{0\}$ for each $g \notin K$, it suffices to show that $V_{B*G}(S) = B*K$. Since $S = C^K$, $V_{B*G}(S) = V_{B*G}(C^K) = V_{B*G}((V_{B*G}(B))^{\bar{K}}) = V_{B*G}(V_{B*G}(B*K))$. Therefore, we only need to show that $B*K$ satifies the double centralizer property in $B*G$. Since $B*G$ is H-separable over B and B-finitely generated projective again, $V_{B*G}(S)$ is a D-S-separable extension of B in $B*G$ by Proposition 5 (for S is a separable C^G-subalgebra of $C(= V_{B*G}(B))$). Next we claim that $B*K$ is a D-S-separable extension of B in $B*G$, and so $V_{B*G}(V_{B*G}(B*K)) = B*K$ by Proposition 5. In fact, since C is a Galois extension of C^G, C is a Galois extension of C^K with the same Galois system. Hence $B*K$ is separable over B by Lemma 3.1-(3) in [5]. Moreover, Since $V_{B*G}(S)$ is a direct summand of a finite direct sum of $B*G$ as a bimodule over $V_{B*G}(S)$ and $V_{B*G}(S) = B*K\oplus\sum_{g\notin K} I_g\, g$, $B*K$ will be a direct summand of a finite direct sum of $B*G$ as a bimodule over $B*K$ if we can show that $\sum_{g\notin K} I_g g$ is a $B*K$-bimodule. In fact, for any $b \in B$ and $k \in K$ and for any $b_g \in I_g$ with $g \notin K$, $(bk)(b_g g) = bk(b_g)(kg)$. Since $k \in K$ and $g \notin K$, $kg \notin K$. Moreover, for any $s \in S$, $(bk(b_g))\,(s - (kg)(s)) = bk(b_g)\,(k(s) - (kg)(s)) = bk(b_g(s - g(s)) = 0$ since $b_g \in I_g$. Hence $bk(b_g) \in I_{kg}$, and so $bk(b_g)(kg) \in \sum_{h\notin K} I_h\, h$. Thus $\sum_{h\notin K} I_h\, h$ is a left $B*K$-module. Similarly, $(b_g g)(bk) = (b_g g(b))(gk)$ with $gk \notin K$ and for any $s \in S$, $(b_g)\, g(b)\,(s - (gk)(s)) = b_g\, g(b)\,(s - g(s)) = (b_g(s - g(s))\, g(b) = 0$ since $b_g \in I_g$. Hence $(b_g\, g(b)) \in I_{gk}$, and so $(b_g\, g)(bk) \in \sum_{h\notin K} I_h\, h$. Thus $\sum_{h\notin K} I_h\, h$ is a right $B*K$-module. Therefore, $\sum_{g\notin K} I_g\, g$ is a $B*K$-bimodule. This completes the proof. ∎

Corollary 7. *Let B be a DeMeyer–Kanzaki Galois extension of B^G, S a separable extension of C^G in C and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $BJ_g^{(S)}$ is a faithful B-module for each $g \notin K$.*

Proof: Since a DeMeyer–Kanzaki Galois extension is a center Galois extension, the Corollary is an immediate consequence of Theorem 6. ∎

Corollary 8. *Let C be a commutative Galois extension of C^G, S a separable extension of C^G in C and $K = \{g \in G \mid g(s) = s$ for all $s \in S\}$. Then, $S = C^K$ if and only if $CJ_g^{(S)}$ is a faithful C-module for each $g \notin K$.*

Proof: Let $B = C$ in Theorem 6 or Theorem 4. ∎

Next, we give a characterization for an invariant separable subring of $B * G$ over $(B * G)^{\bar{G}}$ for some subgroup K of G.

Theorem 9. *Let B be a center Galois extension of B^G with Galois group G of order n invertible in B, S a D-S-separable extension of $(B * G)^{\bar{G}}$ in $B * G$, and $K = \{g \in G \mid \bar{g}(s) = s$ for all $s \in S\}$. Then, $S = (B * G)^{\bar{K}}$ if and only if $C^G J_g^{(S)}$ is a faithful C^G-module for each $g \notin K$.*

Proof: ($\Rightarrow$) Since B is a Galois extension of B^G, $B * G$ is a Galois extension of $(B * G)^{\bar{G}}$ with the same Galois system for B. Hence $B * G$ is right $(B * G)^{\bar{G}}$-finitely generated projective. Moreover, since elements in $\bar{G}$ are inner, $B * G$ is H-separable over $(B * G)^{\bar{G}}$ by Corollary 3 in [4]. Noting that K is a subgroup of G, we have that $C^G K \subset C^G G \subset V_{B*G}(V_{B*G}(C^G G)) = V_{B*G}((B * G)^{\bar{G}})$. Hence, $C^G K$ is a separable C^G-subalgebra of $V_{B*G}((B * G)^{\bar{G}})$. Thus $C^G K$ satisfies the double centralizer property in $B * G$ by Proposition 5. Now, since $S = (B * G)^{\bar{K}}$, $V_{B*G}(S) = V_{B*G}((B * G)^{\bar{K}}) = V_{B*G}(V_{B*G}(C^G K)) = C^G K$. For any $c \in C^G$ such that $cJ_g^{(S)} = \{0\}$, we have $cs = cg(s)$, and so $s(cg) = (sc)g = csg = cg(s)g = (cg)s$ for all $s \in S$. Hence $cg \in V_{B*G}(S)$. But $V_{B*G}(S) = C^G K$, so $c = 0$ for each $g \notin K$. This implies that $C^G J_g^{(S)}$ is a faithful C^G-module for each $g \notin K$.

($\Leftarrow$) By the above argument, $B * G$ is H-separable over $(B * G)^{\bar{G}}$ and right $(B * G)^{\bar{G}}$-finitely generated projective. Since S is a D-S-separable extension of $(B * G)^{\bar{G}}$ in $B * G$, S satisfies the double centralizer property in $B * G$ by Proposition 5. Since B is a center Galois extension of B^G, $V_{B*G}(B) = C$, so $V_{B*G}(B * G) = (V_{B*G}(B))^{\bar{G}} = C^G$, that is, C^G is the center of $B * G$. But, n is invertible in C^G, so $C^G G$ is C^G-separable subalgebra of $V_{B*G}((B * G)^{\bar{G}})$. Hence $V_{B*G}(V_{B*G}(C^G G)) = C^G G$ by Proposition 5. Now, by hypothesis, $(B * G)^{\bar{G}} \subset S$. Hence $V_{B*G}(S) \subset V_{B*G}((B * G)^{\bar{G}}) = V_{B*G}(V_{B*G}(C^G G)) = C^G G$. Therefore, $V_{B*G}(S) = V_{C^G G}(S) = C^G K \oplus \sum_{g \notin K} I_g\, g$ where $I_g = \mathrm{Ann}_{C^G}(J_g^{(S)})$. Since $J_g^{(S)}$ is a faithful C^G-module for each $g \notin K$, $V_{B*G}(S) = C^G K$. Therefore, $S = V_{B*G}(V_{B*G}(S)) = V_{B*G}(C^G K) = (B * G)^{\bar{K}}$. This completes the proof. ∎

3 – Examples

In this section, we give two examples to demonstrate our results and show that Theorem 4 does not hold for a center Galois extension B in general.

Example 1. Let Q be the rational field, $C = Q \oplus Q \oplus Q \oplus Q \oplus Q \oplus Q$, $B = C[i,j,k]$ the quaternion algebra over C, and $G = \langle g \rangle$, the cyclic group generated by g, where $g(a_1,a_2,a_3,a_4,a_5,a_6) = (a_2,a_3,a_4,a_5,a_6,a_1)$ for $a_i \in Q$ and $g(c_1+c_2i+c_3j+c_4k) = g(c_1)+g(c_2)i+g(c_3)j+g(c_4)k$ for $c_1+c_2i+c_3j+c_4k \in B$. Then

(**1**) The center of B is C.

(**2**) $C^G = \{(a,a,a,a,a,a) \mid a \in Q\} \cong Q$.

(**3**) $B^G = C^G[i,j,k] \cong Q[i,j,k]$.

(**4**) B is an Azumaya C-algebra.

(**5**) C is a Galois extension of C^G with Galois group $G|_C \cong G$ with a Galois system $\{e_l, \frac{1}{6}e_l \mid l = 1,2,3,4,5,6\}$ where e_l is the element in C with l^{th} component 1 and elsewhere 0.

(**6**) By (4) and (5), B is a DeMeyer–Kanzaki Galois extension of B^G with Galois group G.

(**7**) The nontrivial subgroups of G are $K_1 = \{1, g^3\}$ and $K_2 = \{1, g^2, g^4\}$.

(**8**) $B^{K_1} = C^{K_1}[i,j,k]$ where $C^{K_1} = \{(a_1,a_2,a_3,a_1,a_2,a_3) \mid a_1,a_2,a_3 \in Q\}$ and $B^{K_2} = C^{K_2}[i,j,k]$ where $C^{K_2} = \{(a_1,a_2,a_1,a_2,a_1,a_2) \mid a_1,a_2 \in Q\}$.

(**9**) $J_g^{(B^{K_1})} = J_{g^2}^{(B^{K_1})} = J_{g^4}^{(B^{K_1})} = J_{g^5}^{(B^{K_1})} = B^{K_1} = C^{K_1}[i,j,k]$ are faithful C-modules. $J_g^{(B^{K_2})} = J_{g^3}^{(B^{K_2})} = J_{g^5}^{(B^{K_2})} = \{(b,-b,b,-b,b,-b) \mid b \in Q[i,j,k]\}$ are faithful C-modules.

(**10**) Let $S = \{(b_1,b_1,b_2,b_1,b_1,b_2) \mid b \in Q[i,j,k]\}$. Then $S (\cong (Q \oplus Q)[i,j,k])$ is separable over $B^G (\cong Q[i,j,k])$, $K = \{g \in G \mid g(s) = s \text{ for all } s \in S\} = K_1 = \{1, g^3\}$, and $S \neq B^{K_1}$ and $J_g^{(S)} = \{(b,0,-b,b,0,-b) \mid b \in Q[i,j,k]\}$ is not a faithful C-module. □

Example 2. Let Q, C, and G acts on C as given in Example 1. Let $A_2(Q) = \{\left(\begin{smallmatrix} q_1 & q_2 \\ 0 & q_3 \end{smallmatrix}\right) \mid q_1,q_2,q_3 \in Q\}$, the ring of all 2 by 2 upper triangular matrices over Q, $B = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) \mid c_1,c_2,c_3 \in C\}$ $(\cong A_2(Q) \otimes_Q C)$, and $g\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} g(c_1) & g(c_2) \\ 0 & g(c_3) \end{smallmatrix}\right)$

for all $\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) \in B$. Then

(**1**) The center of B is $\{\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in C\} \cong C$.

(**2**) $C^G = \{(a, a, a, a, a, a) | \ a \in Q\} \cong Q$ as given in Example 1-(2).

(**3**) C is a Galois extension of C^G with Galois group $G|_C \cong G$ as shown in Example 1-(5). Hence B is a center Galois extension.

(**4**) By the argument in Example 4.3-(8) in [6], B is not an Azumaya C-algebra. Hence B is not a DeMeyer–Kanzaki Galois extension of B^G with Galois group G.

(**5**) $B^G = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) | \ c_1, c_2, c_3 \in C^G\} \cong A_2(Q)$.

(**6**) Let $S_C = \{(q_1, 2q_1, q_2, q_1, 2q_1, q_2) | \ q_1, q_2 \in Q\}$ $(\cong Q \oplus Q)$ and $S = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) | \ c_1, c_2, c_3 \in S_C\}$ $(\cong A_2(Q) \otimes_Q (Q \oplus Q))$. Then S is separable over B^G, $K = \{g \in G \mid g(s) = s$ for all $s \in S\} = \{1, g^3\}$, $S \neq B^K = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) | \ c_1, c_2, c_3 \in C^K\}$ $(\cong A_2(Q) \otimes_Q (Q \oplus Q \oplus Q))$ where $C^K = \{(q_1, q_2, q_3, q_1, q_2, q_3) | \ q_1, q_2, q_3 \in Q\} \cong (Q \oplus Q \oplus Q)$. But $J_g^{(S)} = J_{g^2}^{(S)} = J_{g^4}^{(S)} = J_{g^5}^{(S)} = \{\left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c_3 \end{smallmatrix}\right) | \ c_1, c_2, c_3 \in J_C\}$ where $J_C = \{(a, b, -a-b, a, b, -a-b) | \ a, b \in Q\}$, so $CJ_g^{(S)} = CJ_{g^2}^{(S)} = CJ_{g^4}^{(S)} = CJ_{g^5}^{(S)} = B$ are faithful C-modules even through $S \neq B^K$. Hence Theorem 4 does not hold for a center Galois extension B in general. □

ACKNOWLEDGEMENTS – The author would like to thank Professor George Szeto for many useful suggestions and discussions. This paper was revised under the suggestions of the referee and written under the support of a Caterpillar Fellowship at Bradley University. The author would like to thank the referee for the valuable suggestions and Caterpillar Inc. for the support.

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