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GLOBAL EXISTENCE FOR THE CONSERVED PHASE FIELD MODEL WITH MEMORY AND QUADRATIC NONLINEARITY

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Dedicated to professor Krzysztof Wilmanski on the occasion of his 60th birthday

Presented by Hugo Beirão da Veiga

Abstract: A nonlinear system for the heat diffusion inside a material subject to phase changes is considered. A thermal memory effect is assumed in the heat conduction law; moreover, on account of thermodynamical considerations, a linear growth is allowed for the latent heat density. The resulting problem couples a second order integrodifferential equation, derived from the balance of energy, with a fourth order parabolic inclusion which rules the evolution of an order parameter χ. Homogeneous Neumann boundary conditions guarantee that the space average of χ is conserved in time. Global existence of solutions is proved in a variational setting.

Introduction

Let us consider a smooth, bounded, and connected domain $\Omega \subset \mathbb{R}^3$ and fix a final time $T > 0$. We also set $\Gamma := \partial\Omega$, $Q_t := \Omega \times (0,t)$ for $0 < t \leq T$, $Q := Q_T$, $\Sigma := \Gamma \times (0,T)$ and suppose that Ω is filled with a homogeneous material where a heat diffusion process takes place, possibly leading to a phase transition. In order to represent the evolution of such a phenomenon, we appeal to the conserved

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phase field model with memory (see [7, 8, 19]) in a quite general framework. Hence, the thermodynamical state of the substance at (x,t) is described by the (relative) temperature θ and the order parameter χ, which in most cases is assumed to attain values in between, say, 0 and 1 and whose spatial mean value is conserved in time. Referring to [7] and, especially, to [19] for more details about the modeling, we point out that a thermal memory effect is also accounted for, by assuming that the heat flux only depends on the past history of the temperature gradient through a (smooth) convolution kernel $k: [0,T] \to \mathbb{R}$. Thus, if $\lambda'(\chi)$ denotes the (possibly nonconstant) latent heat of the fusion-solidification process, the resulting differential system reads as follows

$$(\theta + \lambda(\chi))' - \Delta(k * \theta) = g \ , \tag{1}$$

$$\chi' - \Delta\Big(-\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\,\theta\Big) \ni 0 \ , \tag{2}$$

in Q, where we have set

$$(k * \theta)(x,t) := \int_0^t k(t-s)\,\theta(x,s)\ ds\,, \qquad (x,t) \in Q \ ,$$

and g is a given source term, β a *maximal monotone graph* in $\mathbb{R}\times\mathbb{R}$, while σ', λ' are Lipschitz continuous functions. To be more precise, the sum $\beta + \sigma'$ stands for the derivative (in a suitable sense) of the double-well part of a Ginzburg–Landau free energy potential [5, 20]. In view of a mathematical analysis, Cauchy and Neumann boundary conditions have to be added to the system (1)–(2). The latter ones will be homogeneous as far as χ and the so-called *chemical potential* $w := -\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\,\theta$ are concerned. Consequently, it is easy to deduce from (2) that the space average of χ remains constant in time.

Phase-field models, possibly accounting for memory effects, have been extensively investigated in recent years (see, e.g., [4, 6, 9, 11, 12, 13, 16] and references therein). For a partial comparative review of the related work, we refer to the Introduction of [7]. In that paper, the above problem was analyzed. In particular, existence and uniqueness were proved in the case of a nonlinearity λ with at most a linear growth at infinity (see Theorem 2.1 in [7]). Instead, in this note, the function λ is allowed to be quadratic. This choice, which seems unusual in the classical framework of Stefan problems, becomes rather appropriate in other modeling contexts (see [14, 15, 20, 21]). Therefore, our goal consists in showing that the existence part of Theorem 2.1 in [7] still holds when λ' is (no longer bounded but) only Lipschitz continuous. The key argument for the proof relies on the choice of a suitable test function (which involves some technical details) combined with a bootstrap procedure.

It is worth mentioning that the *nonconserved* phase-field model with memory and quadratic nonlinearity (which basically differs from (1)–(2) because of a second order dynamics for χ) has been already deeply investigated. In [2], existence and uniqueness of the solution are proved when the additional diffusion term $-k_0\,\Delta\theta$, $k_0>0$, is present on the left hand side of (1) (see also [12] for the long-time behaviour and the existence of attractors). On the other hand, the system related to (1) has been analyzed in [9]. There, the authors show the existence of a solution to the corresponding initial and boundary value problem as well as they discuss the asymptotic behaviour in time. Subsequently, in [10], the same authors also derive a uniqueness result via a maximum principle argument which is established by a Moser-type technique. Apparently, this procedure cannot be applied to our fourth order kinetic equation (2) and we let the uniqueness issue for our model remain open.

Here is the plan of the paper. In the next Section 2, a precise variational formulation of the initial and boundary value problem associated with (1)–(2) is given and the related existence theorem is stated. Then, we introduce an approximating problem to which the existence result of [7] applies. In Section 3, we derive some basic a priori estimates, which partly follow the ones performed in the quoted paper. Finally, in Section 4, we are able to pass to the limit and achieve the existence proof.

2 – Main result and approximation

We start by listing our hypotheses on β, k, λ, and σ. Let

$$j\colon \mathbb{R}\to[0,+\infty] \quad \text{be proper, convex, and lower semicontinuous}\,, \tag{3}$$

$$j(0)=0\,,\qquad \beta=\partial j \quad\text{and}\quad \beta(0)\ni 0\,, \tag{4}$$

$$k\in W^{2,1}(0,T) \quad\text{and}\quad k(0)>0\,, \tag{5}$$

$$\lambda,\sigma\in C^1(\mathbb{R})\,,\qquad \lambda' \text{ and } \sigma' \text{ be Lipschitz continuous}\,. \tag{6}$$

Then, we indicate by $D(j)$ and $D(\beta)$ the effective domains of j and β, respectively.

As usual, the introduction of a variational formulation for (1)–(2) requires some machinery. First of all, we define $V:=H^1(\Omega)$ and $H=H':=L^2(\Omega)$, in order that (V,H,V') forms a Hilbert triplet. Moreover, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V, while $\|\cdot\|$ and $\|\cdot\|_*$ are the standard norms in V and V', respectively. Also, $|\cdot|$ stands for the norm in H or $H^3=(L^2(\Omega))^3$ and $(\cdot,\cdot)$ is the corresponding scalar product. Besides the notation, we also need to introduce the variational form of the Laplacian with homogeneous Neumann boundary conditions as follows

$$\langle Au, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \text{for all } \ u, v \in V \ . \tag{7}$$

Observe that $A\colon V \to V'$ is not invertible; however, if we set

$$V_0 := \Big\{ v \in V\colon \langle v, 1\rangle = 0 \Big\} \quad \text{and} \quad V_0' := \Big\{ u \in V'\colon \langle u, 1\rangle = 0 \Big\} \ , \tag{8}$$

then it is straightforward to see that the restriction of A to V_0 is an isomorphism of V_0 onto V_0'. In this case, let us call $\mathcal{N}$ the inverse of A restricted to V_0.

In order to state our existence theorem, let us make a change of unknowns. Setting $u := 1 * \theta$, we can rewrite equation (1) in terms of (u, χ); on account of well-known properties of convolutions and taking advantage of (6), the obtained equation turns out to have a hyperbolic character (cf. [6] for more details). We are now able to present the main result.

Theorem 2.1. *Assuming (3)–(6) and*

$$g \in L^1(0, T; H) + W^{1,1}(0, T; V') \ , \tag{9}$$

$$\theta_0 \in H \, , \quad \chi_0 \in V \, , \quad j(\chi_0) \in L^1(\Omega) \ , \tag{10}$$

$$m(\chi_0) := |\Omega|^{-1} \int_\Omega \chi_0 \, dx \in \operatorname{Int} D(\beta) \ , \tag{11}$$

there exists one quadruplet of functions (u, χ, w, ξ) enjoying the regularity properties

$$u \in C^1([0, T]; H) \cap C^0([0, T]; V) \ , \tag{12}$$

$$\chi \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \ , \tag{13}$$

$$w \in L^2(0, T; V) \ , \tag{14}$$

$$\xi \in L^2(0, T; H) \tag{15}$$

and satisfying the relations

$$(u' + \lambda(\chi))' + k(0)\, Au = g - A(k' * u) \quad \text{in } V', \ \text{a.e. in } (0, T) \ , \tag{16}$$

$$\chi' + Aw = 0 \quad \text{in } V', \ \text{a.e. in } (0, T) \ , \tag{17}$$

$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\, u' \quad \text{in } V', \ \text{a.e. in } (0, T) \ , \tag{18}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } Q \ , \tag{19}$$

$$u(0) = 0, \quad u'(0) = \theta_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega \ . \ \blacksquare \tag{20}$$

Let us observe that this result is perfectly analogous to the corresponding Theorem 2.1 of [7], where λ was also assumed to be Lipschitz continuous. Consequently, in order to apply that theorem, we shall approach the problem by operating a suitable truncation of λ'. However, if only this approximation is done, we are not able to perform in a rigorous way the a priori estimates that are needed to generalize the existence part of the quoted theorem. For this reason, rather than proceeding formally, we prefer to regularize β as well. This choice allows us to estimate ξ by a new argument, which is different and more delicate than the one used in [7].

Thus, we substitute β with its Yosida regularization β_ε (cf. [3]) and we approximate λ, for instance, in this way

$$(21) \quad \lambda_\varepsilon(r) = \lambda(0) + \int_0^r \lambda'_\varepsilon(s)\,ds\,, \qquad \lambda'_\varepsilon(r) := \begin{cases} \lambda'(-\varepsilon^{-1}) & \text{if } r \le -1/\varepsilon, \\ \lambda'(r) & \text{if } -1/\varepsilon < r < 1/\varepsilon, \\ \lambda'(\varepsilon^{-1}) & \text{if } r \ge 1/\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is the approximation parameter, intended to go to 0 in the limit. Replacing now β, λ by $\beta_\varepsilon, \lambda_\varepsilon$ into (16)–(20), we obtain a system to which Theorem 2.1 of [7] can be applied. This yields the ε-solution $(u_\varepsilon, \chi_\varepsilon, w_\varepsilon, \xi_\varepsilon)$ enjoying the regularity properties (12)–(15). In addition, the Lipschitz continuity of β_ε also implies $\xi_\varepsilon \in C^0([0,T]; H)$. In the next section, our goal will be that of deriving some a priori estimates, independent of ε, for the ε-solution.

3 – A priori estimates

Throughout this section, whenever we mention relations (16)–(20), we always refer to their approximate formulations involving the ε-solution. Moreover, C will denote a positive constant which may depend on data, but it is always independent of ε. This generic constant can vary from line to line.

First estimate. Test (16) by u'_ε, (17) by $\mathcal{N}\chi'_\varepsilon$, and (18) by χ'_ε. Integrate over $(0,t)$, with $0 < t \le T$, and add the results together. Proceeding exactly as in [7], we find out that

$$(22) \qquad \|u_\varepsilon\|_{L^\infty(0,T;V)\cap W^{1,\infty}(0,T;H)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;V')} \le C\,.$$

Notice that λ does not play any role in this step, since the two related terms in (16) and (18) cancel out. Furthermore, observe that the procedure is just formal, due to the low regularity of the test functions; for a rigorous argument we should argue as, e.g., in [7], where this estimate is obtained in a Faedo–Galerkin approximation scheme.

Second estimate. Let us consider the function $\phi \in C^1(\mathbb{R})$ specified by

$$\phi(r) := \begin{cases} \dfrac{r}{2} + \dfrac{1}{2}\sin\Big(\dfrac{\pi r}{2}\Big) & \text{if } |r| \le 1, \\ |r|^{1/2}\,\mathrm{sign}(r) & \text{if } |r| > 1 . \end{cases} \tag{23}$$

Moreover, for any $t \in (0,T]$, define $x_\varepsilon(t)$ as the unique solution of the equation

$$\Phi(t, x_\varepsilon(t)) = 0, \quad \text{where } \Phi(t,r) := \int_\Omega \phi\Big(\xi_\varepsilon(x,t) - r\Big)\,dx, \quad r \in \mathbb{R} . \tag{24}$$

As Φ is continuous in $[0,T]\times\mathbb{R}$, note that the existence and uniqueness of $x_\varepsilon(t)$ are guaranteed by the behaviour at infinity and the strict monotonicity of Φ with respect to the second variable. Furthermore, since $\xi_\varepsilon \in C^0([0,T];H)$, then $\partial_r\Phi$ is continuous and the implicit function theorem easily yields $x_\varepsilon \in C([0,T])$.

Now, thanks to the Lipschitz continuity of ϕ, we see that $\phi(\xi_\varepsilon - x_\varepsilon)$ is an admissible test function for (18). Moreover, its spatial mean value is 0. Then, we can use (17) and derive

$$\begin{aligned} \int_\Omega \phi'(\xi_\varepsilon - x_\varepsilon)(t)\,\beta_\varepsilon'(\chi_\varepsilon(t))\,|\nabla\chi_\varepsilon(t)|^2 + \Big(\xi_\varepsilon(t), \phi(\xi_\varepsilon - x_\varepsilon)(t)\Big) &= \\ = \Big(F_\varepsilon(t), \phi(\xi_\varepsilon - x_\varepsilon)(t)\Big) , \end{aligned} \tag{25}$$

where we have set

$$F_\varepsilon := -\mathcal{N}\chi_\varepsilon' - \sigma'(\chi_\varepsilon) + \lambda'(\chi_\varepsilon)\,u_\varepsilon' .$$

Owing to (6), to (22), and to the continuous embedding $V \subset L^6(\Omega)$, we easily deduce that F_ε is bounded in $L^2(0,T;L^{3/2}(\Omega))$ independently of ε. Hence, from (25) and Young's inequality, we derive

$$\begin{aligned} \Big(\xi_\varepsilon(t), \phi(\xi_\varepsilon(t) - x_\varepsilon(t))\Big) \ &\le \\ &\le \int_\Omega |F_\varepsilon(t)|\,\Big(1 + |\xi_\varepsilon(t) - x_\varepsilon(t)|^{1/2}\Big) \\ &\le \ \|F_\varepsilon(t)\|_{L^1(\Omega)} + \frac{2}{3}\,\|F_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} + \frac{1}{3}\,\|\xi_\varepsilon(t) - x_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} . \end{aligned}$$

Therefore, we also have that

$$\begin{aligned}
\|\xi_\varepsilon(t) - x_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} \ &\leq \\
&\leq \int_\Omega \Big((\xi_\varepsilon - x_\varepsilon)(t)\, \phi(\xi_\varepsilon - x_\varepsilon)(t) + 1 \Big) \\
&\leq \Big(\xi_\varepsilon(t), \phi(\xi_\varepsilon(t) - x_\varepsilon(t)) \Big) + |\Omega| \\
&\leq |\Omega| + \|F_\varepsilon(t)\|_{L^1(\Omega)} + \frac{2}{3}\, \|F_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} + \frac{1}{3}\, \|\xi_\varepsilon(t) - x_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} \ ,
\end{aligned}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. Now, from (27) it is a standard matter to infer the bound

$$\|\xi_\varepsilon - x_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \leq C \ . \tag{28}$$

At this point, we can repeat the argument of Kenmochi, Niezgódka, and Pawlow [17] (already used and detailed in [7]) to deduce, for a suitable $\delta > 0$,

$$\begin{aligned}
\delta\, \|\xi_\varepsilon(t)\|_{L^1(\Omega)} \ &\leq \int_\Omega (\xi_\varepsilon(t) - x_\varepsilon(t))\, (\chi_\varepsilon(t) - m_0) + C \\
&\leq \ \|\xi_\varepsilon(t) - x_\varepsilon(t)\|_{L^{3/2}(\Omega)}\, \|\chi_\varepsilon(t) - m_0\|_{L^3(\Omega)} + C \ ,
\end{aligned} \tag{29}$$

where δ is fixed and depends on the position of m_0 inside the domain $D(\beta)$. Thus, on account of (22) and (28), we derive the bound

$$\|\xi_\varepsilon\|_{L^2(0,T;L^1(\Omega))} \leq C \ . \tag{30}$$

The above estimate can be improved by simply using the fact that $x_\varepsilon(t)$ is constant in Ω for almost any $t \in (0,T)$. Indeed, we have that

$$\begin{aligned}
\int_\Omega |\xi_\varepsilon(t)|^{3/2} \ &\leq \ \sqrt{2} \int_\Omega |\xi_\varepsilon(t) - x_\varepsilon(t)|^{3/2} \ + \ \sqrt{2} \int_\Omega |x_\varepsilon(t)|^{3/2} \\
&\leq \ \sqrt{2} \int_\Omega |\xi_\varepsilon(t) - x_\varepsilon(t)|^{3/2} \ + \ \sqrt{2}\, |\Omega|^{-1/2} \Big(\int_\Omega |x_\varepsilon(t)| \Big)^{3/2} \\
&\leq \ \sqrt{2}\, \|\xi_\varepsilon(t) - x_\varepsilon(t)\|^{3/2}_{L^{3/2}(\Omega)} \\
&\qquad + \ \sqrt{2}\, |\Omega|^{-1/2} \Big(\int_\Omega |\xi_\varepsilon(t) - x_\varepsilon(t)| \ + \int_\Omega |\xi_\varepsilon(t)| \Big)^{3/2}
\end{aligned} \tag{31}$$

whence it is easy to infer

$$\begin{aligned}
\|\xi_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \ &\leq \\
&\leq \ C\, \|\xi_\varepsilon - x_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} + C\, \|\xi_\varepsilon - x_\varepsilon\|_{L^2(0,T;L^1(\Omega))} + C\, \|\xi_\varepsilon\|_{L^2(0,T;L^1(\Omega))}
\end{aligned}$$

so that, recalling (28) and (30), we obtain the desired estimate

$$\|\xi_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \le C \; . \tag{32}$$

Third estimate. We test both (17) and (18) by w_ε. Adding and integrating over $(0,t)$ for $0<t\le T$, we get

$$\begin{aligned} \int_0^t \|w_\varepsilon(s)\|^2\,ds \;&=\; -\int_0^t (\chi_\varepsilon'(s), w_\varepsilon(s))\,ds + \int_0^t (\nabla\chi_\varepsilon(s), \nabla w_\varepsilon(s))\,ds \\ &\quad + \int_0^t \Big(\Big(\xi_\varepsilon + \sigma'(\chi_\varepsilon) - \lambda_\varepsilon'(\chi_\varepsilon)\,u_\varepsilon'\Big)(s),\, w_\varepsilon(s)\Big)\,ds \\ &\le\; C + \frac{1}{2}\int_0^t \|w_\varepsilon(s)\|^2\,ds \;, \end{aligned} \tag{33}$$

where the last inequality follows as a consequence of (22) and (32), provided we take the continuous embedding $V \subset L^6(\Omega)$ into account. We finally see that

$$\|w_\varepsilon\|_{L^2(0,T;V)} \le C \; . \tag{34}$$

Fourth estimate. By comparison in (18), we easily infer that $A\chi_\varepsilon$ is bounded in $L^2(0,T;L^{3/2}(\Omega))$ independently of ε, whence classical elliptic regularity results yield

$$\|\chi_\varepsilon\|_{L^2(0,T;W^{2,3/2}(\Omega))} \le C \; . \tag{35}$$

Fifth estimate. It is now possible to rewrite (18) in the form

$$\chi_\varepsilon + A\chi_\varepsilon + \xi_\varepsilon = \widetilde{F}_\varepsilon \quad \text{in } V', \quad \text{a.e. in } (0,T) \;, \tag{36}$$

where

$$\widetilde{F}_\varepsilon := \chi_\varepsilon + w_\varepsilon - \sigma'(\chi_\varepsilon) + \lambda_\varepsilon'(\chi_\varepsilon)\,u_\varepsilon' \; . \tag{37}$$

Recalling (6), (22), (34)–(35) and taking advantage of the inclusion $W^{2,3/2}(\Omega) \subset L^p(\Omega)$, which holds for any $p \in [1,+\infty)$, one can easily verify that

$$\|\widetilde{F}_\varepsilon\|_{L^2(0,T;L^{2-\delta}(\Omega))} \le C_\delta \quad \text{for all } \delta \in (0,1] \; . \tag{38}$$

We now apply to the elliptic equation (36) the following monotonicity argument. If $\widetilde{F}_\varepsilon \in L^2(0,T;L^q(\Omega))$ for some $q \in (1,\infty)$, then χ_ε and ξ_ε belong to $L^2(0,T;L^q(\Omega))$; whence also $A\chi_\varepsilon \in L^2(0,T;L^q(\Omega))$. Therefore, we conclude that

$$\|\xi_\varepsilon\|_{L^2(0,T;L^{2-\delta}(\Omega))} + \|\chi_\varepsilon\|_{L^2(0,T;W^{2,2-\delta}(\Omega))} \le C_\delta \; . \tag{39}$$

Then, since $W^{2,2-\delta}(\Omega) \subset L^\infty(\Omega)$ for $\delta < 1/2$, going back to (37), we can improve (38) as follows

$$\|\widetilde{F}_\varepsilon\|_{L^2(0,T;H)} \leq C \ , \tag{40}$$

so that (36) implies

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} + \|\chi_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C \ . \tag{41}$$

We eventually observe that estimates (22), (34), and (41) are the same as the ones entailed by (4.38), (4.47)–(4.48) of [7]. In particular, such regularity properties still hold for the solution to the original problem which is obtained in the next section by passing to the limit as $\varepsilon \to 0$.

4 – Passage to the limit

First of all, recalling (6) and (21) we observe that

$$\lambda_\varepsilon \to \lambda \ \text{ and } \ \lambda'_\varepsilon \to \lambda' \quad \text{uniformly on compact subsets of } \mathbb{R}. \tag{42}$$

Moreover, in view of (22), (34), and (41), there exist functions u, χ, w, ξ such that, at least on a subsequence of $\varepsilon \to 0$,

$$u_\varepsilon \to u \quad \text{weakly star in } \ L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H) \ , \tag{43}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } \ L^\infty(0,T;V) \ , \tag{44}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly in } \ H^1(0,T;V') \cap L^2(0,T;H^2(\Omega)) \ , \tag{45}$$

$$w_\varepsilon \to w \quad \text{weakly in } \ L^2(0,T;V) \ , \tag{46}$$

$$\xi_\varepsilon \to \xi \quad \text{weakly in } \ L^2(0,T;H) \ . \tag{47}$$

We aim to show that the quadruplet (u, χ, w, ξ) fulfills the conditions required in Theorem 2.1. Now, (13)–(15) follow at once, while the regularity (12) is not simply ensured by (43) and will be discussed later on. On the other hand, to show the validity of (16)–(20), we start by observing that, thanks to (43)–(45), the Aubin compactness lemma (see, e.g., [18, p. 58]) gives

$$u_\varepsilon \to u \quad \text{strongly in } \ C^0([0,T];H) \ , \tag{48}$$

$$\chi_\varepsilon \to \chi \quad \text{strongly in } \ C^0([0,T];H) \cap L^2(0,T;V) \ . \tag{49}$$

Hence, (6) immediately yields

$$\sigma'(\chi_\varepsilon) \to \sigma'(\chi) \quad \text{strongly in } \ C^0([0,T];H) \ . \tag{50}$$

Instead, the term $\lambda_\varepsilon(\chi_\varepsilon)$ deserves a more careful analysis, because of the quadratic growth of λ. We first remark that (6) implies the existence of a constant $L > 0$ such that

$$|\lambda_\varepsilon(r)| \le L(1 + |r|^2) \quad \text{for all } r \in \mathbb{R} \text{ and } \varepsilon > 0 \tag{51}$$

(of course, this also holds for λ). In addition, with the help of (42), it is not difficult to verify that

$$\chi_\varepsilon \to \chi \text{ and } \lambda_\varepsilon(\chi_\varepsilon) \to \lambda(\chi) \quad \text{a.e. in } Q \ , \tag{52}$$

possibly up to the extraction of another subsequence. Moreover, by virtue of (44), (51) and exploiting the continuous embedding $V \subset L^6(\Omega)$, we can deduce the bound

$$\|\lambda_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^3(\Omega))} \le C \ . \tag{53}$$

Thus, (52) and (53) entail (cf., e.g., [18, Lemme 1.3, p. 12])

$$\lambda_\varepsilon(\chi_\varepsilon) \to \lambda(\chi) \quad \text{weakly star in } L^\infty(0,T;L^3(\Omega)) \text{ and strongly in } L^2(0,T;H) \ . \tag{54}$$

The same argument applied to λ' yields in particular

$$\lambda'_\varepsilon(\chi_\varepsilon) \to \lambda'(\chi) \quad \text{strongly in } L^2(0,T;H) \ , \tag{55}$$

which, combined with (43), entails

$$\lambda'_\varepsilon(\chi_\varepsilon)\, u'_\varepsilon \to \lambda'(\chi)\, u' \quad \text{weakly in } L^1(Q) \ .$$

Recalling (43)–(50) and (54)–(55), we now have enough information to pass to the limit in the ε-approximation of (16)–(18) and also to deduce the first and third conditions in (20). In order to complete the proof, it is convenient to take the limit of the integrated version of (16), as well. This gives

$$u' + \lambda(\chi) + k(0)\, A(1 * u) = \theta_0 + \lambda(\chi_0) + 1 * g - A(k' * 1 * u) \tag{56}$$

in V', a.e. in $(0,T)$. Now, (13) and (6) entail $\lambda(\chi) \in L^2(0,T;V)$ and $\chi \in C^0([0,T];L^4(\Omega))$, whence the inequality

$$|\lambda(r_1) - \lambda(r_2)| \le \left| \int_{r_1}^{r_2} \lambda'(\eta)\, d\eta \right| \le L_1 \Big(1 + |r_1| + |r_2|\Big) |r_1 - r_2| \ ,$$

holding for all $r_1, r_2 \in \mathbb{R}$ and for some constant $L_1 > 0$, leads to $\lambda(\chi) \in C^0([0,T];H)$. Then, by comparison in (56) we see that $u' \in C^0([0,T];V')$ and this implies $u'(0) = \theta_0$. Moreover, the regularity (12) can now be shown arguing as in the Conclusion of the proof of [7, Lemma 4.2]. Finally, we have to check (19). To this purpose, we exploit (47), (49), and apply the standard monotonicity argument of [1, Prop. 1.1, p. 42]. ∎

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