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VIABLE SOLUTIONS OF DIFFERENTIAL INCLUSIONS WITH MEMORY IN BANACH SPACES *

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Abstract: In this paper we study functional differential inclusions with memory and state constraints. We assume the state space to be a separable Banach space and prove existence results for an u.s.c. orientor field; we consider both the case of a globally measurable orientor field and the case of a Caratheodory one.

1 – Introduction

Let E be a separable Banach space, r and T two positive numbers, and put $I=[0,T]$. If $u\colon I\to E$ is a given function, we define the function $u_t\colon[-r,0]\to E$ in the following way:

$$u_t(s)=u(t+s)\qquad\text{for all}\ \ s\in[-r,0]\ .$$

Let $\mathcal{C}_r=C([-r,0];E)$ be the space of all continuous functions from $[-r,0]$ into E, endowed with the topology of uniform convergence, and consider two multi-functions $F: I\times\mathcal{C}_r\rightrightarrows E$ and $K\colon I\rightrightarrows E$ with non-empty values. For every fixed $\phi\in\mathcal{C}_r$ such that $\phi(0)\in K(0)$, we are looking for a function $u:\ [-r,T]\to E$ which is absolutely continuous on I, and fulfils the following conditions:

$$\begin{aligned}
&u'(t)\in F(t,u_t)\qquad\text{a.e. on}\ \ I\ ,\\
&u(t)\in K(t)\qquad\text{for all}\ \ t\in I\ ,\\
&u(t)=\phi(t)\qquad\text{for all}\ \ t\in[-r,0]\ .
\end{aligned}\tag{1.1}$$

Such problems are often mentioned as *differential inclusions with memory and state constraints*. Usually, some property of semicontinuity is required on F

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(with respect to the pair (t, x), or even only to x): existence results for (1.1) were obtained both on lower and upper semicontinuity assumptions. As regards the first kind of results, we quote e.g. [10] and the references contained therein. In the present paper, we are interested in the second kind of assumptions. As far as we know, the first results in this direction were given by Haddad [12], [13], in the case in which F is globally u.s.c. and takes convex compact values, and K is fixed (see also [1] and [8]). In [12] E is the euclidean space $\mathbb{R}^n$, and F does not depend on t, while in [13] E is a Hilbert space and K is convex. In both cases Haddad gives necessary and sufficient conditions in order to solve (1.1).

As regards the existence of solutions in a separable Banach space, we refer to Syam [15], where two results are given on this subject: the first one deals with a fixed convex constraint K and a globally measurable multifunction F which is u.s.c. on its second argument, while in the second one the constraint is convex and moving, but F is globally u.s.c..

Some authors also studied differential inclusions where the right-hand side contains, together with a perturbation with memory, a term which is a "maximal monotone operator" with respect to the state variable x, for instance the so-called sweeping process: on this subject we recall Castaing and Monteiro Marques [6], Avgerinos [2] and also, as regards the sweeping process of second order, Duc Ha and Monteiro Marques [9].

In the present paper we get existence results in a separable Banach space with a not necessarily convex constraint which depends on time. Such results are contained in §4 (Theorems 4.1, 4.2, 4.3) and extend corresponding results without delay which are presented in §3 (Theorems 3.1, 3.2, 3.3). In particular, Thm. 3.1 is given by Gavioli in [11], Thm. 3.2 improves a recent result due to Malaguti [14], and Thm. 3.3 is essentially due to Benabdellah, Castaing and Gamal Ibrahim [3].

In order to get the results of §4 from the corresponding results of §3 we adopt the same technique as in [12], so as to reduce the original problem (1.1) to a suitable sequence of problems without delay. The main differences in the assumptions of the three theorems concern the measurability of the multifunction F and the tangential conditions: in particular, in Theorem 4.1 F is globally measurable, while in Theorems 4.2 and 4.3 F is supposed to be measurable only with respect to t. As is known, this case can present some difficulties, in particular when the constraint K depends on time, and F is only defined on the graph Γ of K: indeed, an example by Bothe in [4] shows that, on these assumptions, there may be no solutions, even at finite dimension, in the single-valued case and in lack of delay. This circumstance, however, can be avoided also in presence of a moving constraint, provided that F is supposed to be defined on the whole space $I \times E$.

2 – Notations and preliminary results

First of all we shortly recall some basic notions about multifunctions: if X and Y are two non-empty sets, a multifunction $\Phi\colon X \rightrightarrows Y$ is a map from X into the family of all subsets of Y. The *graph* of Φ is the set $\mathrm{Gr}(\Phi)$ made up by those pairs (x, y) such that $x \in X$ and $y \in \Phi(x)$. A *selection* of Φ is a map $\phi\colon X \to Y$ such that, for every $x \in X$, $\phi(x) \in \Phi(x)$. If $V \subseteq Y$, $\Phi^{-1}(V)$ is the set of those points $x \in X$ such that $\Phi(x) \cap V \neq \emptyset$. If X is a measurable space, endowed with a σ-field $\mathcal{T}$, and Y is a topological space, Φ is said to be *measurable* if $\Phi^{-1}(V) \in \mathcal{T}$ whenever V is open in Y. If X and Y are both topological spaces, Φ is said to be *upper semicontinuous* if $\Phi^{-1}(V)$ is closed in X whenever V is closed in Y.

From now on I will denote an interval of the kind $[0, T]$, with $0 < T < +\infty$, $\mathcal{L}$ its Lebesgue σ-field, λ the Lebesgue measure on I. If Y is a topological space, the script $\mathcal{B}(Y)$ stands for the Borel σ-field on Y. Now we are going to present a multivalued version of Scorza–Dragoni Theorem, which is due to Castaing–Monteiro Marques [5]. We give it in a less general form, which is sufficient to our purposes.

Theorem 2.1. *Let E be a separable Banach space, Y be a convex compact metrizable subset of a Hausdorff locally convex space. Let $F : I \times X \rightrightarrows Y$ a multifunction with nonempty convex compact values which fulfils the following conditions:*

(i) *for any $t \in I$, the set $\{(x, y) \in E \times Y\colon\ y \in F(t, x)\}$, that is the graph of $F(t, \cdot)$, is closed in $X \times Y$;*

(ii) *for any $x \in E$, $F(\cdot, x)$ admits a measurable selection.*

Then, there exists a measurable multifunction $F_0 : I \times E \rightrightarrows Y$ with (possibly empty) convex compact values which enjoys the following properties:

(p_1) *there is a λ-null set N, not depending on x, such that*

$$F_0(t, x) \subset F(t, x), \qquad \text{for all } t \notin N \text{ and } x \in E\ ;$$

(p_2) *if $u : I \to E$ and $v : I \to D$ are measurable functions such that $v(t) \in F(t, u(t))$ a.e., then $v(t) \in F_0(t, u(t))$ a.e.;*

(p_3) *for every $\epsilon > 0$, there is a compact subset $I_\epsilon \subset I$ such that $\lambda(I \backslash I_\epsilon) < \epsilon$, the graph of the restriction $F_{0|I_\epsilon \times E}$ is closed and $\emptyset \neq F_0(t, x) \subset F(t, x)$, for all $(t, x) \in I_\epsilon \times E$.* ∎

The three following results will also be useful.

Proposition 2.2. *Let X be a topological space, E be a separable Banach space, and suppose that the multifunction $\Phi\colon X \rightrightarrows E$ turns out to be u.s.c. when E is endowed with its weak topology $\sigma(E,E^*)$. Then $\Phi^{-1}(V) \in \mathcal{B}(X)$ whenever V is open in the strong topology of E.*

Proof: Let $V \subseteq E$ be strongly open in E. Given a countable, dense subset Λ of E, let $\mathcal{F}$ be the family of all closed balls $U = B(x;r)$ such that $x \in \Lambda$, $r \in \mathbb{Q}^+$, $B(x;r) \subseteq V$. Then $\mathcal{F}$ is countable, and $V = \bigcup \mathcal{F}$. Furthermore, every ball $U \in \mathcal{F}$ is also weakly closed, so that $F^{-1}(U)$ is closed in X. Then $F^{-1}(V)$ is the union of a countable family of closed sets, since $F^{-1}(V) = \bigcup\{F^{-1}(U);\, U \in \mathcal{F}\}$. ∎

Proposition 2.3. *Let $(A,\mathcal{T})$ be a measurable space, X and Y be topological spaces $F : T{\times}X \rightrightarrows Y$ be a multifunction with non-empty closed values and $\phi : T \to X$ a measurable function. Suppose that, for every $x \in X$, $F(\cdot,x)$ is measurable, and $\phi(T)$ is countable. Then the multifunction $t \mapsto \Psi(t) = F(t,\phi(t))$ is measurable.*

Proof: Let us order $\phi(T)$ in a sequence $(\xi_n)_n$, and take an open set $V \subseteq Y$. According to our assumptions, for every $n \in N$ the sets $A_n = \phi^{-1}(\{\xi_n\})$, $C_n = F(\cdot,\xi_n)^{-1}(V)$ are measurable in A. Now it is enough to remark that

$$\Psi^{-1}(V) = \bigcup_{n=1}^{+\infty} \Big(C_n \cap A_n\Big) \;. \blacksquare$$

Proposition 2.4. *Let $(E,\|\cdot\|)$ be a Banach space, J a compact interval of the real line, $\mathcal{C}(J;E)$ the space of all continuous functions $u\colon J \to E$, endowed with the norm $\|\cdot\|_\infty$ of uniform convergence. Let $\psi\colon J{\times}E \to E$ be a continuous function, $\mu\colon E \to \mathcal{C}(J;E)$ the map $x \mapsto \psi(\cdot,x)$. Then μ is continuous.*

Proof: Let $K \subseteq E$ be compact: thanks to Heine's theorem, ψ is uniformly continuous on $J{\times}K$. In particular, for every $\varepsilon > 0$ we can find $\delta > 0$ such that, for every $s \in J$, $x,y \in E$, with $\|x-y\| \le \delta$,

$$\Big\|\psi(s,x) - \psi(s,y)\Big\| \le \varepsilon \;.$$

Then μ is continuous on K. In particular, for any convergent sequence in E, we can take K as the set of all its points, together with its limit. Then μ is sequentially continuous: since E is a normed space, the assertion is proved. ∎

3 – Viability results without memory

In this section we deal with the case $r = 0$, that is without delay. Hence the multifunction F is now defined on $I \times E$ (or also on a subset of this space), and problem (1.1) takes the following, more simple form:

$$
\begin{aligned}
& x'(t) \in F(t, x(t)) \quad \text{a.e. on I} \quad , \\
(3.1) \qquad & x(t) \in K(t) \quad \text{for all } \ t \in I \ , \\
& x(0) = x_0 \ .
\end{aligned}
$$

On this subject, we are going to recall some well-known existence theorems, and also give new ones. From now on, $K\colon I \rightrightarrows E$ is a given set-valued map, whose graph will be denoted by Γ. According to the general definition given at the beginning of §2, we say that K is *upper semicontinuous from the left* if $K^{-1}(V)$ is closed from the left in I whenever V is closed in E. For every $z = (t, x) \in \Gamma$ we denote by $T_\Gamma(z)$ the *Bouligand contingent cone* of Γ at z (see, for instance [1]), and put

$$
Q_\Gamma(t, x) = \Big\{ y \in E \colon \ (1, y) \in T_\Gamma(t, x) \Big\} \, , \qquad (t, x) \in \Gamma \ .
$$

We denote by $\sigma_w = \sigma(E, E^*)$ the weak topology on E. From now on $\mathcal{B}(E)$ will stand for the Borel σ-field on E, with respect to the strong topology.

In order to compare more easily the results we are going to explain, we sum up all their assumptions now. Conditions (a), (b) and (c) are common to all theorems, while conditions (e) and (g) are alternative to (d) and (f) respectively. Furthermore, conditions (i) and (l) are alternative to (h).

Basic assumptions

(**a**) The multifunction $K\colon I \rightrightarrows E$ takes compact non-empty values and is upper semicontinuous from the left.

(**b**) The sets $F(t, x)$ are non-empty, convex and weakly compact.

(**c**) For every $t \in I$, $F(t, \cdot)$ is u.s.c. from $(E, \|\cdot\|)$ to (E, σ_w).

Measurability conditions on F

(**d**) For every $x \in E$ the multifunction $F(\cdot, x)$ is measurable on I.

(**e**) F is measurable with respect to $\mathcal{L} \otimes \mathcal{B}(E)$.

Bounds from the exterior on the sets $F(t,x)$

(**f**) There exist a function $a \in L^1(I)$ and a set $J \subseteq I$, with $|I \backslash J| = 0$, such that, for every $(t,x) \in \Gamma$, with $t \in J$, it is $\|F(t,x)\| \leq a(t)\,(1+\|x\|)$.

(**g**) There exists a convex weakly compact set $D \subset E$ such that, for all $(t,x) \in I \times E$, $F(t,x) \subseteq D$.

Tangential conditions

(**h**) There exists a set $J \subseteq I$, with $|I \backslash J| = 0$, such that: if $(t,x) \in \Gamma$ and $t \in J$, then $F(t,x) \cap Q_\Gamma(t,x) \neq \emptyset$; otherwise, $Q_\Gamma(t,x) \neq \emptyset$.

(**i**) For every $(t,x) \in \Gamma$ it is $Q_\Gamma(t,x) \neq \emptyset$. Furthermore, there exists a set $J' \subseteq I$, with $|I \backslash J'| = 0$, such that, for every $(t,x) \in \Gamma$, with $t \in J'$, it is $F(t,x) \subseteq Q_\Gamma(t,x)$.

(**l**) For every $(t,x) \in \Gamma$ with $t < T$ and every $\epsilon > 0$ there exists $(t_\epsilon, x_\epsilon) \in \Gamma$ such that

$$t_\epsilon \in [t, t+\epsilon] \quad \text{and} \quad \frac{x_\epsilon - x}{t_\epsilon - t} \in \frac{1}{t_\epsilon - t} \int_t^{t_\epsilon} F(s,x)\, ds + \epsilon B\,.$$

Remark. As is known [7, Thm. II.20] condition (c) is equivalent to the following one: for any $t \in I$, $p \in E^*$, the function

$$\phi \mapsto \delta^*(p; F(t,\phi)) \doteq \sup\Big\{\langle p, v\rangle;\ v \in F(t,\phi)\Big\}$$

is upper semicontinuous. Furthermore, we point out that obviously $(d) \Rightarrow (e)$, $(f) \Rightarrow (g)$, and $(i) \Rightarrow (h)$. □

The first theorem which we present is given in [11].

Theorem 3.1. *Let E be a separable Banach space, and consider two multifunctions $K\colon I \rightrightarrows E$ and $F\colon \Gamma \rightrightarrows E$, where Γ is the graph of K. Suppose that conditions (a), (b), (c), (e), (f), (h) hold: then, for every $x_0 \in K(0)$, there exists an absolutely continuous function $x\colon I \to E$ satisfying (3.1).* ■

Now we are going to deduce, from the previous theorem and Thm. 2.1, a result which holds even when $F(t,x)$ is supposed to be measurable only with respect to t. As we told at the end of the introduction, in this case particular difficulties can arise, as is shown by an example in [4]. For this reason, we shall assume that F is defined on the whole space $I \times E$. Furthermore, we also need to strengthen conditions (f) and (h).

Theorem 3.2. *Let E be a separable Banach space, $K\colon I \rightrightarrows E$ and $F\colon I\times E \rightrightarrows E$ two multifunctions satisfying conditions (a), (b), (c), (d), (g), (i). Then, for every $x_0 \in K(0)$, there exists an absolutely continuous function $x : I \to E$ satisfying (3.1).*

Proof: It is easy to check that F fulfils the assumptions of Thm. 2.1: indeed, the weak topology σ_w is metrizable on D, because E is separable. Furthermore, for every $t \in I$, the multifunction $F(t,\cdot)$ has a closed graph, since it is upper semicontinuous from E into (D,σ_w) [1, Prop. 2, p. 41]. On the other hand, for every $x \in E$, $F(\cdot,x)$ is measurable, so that it admits a measurable selection [7, Thm. III.6]. Since all sets $F_0(t,x)$ are contained in the same weakly compact set D, property (p_3) ensures that F_0 is u.s.c. from $I_\epsilon\times E$ to (E,σ_w) (see i.e. [1], Corollary 1, p. 42). Now, let $\epsilon_n \downarrow 0$, apply property (p_3) with $\epsilon = \epsilon_n$ and put $J_n = I_{\epsilon_n}$, $J = \bigcup_n J_n$, $S = N\bigcup(I\backslash J)$: of course, it is right to suppose that the sequence $(J_n)_n$ is increasing, and we also get obviously $\lambda(S) = 0$. Now, let us define a multifunction $\tilde{F}\colon\ I\times E \rightrightarrows D$ as follows:

$$\tilde{F}(t,x) \;=\; \begin{cases} F_0(t,x) & \text{if } (t,x) \in (I\backslash S)\times E, \\ D & \text{if } (t,x) \in S\times E\ . \end{cases}$$

Now we are going to check that $\tilde{F}$ fulfils the same conditions as F in Theorem 3.1, that is (b), (c), (e), (f), (h). As regards (b), we only need to remark that F takes non-empty values, thanks to the construction of $I_0 = I\backslash S$. Also condition (c) holds obviously: indeed, if $t \notin I_0$, $\tilde{F}(t,\cdot) \equiv D$; otherwise, for some $n \in \mathbb{N}$, it is $t \in J_n$, so that $\tilde{F}(t,\cdot)$ agrees with the section of a globally u.s.c. multifunction, namely the restriction of F_0 to the set $J_n\times E$. Now, let us prove (e): to this end, let $A \subseteq E$ be open, $C = \tilde{F}^{-1}(A)$, and suppose that $A \cap D \neq \emptyset$ (otherwise, $C = \emptyset$). Let us put, for every $n \in \mathbb{N}$, $C_n = C \cap (J_n\times E)$, and $C' = \bigcup_n C_n$. Then $C = C' \cup (N_0\times E)$, with $N_0 = S \cap C$, and we only need to prove that, for every $n \in \mathbb{N}$, it is $C_n \in \mathcal{L} \times \mathcal{B}(E)$. But $C_n = F_n^{-1}(A)$, where F_n is the restriction of F_0 to $W_n = J_n\times E$, and F_n is u.s.c. from $(W_n, \|\cdot\|)$ to (E,σ_w): then Prop. 2.2 ensures that $C_n \in \mathcal{L}\times\mathcal{B}(E)$, and $\tilde{F}$ actually satisfies (e). Since (f) holds obviously, we only need to prove (h): to this end, let us put $J = J' \cap I_0$, and let $(t,x) \in \Gamma$, with $t \in J'$. Now, since $t \in J'$, it is $F(t,x) \subseteq Q_\Gamma(t,x)$. On the other hand, since $t \in I_0$, we get $\emptyset \neq F_0(t,x) \subseteq F(t,x) = \tilde{F}(t,x)$. Hence $\tilde{F}(t,x) \cap Q_\Gamma(t,x) \neq \emptyset$.

For every $x_0 \in K(0)$ it is then possible to find an absolutely continuous function $x\colon I \to E$ such that conditions (3.1) hold, with $\tilde{F}$ in place of F. Thanks to property (p_2), it is easy to see that the same conditions are satisfied by F. ∎

Now we are going to state a third result, which still concerns the case in which F is measurable only with respect to t, and can be essentially deduced, through some changes in the proof, from Prop. 6.2 of [3], where the authors deal with a globally measurable multifunction F.

Theorem 3.3. *Let E be a separable Banach space, $K\colon I \rightrightarrows E$ and $F\colon I{\times}E \rightrightarrows E$ two multifunctions satisfying conditions (a), (b), (c), (d), (f), (l). Then for every $x_0 \in K(0)$, there exists an absolutely continuous function $x : I \to E$ satisfying (3.1).*

The proof of Theorem 3.3 needs the following result about approximate solutions, which is essentially given in [3], although under slightly different assumptions.

Proposition 3.4. *Let E be a separable Banach space, $K : I \rightrightarrows E$ and $F : I{\times}E \rightrightarrows E$ two multifunctions satisfying conditions (a), (b), (d), (f), (l). Then, for every $x_0 \in K(0)$ it is possible to find a constant $m > 1$ such that for any $\epsilon \in \,]0,1]$, there are an increasing, right continuous function $\theta\colon I \to I$ and an absolutely continuous function $x\colon I \to E$ with the following properties:*

- **(i)** $\theta(0) = 0,\ \ \theta(T) = T$ *and* $\theta(t) \in [t-\epsilon, t] \cap I$ *for all* $t \in I$;
- **(ii)** *for all* $t \in I$, $x(\theta(t)) \in K(\theta(t))$;
- **(iii)** $\|x'(t)\| \le m\, c(t) + 1$ λ*-a.e. on* I;
- **(iv)** $x'(t) \in F(t, x(\theta(t))) + \epsilon B$ *a.e. on* I;
- **(v)** $\theta(t)$ *takes at most a countable set of values.*

Proof: We apply Prop. 6.1 of [3], in the particular case in which μ agrees with the Lebesgue measure λ on I. We remark that in [3] F is supposed to take strongly compact values, but it is easy to check that the given proof still works when the sets $F(t,x)$ are weakly compact. Then we infer the existence of an absolutely continuous function $x\colon I \to E$ enjoying properties (i)–(iv). Now, the techniques used in [3] also show how it is right to suppose that (v) holds as well. ∎

Proof of Theorem 3.3: Given $x(0) \in K(0)$, let us apply the previous proposition with $\epsilon = \epsilon_n$, where $\epsilon_n \downarrow 0$ as $n \to +\infty$: then we find a constant $m > 1$, and functions $\theta_n : I \to I$, $x_n : I \to E$ which enjoy the same properties as θ and x in (i)–(v). As a first step we show that $(x_n)_n$ admits a subsequence which converges uniformly on I to an absolutely continuous function x. To this end let us put, for every $t \in I$, $\beta(t) = m\,a(t) + 1$, and, for every $n \in N$,

$$\xi_n = x_n \circ \theta_n\,, \qquad \sigma_n(t) = e\Big(K(\theta_n(t)), K(t)\Big)\,,$$
$$\delta_n = \sup\Big\{\int_A \beta(\tau)\,d\tau;\ A \in \mathcal{L},\ \lambda(A) \le \epsilon_n\Big\}\,,$$

where $e(C, D) \doteq \sup\{\text{dist}(x; D);\ x \in C\}$ is the *excess* of C over D. Then from conditions (i) and (iii) (in which we put $x = x_n$, $\theta = \theta_n$) we get, for every $t \in I$,

$$\|\xi_n(t) - x_n(t)\| \le \int_{\theta_n(t)}^{t} \|x_n'(\tau)\|\,d\tau \le \delta_n\,. \tag{3.2}$$

On the other hand, according to (ii) (with $\theta = \theta_n$), it is $\xi_n(t) \in K(t) + \sigma_n(t)B$, so that, by virtue of (3.2),

$$x_n(t) \in K(t) + (\sigma_n(t) + \delta_n)\,B \quad \text{for all } t \in I \text{ and } n \in N\,.$$

Now, let us fix $t \in I$; since K is u.s.c. from the left, and $\theta_n(t) \in [t - \epsilon_n, t] \cap I$ for all n, we get $\sigma_n(t) \to 0$ as $n \to +\infty$; furthermore, $\beta \in L^1(I)$, so that $\delta_n \to 0$ as well. Hence the set $X(t) = \{x_n(t) :\ n \in N\}$ is relatively strongly compact in E. As a consequence of (iii) (where we put $x = x_n$) the sequence $(x_n)_n$ is also equicontinuous. Then it is possible to extract a subsequence, again denoted $(x_n)_n$, uniformly convergent in I to an absolutely continuous function $x : I \to E$.

By virtue of (3.2), also the set $Y(t) = \{\xi_n(t) :\ n \in N\}$ is, for every $t \in I$, relatively compact in E with respect to the norm topology; furthermore, since $x = x_n$ and $\theta = \theta_n$ fulfil (iv), we get

$$x_n'(t) \in \Phi(t) + \epsilon_n B \quad \text{a.e. on } I\,,$$

where $\Phi(t)$ denotes the closed convex hull of $F(t, Y(t))$, which is a convex weakly compact subset of E. On the other hand, (iii) holds with $x = x_n$, so that the sequence $(x_n')_n$ is bounded and uniformly integrable in $L^1(I; E)$: then, according to [3], Thm. 5.4, it admits a subsequence, still denoted as $(x_n')_n$, which converges to a function $v \in L^1(I; E)$ in the $\sigma(L^1, L^\infty)$-topology. Indeed, it is easy to show that $v(t) = x'(t)$ for almost every $t \in I$.

Now we are going to prove that $x(t)$ is a solution of (3.1): since, for every n, ξ_n is a right continuous map, it is also measurable; furthermore, since $\theta = \theta_n$ fulfils (v), ξ_n takes at most a countable set of values: then Prop. 2.3 ensures that the multifunction $t \to F(t, \xi_n(t))$ is measurable. Now, for every $n \in N$, $t \in I$, let $\Psi_n(t) = (x_n'(t) + \epsilon_n B) \cap F(t, \xi_n(t))$: then $\Psi_n : I \rightrightarrows E$ is a measurable multifunction with non-empty closed values, so that it admits a measurable selection

z_n (see e.g. [7], Thm. III.6). Since $\|z_n(t) - x'_n(t)\| \leq \epsilon_n$ for every $n \in \mathbb{N}$, $t \in I$, and $x'_n \rightharpoonup x'$, we easily argue that $z_n \rightharpoonup x'$ in the $\sigma(L^1, L^\infty)$-topology; moreover

$$z_n(t) \in F(t, \xi_n(t)) \quad \text{for every} \quad t \in I \ .$$

By (3.2) the sequence $\xi_n \to x$ uniformly in I. Therefore, according to [7], Thm. VI.4, we obtain

$$x'(t) \in F(t, x(t)) \quad \text{a.e. in} \quad I \ .$$

Finally, let us recall (ii) (with $x = x_n$, $\theta = \theta_n$), so as to get, for every $n \in \mathbb{N}$, $t \in I$, $\xi_n(t) \in K(\theta_n(t))$: as $n \to +\infty$, $\xi_n(t) \to x(t)$, while $\theta_n(t) \to t$ from the left. Since K is u.s.c. from the left, we get $x(t) \in K(t)$ for every $t \in I$. ∎

4 – Viability results with memory

In the sequel we are going to show the existence of viable solutions for the functional differential inclusion (1.1). First of all, we sum up all the assumptions which are used in such theorems. They all correspond, in a natural way, to the assumptions we gave at the beginning of §3, and indeed they are pointed out by the same letters, but with an apex. We introduce the following set:

$$\Lambda = \Big\{(t, w) \in I \times \mathcal{C}_r \colon \ w(0) \in K(t)\Big\} \ . \tag{4.1}$$

Basic assumptions

(**a**$'$) the multifunction $K\colon I \rightrightarrows E$ takes compact non-empty values and is upper semicontinuous from the left.

(**b**$'$) the sets $F(t, w)$ are non-empty, convex and weakly compact.

(**c**$'$) for every $t \in I$, $F(t, \cdot)$ is u.s.c. from $(\mathcal{C}_r, \|\cdot\|_\infty)$ to (E, σ_w).

Measurability conditions on F

(**d**$'$) For every $w \in \mathcal{C}_r$ the multifunction $F(\cdot, w)$ is measurable on I.

(**e**$'$) F is measurable with respect to $\mathcal{L} \otimes \mathcal{B}(\mathcal{C}_r)$.

Bounds from the exterior on the sets $F(t, w)$

(**f**$'$) there exist a function $a \in L^1(I)$ and a set $J \subseteq I$, with $|I \backslash J| = 0$, such that, for every pair $(t, w) \in \Lambda \cap (J \times E)$ it is $\|F(t, w)\| \leq a(t)\,(1 + \|w(0)\|)$.

($\mathbf{g}'$) there exists a convex weakly compact set $D \subset E$ such that, for every pair $(t,w) \in \Lambda$ it is $F(t,w) \subseteq D$.

Tangential conditions

($\mathbf{h}'$) there exists a set $J \subseteq I$, with $|I \backslash J| = 0$, such that: if $(t,w) \in \Lambda$ and $t \in J$, then $F(t,w) \cap Q_\Gamma(t,w(0)) \neq \emptyset$; otherwise, $Q_\Gamma(t,x) \neq \emptyset$;

($\mathbf{i}'$) for every $(t,x) \in \Gamma$ it is $Q_\Gamma(t,x) \neq \emptyset$. Furthermore, there exists a set $J' \subseteq I$, with $|I \backslash J'| = 0$, such that, for every $(t,w) \in \Gamma$, with $t \in J'$, it is $F(t,w) \subseteq Q_\Gamma(t,w(0))$.

($\mathbf{l}'$) for every $(t,w) \in \Lambda$ with $t < T$ and every $\epsilon > 0$ there exists $(t_\epsilon, x_\epsilon) \in \Gamma$ such that

$$t_\epsilon \in [t, t+\epsilon] \quad \text{and} \quad \frac{x_\epsilon - w(0)}{t_\epsilon - t} \in \frac{1}{t_\epsilon - t}\int_t^{t_\epsilon} F(s,w)\,ds + \epsilon B\ ;$$

Theorem 4.1. *Let E be a separable Banach space, and consider two multifunctions $K\colon I \rightrightarrows E$ and $F : \Lambda \rightrightarrows E$, where Λ is given by (4.1). Suppose that conditions (a′), (b′), (c′), (e′), (f′), (h′) hold: then for every $\phi \in \mathcal{C}_r$ with $\phi(0) \in K(0)$, there exists an absolutely continuous function $x\colon I \to E$ satisfying (1.1).*

Theorem 4.2. *Let E be a separable Banach space, $K : I \rightrightarrows E$ and $F : I \times \mathcal{C}_r \rightrightarrows E$ two multifunctions satisfying conditions (a′), (b′), (c′), (d′), (g′), (i′). Then, for every $\phi \in \mathcal{C}_r$ with $\phi(0) \in K(0)$, there exists an absolutely continuous function $x\colon I \to E$ satisfying (1.1).*

Theorem 4.3. *Let E be a separable Banach space, $K : I \rightrightarrows E$ and $F : I \times \mathcal{C}_r \rightrightarrows E$ two multifunctions satisfying conditions (a′), (b′), (c′), (d′), (f′), (l′). Then for every $\phi \in \mathcal{C}_r$ with $\phi(0) \in K(0)$, there exists an absolutely continuous function $x\colon I \to E$ satisfying (1.1).*

In order to prove the previous theorems we need the following result.

Lemma 4.4. *Let $K\colon I \rightrightarrows E$ and $F\colon I \times \mathcal{C}_r \rightrightarrows E$ be two given multifunctions, $\psi\colon [-r,0] \times E \to E$ be a continuous function such that, for every $x \in E$, $\psi(0,x) = x$. For every $t \in I$, $x \in K(t)$, let us put $G(t,x) = F(t, \psi(\cdot, x))$. Suppose that K and F fulfil one of the conditions from (b′) to (l′) at the beginning of this section: then K and G fulfil the corresponding condition at the beginning of §3.*

Proof: We only show that (c′)⇒(c) and (e′)⇒(e), since the other implications can be checked directly. So, let us suppose that F fulfils (c′), and define μ as in Prop. 2.4. Let $t \in I$, V a weakly closed subset of E: then $C = F(t,\cdot)^{-1}(V)$ is closed in E, so that Prop. 2.4 ensures that $\mu^{-1}(C) = G^{-1}(V)$ is closed in $\mathcal{C}_r$, since μ is continuous: then G satisfies condition (c). Now, let us suppose that (e′) holds, and take an open set $V \subseteq E$: then $S \doteq F^{-1}(V) \in \mathcal{L}\otimes\mathcal{B}(\mathcal{C}_r)$. On the other hand, again thanks to the continuity of μ, the map $\nu\colon (t,x) \mapsto (t,\mu(x))$ is certainly measurable from $(I\times E,\, \mathcal{L}\otimes\mathcal{B}(E))$ to $(I\times\mathcal{C}_r,\, \mathcal{L}\otimes\mathcal{B}(\mathcal{C}_r))$: hence $G^{-1}(V) = \nu^{-1}(S) \in \mathcal{L}\times\mathcal{B}(E)$, and G fulfils (e). ∎

Proof of Theorems 4.1, 4.2 and 4.3: Let $n \in \mathbb{Z}^+$: for every $i \in \{1,...,n\}$ we put $t_n^i = \frac{i}{n}T$, $J_n^i = [t_n^{i-1}, t_n^i]$. By induction on the index i, we are going to define the maps $\psi_n^i\colon [-r,0]\times E \to E$, the multifunctions $G_n^i\colon I\times E \to E$, and the functions $x_n^i\colon [-r,t_n^i] \to E$. If $i=1$, we put

$$\psi_n^1(s,x) \;=\; \begin{cases} \phi\Big(s+\dfrac{T}{n}\Big) & \text{if } -r \le s \le -\dfrac{T}{n}\,, \\[2ex] \phi(0) + \Big(1+\dfrac{ns}{T}\Big)\Big(x-\phi(0)\Big) & \text{if } -\dfrac{T}{n} \le s \le 0\,. \end{cases}$$

Then, for every $t \in I$, $x \in E$, we put $G_n^1(t,x) = F(t,\psi_n^1(\cdot,x))$. We remark that ψ_n^1 is continuous on $[-r,0]\times E$, and that $\psi_n^1(0,x) = x$, hence Lemma 4.4 and Thm. 3.1 or Thm. 3.2 or Thm. 3.3 ensure the existence of a solution $\xi_n^1\colon [0,T] \to E$ of the differential inclusion $\xi'(t) \in G_n^1(t,\xi(t))$ such that $\xi_n^1(0) = \phi(0)$, and $\xi_n^1(t) \in K(t)$ for every $t \in [0,T]$. Let

$$x_n^1(t) = \begin{cases} \phi(t) & \text{if } -r \le t \le 0\,, \\[1ex] \xi_n^1(t) & \text{if } t \in J_n^1\,. \end{cases}$$

Now, let us suppose that ψ_n^k, G_n^k and x_n^k are already defined for $k = 1,...,i-1$. Then we put

$$\psi_n^i(s,x) = \begin{cases} x_n^{i-1}(s+t_n^i) & \text{if } -r \le s \le -\dfrac{T}{n}\,, \\[2ex] x_n^{i-1}(t_n^{i-1}) + \Big(1+\dfrac{ns}{T}\Big)\Big(x - x_n^{i-1}(t_n^{i-1})\Big) & \text{if } -\dfrac{T}{n} \le s \le 0\,, \end{cases}$$

and for every $t \in I$, $x \in E$, $G_n^i(t,x) = F(t,\psi_n^i(\cdot,x))$. By the same argument as in the case $i=1$, we find a viable solution $\xi_n^i\colon [t_n^{i-1},T] \to E$ of the differential inclusion $\xi'(t) \in G_n^i(t,\xi(t))$ such that $\xi_n^i(t_n^{i-1}) = x_n^{i-1}(t_n^{i-1})$ and $\xi_n^i(t) \in K(t)$ for all $t \in [t_n^{i-1}, T]$. Let us put

$$x_n^i(t) = \begin{cases} x_n^{i-1}(t) & \text{if } -r \le t \le t_n^{i-1} \,, \\ \xi_n^i(t) & \text{if } t \in J_n^i \,. \end{cases}$$

Now we can suppose that the function $x_n = x_n^n$ is defined on the whole interval $[-r, T]$. We notice that, by construction, $x_n \equiv x_n^i$ on $[-r, t_n^i]$, and

$$x_n'(t) \in \Phi_n(t) = F(t, \lambda_n(\cdot, t)) \,,$$

where we put $\lambda_n(s,t) = \psi_n^i(s, x_n(t))$ whenever $t \in J_n^i$. In particular, $\lambda_n(0,t) = x_n(t)$, so that, by virtue of condition (e′) or (e′$_1$) we get $\|\Phi_n(t)\| \le a(t)\,(1 + \|x_n(t)\|)$ a.e. on I (in particular, when (e′$_1$) holds, we can take $a(t) \equiv \|D\|$). Since $\|x_n'(t)\| \le \|\Phi_n(t)\|$ and $x_n(0) = \phi(0)$ for every n, Gronwall's Lemma ensures the existence of a positive constant $M = (1 + \|\phi(0)\|)\exp(\int_I a(\tau)\,d\tau)$ such that

$$\text{(4.2)} \qquad \|x_n\|_\infty \le M, \quad \|x_n'(t)\| \le M a(t) \quad \text{a.e.}, \qquad n \in \mathbb{Z}^+ \,.$$

In particular, the functions x_n are equibounded and equicontinuous on I. Since, for every $t \in I$, the points $x_n(t)$ lie in the same compact set $K(t)$, thanks to Ascoli–Arzelà theorem it is right to suppose that the sequence $(x_n)_n$ converges uniformly on I to a continuous function u (indeed, this happens on a suitable subsequence). Since all functions x_n agree with ϕ on $[-r, 0]$, we can obviously say that $x_n \to u$ on $[-r, T]$, if we extend u in such a way that $u \equiv \phi$ on $[-r, 0]$. Furthermore, it is easy to see that u is absolutely continuous on I. Now we are going to show that, for every $t \in I$, the functions $\lambda_n(\cdot, t)$ converge uniformly on $[-r, 0]$, as $n \to +\infty$, to the function $s \mapsto u(t+s)$. To this end, we put

$$\omega(\rho) = \sup\Big\{|x_n(t) - x_n(\tau)|;\ n \in \mathbb{Z}^+,\ |t - \tau| \le \rho\Big\}, \qquad \rho > 0 \,,$$

$$\delta_n = \|x_n - u\|_\infty, \quad r_n = 2\,\omega(T/n) + \delta_n \,, \qquad n \in \mathbb{Z}^+ \,.$$

We already know that $\delta_n \to 0$ as $n \to +\infty$. Furthermore, since the functions x_n are equicontinuous, we argue that $\omega(\rho) \to 0$ as $\rho \to 0$, so that $r_n \to 0$ too, as $n \to +\infty$. Let $n \in \mathbb{Z}^+$, $t \in I$ be fixed, and let $i \in \{1, ..., n\}$ be such that $t \in J_n^i$: then $\lambda_n(s,t) = \psi_n^i(s, x_n(t))$. If n is large enough, we may suppose that $T/n < r$ and consider two cases, according to whether it is $-r \le s \le -T/n$ or $-T/n \le s \le 0$. Now, in the first case, let us combine the equality $\lambda_n(s,t) = x_n(s + t_n^i)$ with the following relations

$$\text{(4.3)} \qquad \Big\|x_n(s + t_n^i) - x_n(t+s)\Big\| \le \omega(T/n) \,, \qquad \Big\|x_n(t+s) - u(t+s)\Big\| \le \delta_n \,.$$

Then we get $\|\lambda_n(s,t) - u(t+s)\| \leq \omega(T/n) + \delta_n \leq r_n$. In the second case, since $\psi_n^i(s,x)$ is linear with respect to s, we can exploit the inequality $\|\psi_n^i(s,x) - x\| \leq \|x_n(t_n^{i-1}) - x\|$, with $x = x_n(t)$, so as to get

$$\|\lambda_n(s,t) - x_n(t_n^{i-1})\| \;\leq\; \|x_n(t) - x_n(t_n^{i-1})\| \;\leq\; \omega(T/n)\;. \tag{4.4}$$

Now, if we combine (4.4) with the inequality $\|x_n(t_n^{i-1}) - x_n(t+s)\| \leq \omega(T/n)$ and with the second inequality in (4.3), we get $\|\lambda_n(s,t) - u(t+s)\| \leq 2\,\omega(T/n) + \delta_n \leq r_n$. Since $r_n \to 0$ as $n \to +\infty$, we get indeed that $\lambda_n(\cdot,t)$ converge uniformly on $[-r,0]$ to the function $s \mapsto u(t+s)$ (that is u_t), as claimed. In particular, for every $t \in I$, the set $W(t)$ made up by the functions $\lambda_n(\cdot,t)$ is relatively compact in $\mathcal{C}_r$. Then, thanks to condition (c$'$) and general properties of u.s.c. multifunctions [1, Prop. 3, p. 42], the set $H(t) = F(t, \mathrm{cl}\, W(t))$ is weakly compact in E. On the other hand, $x_n'(t) \in H(t)$ almost everywhere, so that from (4.2) and Thm. 5.4 of [3] we get that the sequence $(x_n')_n$ is relatively weakly compact in $L^1(I;E)$: then we can suppose that it converges to an integrable function v (up to a subsequence, as usual). It is easy to see that actually $v = u'$: then we can apply Theorem VI.4 of [7], and argue that $u'(t) \in F(t, u_t)$ almost everywhere in $[0,T]$. Finally, since the sets $K(t)$ are closed, and it is $x_n(t) \in K(t)$ for every $n \in \mathbb{Z}^+$, $t \in I$, we get obviously $u(t) \in K(t)$ for every $t \in I$. ∎

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