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NONLINEAR FILTERING WITH AN INFINITE DIMENSIONAL SIGNAL PROCESS

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Abstract: In this paper, we investigate a nonlinear filtering problem with correlated noises, bounded coefficients and a signal process evolving in an infinite dimensional space. We derive the Kushner–Stratonovich and the Zakai equation for the associated filter respectively unnormalized filter. A robust form of the Zakai equation is established for an uncorrelated filtering problem.

0 – Introduction

Consider a diffusion process $X = \{X^i_t,\ i \in \mathbb{Z};\ t \in [0,T]\}$ solution of the infinite dimensional system of stochastic equations

$$dX^i_t = \sum_{j\in\mathbb{Z}} \sigma^i_j(t,X_t)\,dw^j_t + b^i(t,X_t)\,dt\ , \tag{0.1}$$

where $\{w^j_t,\ j \in \mathbb{Z};\ t \in [0,T]\}$ is an infinite dimensional Brownian motion with variance $\gamma_j\,t$, γ_j being positive real numbers such that $\sum_{j\in\mathbb{Z}} \gamma_j < +\infty$. These equations have been considered by several authors (see e.g. [6], [23], [14], [19]). They are related with certain continuous state Ising-type models in statistical mechanics, as for instance the "plane rotor model", and also with models arising in population genetics.

In this paper we take a diffusion of a similar type as signal process of a nonlinear filtering problem. Our aim is to prove that the unnormalized filter associated with a nonlinear filtering problem with correlated noises, bounded coefficients and a signal process evolving in an infinite dimensional space, solves a Zakai equation. Moreover a Kushner–Stratonovich equation for the filter is deduced by usual arguments form the Kallianpur–Striebel formula and a robust form of the Zakai equation is established in the case of independent noises.

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This problem has already been investigated when the signal process is finite dimensional by many authors. It has been known for a long time in nonlinear filtering theory that the filter solves the Kushner–Stratonovich equation (see e.g. [13] or [7]) but the nonlinearity of that equation prevents any progress in the study of the properties of its solution. Later, M. Zakai [24] has showed that the unnormalized filter associated with an uncorrelated nonlinear filtering problem is, if it exists, solution of a stochastic partial differential equation of parabolical type. Then M. Davis [4], M. Davis and S.I. Marcus [5] and E. Pardoux [20] have extended Zakai's method to the case of nonlinear filtering problems with correlated noises.

In [11], P. Florchinger has proved that the unnormalized filter associated with a nonlinear filtering problem with independent noises and bounded coefficients solves a Zakai equation, even if the signal process is infinite dimensional.

The robust form of the Zakai equation has been introduced by J. Clark [3] to define a "robust" filter associated with a non correlated filtering system with bounded observation coefficients. The idea is to reduce the Zakai equation to a deterministic differential equation with random coefficients, by means of a multiplicative transformation. W. Hopkins [15] then established, by means of an analogous method, a robust form for the Zakai equation for an uncorrelated nonlinear filtering system with unbounded observation coefficients.

This paper is divided in five sections organized as follows. In the first section, we introduce the nonlinear filtering problem studied in this paper and we show that it has a unique strong solution with a.s. continuous paths. In section two, we define an unnormalized filter linked with the filter defined in the previous section by means of a Kallianpur–Striebel formula. In the third and fourth sections we derive the Zakai and Kushner–Stratonovich equations associated with our nonlinear filtering problem. The fifth section, finally, is devoted to the proof of a robust form for the Zakai equation under the hypothesis that the noises are independent.

1 – Setting of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\gamma = \{\gamma_i,\ i \in \mathbb{Z}\}$ a summable sequence of strictly positive real numbers. Set

$$L^2(\gamma) = \left\{ x = (x_i) \in \mathbb{R}^{\mathbb{Z}} :\ \|x\|_\gamma^2 = \sum_{i \in \mathbb{Z}} \gamma_i \, |x_i|^2 < +\infty \right\} ,$$

$$L^2(\gamma\times\gamma) = \Big\{x = (x^i_j) \in \mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}\colon\ \|x\|^2_{\gamma\times\gamma} = \sum_{i,j\in\mathbb{Z}} \gamma_i\,\gamma_j\,|x^i_j|^2 < +\infty\Big\}$$

and

$$L^2(\gamma\times p) = \Big\{x = (x^i_k) \in \mathbb{R}^{\mathbb{Z}\times p}\colon\ \|x\|^2_{\gamma\times p} = \sum_{i\in\mathbb{Z}}\sum_{k=1}^{p} \gamma_i\,|x^i_k|^2 < +\infty\Big\}\ .$$

On the other hand, consider the nonlinear filtering problem associated with the signal-observation pair $(x_t, y_t) \in L^2(\gamma)\times\mathbb{R}^p$ solution of the stochastic differential system

$$(1.1)\quad \begin{cases} x^i_t = x^i_0 + \displaystyle\int_0^t b_i(s,x_s)\,ds + \int_0^t \sum_{j\in\mathbb{Z}} \sigma^i_j(s,x_s)\,dw^j_s + \sum_{k=1}^{p} g^i_k(s,x_s)\,dv^k_s\,, \qquad i\in\mathbb{Z}\ , \\ y_t = \displaystyle\int_0^t h(s,x_s)\,ds + v_t\ , \end{cases}$$

where

1) $W = \{w^i_t,\ t\in[0,T],\ i\in\mathbb{Z}\}$ is a family of independent Brownian motions with variances $\gamma_i\,t$.

2) $V = \{v_t,\ t\in[0,T]\}$ is a p-dimensional standard Brownian motion independent of W.

3) x_0 is an $L^2(\gamma)$-valued random variable independent of W and V.

4) The maps $b\colon [0,T]\times\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$, $\sigma\colon [0,T]\times\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}$ and $g\colon [0,T]\times\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}\times p}$, as well as their partial derivatives, are uniformly bounded. In particular there exists a strictly positive constant K for which

$$(H_1)\qquad \begin{aligned} &\forall\,t\in[0,T],\quad \forall\,x\in L^2(\gamma),\\ &\|b(t,x)\|^2_\gamma + \|\sigma(t,x)\|^2_{\gamma\times\gamma} + \|g(t,x)\|^2_{\gamma\times p} \le K\,(1+\|x\|^2_\gamma) \end{aligned}$$

and

$$(H_2)\qquad \begin{aligned} &\forall\,t\in[0,T],\quad \forall\,x,y\in L^2(\gamma),\\ &\Big\|b(t,x)-b(t,y)\Big\|^2_\gamma + \Big\|\sigma(t,x)-\sigma(t,y)\Big\|^2_{\gamma\times\gamma} + \Big\|g(t,x)-g(t,y)\Big\|^2_{\gamma\times p} \le \\ &\hspace{20em}\le\ K\,\|x-y\|^2_\gamma\ . \end{aligned}$$

5) $h\colon [0,T]\times\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^p$ is a bounded Lipschitz function.

Then, the stochastic differential system (1.1) is well defined for the stochastic processes x in $L^2(\Omega\times[0,T];L^2(\gamma))$ and y in $L^2(\Omega\times[0,T];\mathbb{R}^p)$. Moreover, if $u=\{u^i_t,\ t\in[0,T],\ i\in\mathbb{Z}\}$ is an $L^2(\gamma)$-valued square integrable and adapted process, we have (cf. [14]):

$$E\left[\left(\int_0^t\sum_{i\in\mathbb{Z}}u^i_s\,dw^i_s\right)^2\right]=E\left(\int_0^t\sum_{i\in\mathbb{Z}}\gamma_i\,|u^i_s|^2\,ds\right)\,, \tag{1.2}$$

which allows us to prove the following existence and unicity theorem:

Theorem 1.1. *For any $L^2(\gamma)$-valued random variable x_0, independent of the Brownian motion $\{W_t,\ t\in[0,T]\}$, the stochastic differential system (1.1) has a unique continuous $L^2(\gamma)$-valued strong solution $\{x_t,\ t\in[0,T]\}$ such that $E(\sup_{0\le t\le T}\|x_t\|^2_\gamma)<+\infty$.*

Proof: We use Picard's iteration method to construct an approximation of the solution of (1.1). Set

$$\begin{cases} x^{(0)}_t=x_0\ , \\ x^{i(n+1)}_t=x^i_0+\displaystyle\int_0^t b_i(s,x^{(n)}_s)\,ds+\int_0^t\sum_{j\in\mathbb{Z}}\sigma^i_j(s,x^{(n)}_s)\,dw^j_s \\ \qquad\qquad +\displaystyle\int_0^t\sum_{k=1}^p g^i_k(s,x^{(n)}_s)\,dv^k_s, \qquad \text{for each}\ \ n\ge 0\ . \end{cases} \tag{1.3}$$

At first, we show by induction on n, that $E\{\sup_{t\in[0,T]}\|x^{(n)}_t\|^2_\gamma\}<+\infty$, to ensure that (1.3) makes sense.

$$\begin{aligned} E\left\{\sup_{t\in[0,T]}\|x^{(n+1)}_t\|^2_\gamma\right\}&=E\left\{\sup_{s\in[0,T]}\sum_{i\in\mathbb{Z}}\gamma_i\left(\left|x^i_0+\int_0^t b_i(s,x^{(n)}_s)\,ds\right.\right.\right.\\ &\quad\left.\left.\left.+\int_0^t\sum_{j\in\mathbb{Z}}\sigma^i_j(s,x^{(n)}_s)\,dw^j_s+\int_0^t\sum_{k=1}^p g^i_k(s,x^{(n)}_s)\,dv^k_s\right|^2\right)\right\}\ . \end{aligned}$$

The relation (1.2), as well as the Minkowski and Burkholder inequalities imply that the latter quantity is smaller than

$$\begin{aligned} C\,E\Bigg(\sum_{i\in\mathbb{Z}}\bigg\{\gamma_i\,|x^i_0|^2&+\int_0^T\gamma_i\,|b_i(t,x^{(n)}_t)|^2\,dt+\int_0^T\sum_{j\in\mathbb{Z}}\gamma_i\,\gamma_j\,|\sigma^i_j(t,x^{(n)}_t)|^2\,dt+\\ &+\int_0^T\sum_{k=1}^p\gamma_i\,|g^i_k(t,x^{(n)}_t)|^2\,dt\bigg\}\Bigg)\le \end{aligned}$$

$$\leq C\Big\{E\,\|x_0\|_\gamma^2 + K\,E\int_0^T \Big(1+\|x_t^{(n)}\|_\gamma^2\Big)\,dt\Big\}\ ,$$

due to condition (H_1),

$$\leq C_1 + C_2\,E\Big\{\sup_{t\in[0,T]}\|x_t^{(n)}\|_\gamma^2\Big\} \leq \ldots \leq C\,E\Big\{\sup_{t\in[0,T]}\|x_t^{(0)}\|_\gamma^2\Big\} < +\infty\ .$$

On the other hand, the same arguments and condition (H_2) show that

$$E\Big\{\sup_{t\in[0,T]}\|x_t^{(n+1)} - x_t^{(n)}\|_\gamma^2\Big\} \leq C\,E\int_0^T \|x_t^{(n)} - x_t^{(n-1)}\|_\gamma^2\,dt\ . \tag{1.4}$$

Consequently, the sequence $(x^{(n)})_{n\geq 0}$ is a Cauchy sequence in the complete space $L^2(\Omega\times[0,T];\,L^2(\gamma))$, so it possesses a limit, which ensures the existence of an a.s. continuous process $x=\{x_t,\ t\in[0,T]\}$ such that

$$E\Big\{\sup_{t\in[0,T]}\|x_t^{(n)} - x_t\|_\gamma^2\Big\} \to 0, \qquad \text{when}\ \ n\to+\infty\ .$$

Furthermore, it is easy to check that x verifies (1.1).

Now, let us prove the uniqueness of the solution.

If $\widetilde{x}=\{\widetilde{x}_t,\ t\in[0,T]\}$ denotes another solution of (1.1), the same arguments that we have used to prove (1.4) imply that

$$E\Big\{\sup_{t\in[0,T]}\|x_t - \widetilde{x}_t\|_\gamma^2\Big\} \leq C\int_0^T E\Big[\sup_{0\leq s\leq t}\|x_s - \widetilde{x}_s\|_\gamma^2\Big]\,dt\ .$$

Therefore, the uniqueness of the solution follows from Gronwall's lemma.

Moreover, since $x_t^{(n)}$ tends almost surely to x_t, we get by Fatou's lemma that $E\{\sup_{t\in[0,T]}\|x_t\|_\gamma^2\} < +\infty$. ∎

To determine the infinitesimal generator associated with the stochastic process $\{x_t,\ t\in[0,T]\}$, we denote for all $t\in[0,T]$, $x\in L^2(\gamma)$, $i,j\in\mathbb{Z}$ and $k=1,\ldots,p$ the matrices $a(t,x)$ and $\alpha(t,x)$ in $\mathcal{M}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{R})$ and $\mathcal{M}_{\mathbb{Z}\times p}(\mathbb{R})$ respectively, defined by

$$a_j^i(t,x) = \sum_{l\in\mathbb{Z}} \gamma_l\,\sigma_l^i(t,x)\,\sigma_l^j(t,x)$$

(1.5) and

$$\alpha_k^i(t,x) = \sum_{l=1}^p g_l^i(t,x)\,g_l^k(t,x)\ .$$

Notice that the matrix $a(t,x)$ exists, since the map σ is uniformly bounded and γ is a summable sequence. Furthermore, under the condition (H_2), the matrices $a(t,x)$ et $\alpha(t,x)$ satisfy the following property:

For every strictly positive constant C, there exists a strictly positive constant K_C such that, for any $t \in [0,T]$,

$$\Big\| a(t,x) - a(t,y) \Big\|^2_{\gamma\times\gamma} + \Big\| \alpha(t,x) - \alpha(t,y) \Big\|^2_{\gamma\times p} \leq K_C \,\|x-y\|^2_\gamma \,, \tag{1.6}$$

for all $x,y \in L^2(\gamma)$ such that $\|x\|^2_\gamma$ and $\|y\|^2_\gamma$ are less than C.

Notation 1.2. Denote by $\mathfrak{D}^2$ the set of functions $f:\ [0,T]\times\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ such that there exists $M \in \mathbb{N}^*$, a subset $\{i_1,\dots,i_M\} \subset \mathbb{Z}$ and a $\mathcal{C}^{1,2}_0$-function $\overline{f}\colon [0,T]\times\mathbb{R}^M \to \mathbb{R}$ such that $f(t,x) = \overline{f}(t,x_{i_1},\dots,x_{i_M})$ for every $t \in [0,T]$ and $x \in \mathbb{R}^{\mathbb{Z}}$.

Hence, we have:

Proposition 1.3 (cf. [14]). *The process $\{x_t,\ t \in [0,T]\}$, solution of the stochastic differential system (1.1) is a Markov diffusion process whose infinitesimal generator L is defined for every function f in $\mathfrak{D}^2$ by*

$$\begin{aligned} L\,f(t,x) = \frac{\partial}{\partial t} f(t,x) &+ \sum_{i\in\mathbb{Z}} b_i(t,x)\,\nabla_i f(t,x) + \frac{1}{2}\sum_{i,j\in\mathbb{Z}} a^i_j(t,x)\,\nabla_{i,j} f(t,x) \\ &+ \frac{1}{2}\sum_{i\in\mathbb{Z}}\sum_{k=1}^{p} \alpha^i_k(t,x)\,\nabla_{i,k} f(t,x)\ . \end{aligned} \tag{1.7}$$

Then, as usually in nonlinear filtering theory we define the filter associated with (1.1) by

Definition 1.4. For all $t \in [0,T]$, denote by Π_t the filter associated with the stochastic differential system (1.1), defined for every function ψ in $\mathfrak{D}^2$ by

$$\Pi_t(\psi) = E\Big[\psi_t(t,x_t)/\mathcal{Y}_t\Big]\ , \tag{1.8}$$

where $\mathcal{Y}_t = \sigma(y_s \,/\, 0 \leq s \leq t)$.

Remark. We could have defined the filter for a larger class of functions, but we have restricted it here to functions in $\mathfrak{D}^2$, because the Zakai and Kushner–Stratonovich equations are only valid for such functions.

2 – The reference probability measure

In order to define an unnormalized filter, we make use of the "reference probability measure" method to transform the stochastic process $\{y_t,\ t \in [0,T]\}$ into a standard Wiener process. With this aim set

Definition 2.1. For all t in $[0,T]$, denote by Z_t the Girsanov exponential defined by

$$Z_t = \exp\left(\int_0^t \sum_{k=1}^p h_k(s,x_s)\, dv_s^k + \frac{1}{2}\int_0^t |h(s,x_s)|^2\, ds\right) . \tag{2.1}$$

Since $(Z_t)_{t\in[0,T]}$ is a martingale we can introduce the reference probability measure as follows.

Definition 2.2. Denote by $\overline{P}$ the reference probability measure, defined on the space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ by the Radon–Nikodym derivative

$$\frac{d\overline{P}}{dP}_{/\,\mathcal{F}_t} = Z_t^{-1} . \tag{2.2}$$

Then, by Girsanov's theorem, the process $\{y_t,\ t \in [0,T]\}$ is a standard Brownian motion on the space $(\Omega, \mathcal{F}, \mathcal{F}_t, \overline{P})$ independent of the Wiener process W.

Hence we can define the unnormalized filter associated with the system (1.1) in the following way

Definition 2.3. For all t in $[0,T]$, denote by ρ_t the unnormalized filter associated with the system (1.1), defined for every function ψ in $\mathfrak{D}^2$ by

$$\rho_t\,\psi = \overline{E}\Big[\psi(t,x_t)\, Z_t \,/\, \mathcal{Y}_t\Big] , \tag{2.3}$$

where $\overline{E}$ denotes the expectation under the probability $\overline{P}$.

Furthermore, we have the following Kallianpur–Striebel formula which links the filter and the unnormalized filter

Proposition 2.4 (cf. [16] or [21]). *For all t in $[0,T]$ and every function ψ in $\mathfrak{D}^2$, we have*

$$\Pi_t\,\psi = \frac{\rho_t\,\psi}{\rho_t\,1} . \tag{2.4}$$

Remark. If the signal process, driven by an infinite dimensional Brownian motion, is itself finite dimensional, then P. Florchinger and J. Schiltz have shown [12] that, under a local Hörmander condition the unnormalized filter has a smooth density with respect to the Lebesgue measure. The essential tool of the proof is the Malliavin Calculus for finite dimensional processes driven by an infinite dimensional Brownian motion (cf. [22]). This result has not yet been extended to the case of infinite dimensional processes, but it should be possible to do so, using the results of P. Cattiaux, S. Roelly and H. Zessin ([2]) who established a one-to-one correspondence between the laws of smooth infinite dimensional Brownian diffusions and the Gibbs states on the path space $\Omega = \mathcal{C}(0,1)^{\mathbb{Z}^d}$ and give an infinite dimensional version of the Malliavin calculus integration by parts formula.

The same problem has already been investigated when the noise appearing in the system process is finite dimensional by several authors. D. Michel [18] and J.M. Bismut and D. Michel [1] have solved this problem in the case of systems with dependent noises and bounded coefficients. The case of independent noises and unbounded observation coefficients has been handled by G.S. Ferreyra [8] whereas P. Florchinger [9] has treated the case of dependent noises.

3 – The Zakai equation

In this section, we show that the unnormalized filter ρ_t defined by (2.3) solves a Zakai equation associated to the system (1.1).

For this, we need the following stochastic Fubini theorem:

Lemma 3.1 (cf. [14]). *If U_t is a stochastic process such that $\overline{E}\int_0^t U_s^2\,ds < +\infty$, then*

$$\overline{E}\Big[\int_0^t \sum_{i\in\mathbb{Z}} U_s^i\,dw_s^i\,/\,\mathcal{Y}_t\Big] = 0\ , \tag{3.1}$$

and

$$\overline{E}\Big[\int_0^t \sum_{k=1}^p U_s^k\,dy_s^k\,/\,\mathcal{Y}_t\Big] = \int_0^t \sum_{k=1}^p \overline{E}\Big[U_s^k\,/\,\mathcal{Y}_t\Big]\,dy_s^k\ . \tag{3.2}$$

Theorem 3.2. *For every function ψ in $\mathfrak{D}^2$, the unnormalized filter ρ_t is a solution of the stochastic partial differential equation*

$$\rho_t(\psi) = \rho_0(\psi) + \int_0^t \rho_s(L\psi)\, ds + \int_0^t \sum_{k=1}^{p} \rho_s(L_k\psi)\, dy_s^k \ , \tag{3.3}$$

where L_k is the first order differential operator defined for any function ψ in $\mathfrak{D}^2$ by

$$L_k\psi(t,x) = \sum_{i\in\mathbb{Z}} g_k^i(t,x)\, \nabla_i\psi(t,x) + h_k(t,x)\, \psi(t,x) \ . \tag{3.4}$$

Remark. In [17] N.V. Krylov gives sufficient conditions to ensure the uniqueness of the solution of such an equation.

Proof: By Itô's formula,

$$\begin{aligned} d\psi(t,x_t) &= L\psi(t,x_t)\, dt + \sum_{i,j\in\mathbb{Z}} \sigma_j^i(t,x_t)\, \nabla_i\psi(t,x_t)\, dw_t^j \\ &\quad + \sum_{i\in\mathbb{Z}} \sum_{k=1}^{p} g_k^i(t,x_t)\, \nabla_i\psi(t,x_t)\, dv_t^k \end{aligned}$$

and

$$\begin{aligned} dZ_t &= \sum_{k=1}^{p} Z_t\, h_k^2(t,x_t)\, dt + \sum_{k=1}^{p} Z_t\, h_k(t,x_t)\, dv_t^k \\ &= \sum_{k=1}^{p} h_k(t,x_t)\, Z_t\, dy_t^k \ . \end{aligned}$$

Hence,

$$\begin{aligned} d(\psi(t,x_t)\, Z_t) &= L\psi(t,x_t)\, Z_t\, dt + \sum_{i,j\in\mathbb{Z}} \sigma_j^i(t,x_t)\, \nabla_i\psi(t,x_t)\, Z_t\, dw_t^j \\ &\quad + \sum_{i\in\mathbb{Z}} \sum_{k=1}^{p} g_k^i(t,x_t)\, \nabla_i\psi(t,x_t)\, Z_t\, dv_t^k + \sum_{k=1}^{p} h_k(t,x_t)\, \psi(t,x_t)\, Z_t\, dy_t^k \\ &\quad + \sum_{i\in\mathbb{Z}} \sum_{k=1}^{p} g_k^i(t,x_t)\, h_k(t,x_t)\, \nabla_i\psi(t,x_t)\, Z_t\, dt \\ &= L\psi(t,x_t)\, Z_t\, dt + \sum_{i,j\in\mathbb{Z}} \sigma_j^i(t,x_t)\, \nabla_i\psi(t,x_t)\, Z_t\, dw_t^j \\ &\quad + \sum_{k=1}^{p} L_k(\psi(t,x_t))\, Z_t\, dy_t^k \ . \end{aligned}$$

Consequently,

$$\psi(t,x_t)\,Z_t = \psi(0,x_0) + \int_0^t L\psi(s,x_s)\,Z_s\,ds + \int_0^t \sum_{i,j\in\mathbb{Z}} \sigma^i_j(s,x_s)\,\nabla_i\psi(s,x_s)\,Z_s\,dw^j_s$$
$$+ \int_0^t \sum_{k=1}^p L_k(\psi(s,x_s))\,Z_s\,dy^k_s\ .$$

Since Z_t is in $L^p(\Omega)$, for all p, the boundedness of the functions σ, g and h implies that we can use Lemma 3.1. Hence

$$\begin{aligned}\rho_t(\psi) &= \overline{E}\Big[\psi(t,x_t)\,Z_t\,/\,\mathcal{Y}_t\Big]\\ &= \overline{E}\Big[\psi(0,x_0)\,/\,\mathcal{Y}_0\Big] + \int_0^t \overline{E}\Big[L\psi(s,x_s)\,Z_s\,/\,\mathcal{Y}_s\Big]\,ds\\ &\quad + \sum_{k=1}^p \int_0^p \overline{E}\Big[L_k(\psi(s,x_s))\,Z_s\,/\,\mathcal{Y}_s\Big]\,dy^k_s\ ,\end{aligned}$$

which gives the equality (3.3). ∎

4 – The Kushner–Stratonovich equation

In this section, we prove that the filter Π_t defined by (1.8) solves a Kushner–Stratonovich equation. With this aim, we show at first:

Proposition 4.1. *For all t in $[0,T]$,*

$$\rho_t\,1 = \exp\left(\int_0^t \sum_{k=1}^p \Pi_s(h_k)\,dy^k_s - \frac{1}{2}\int_0^t \sum_{k=1}^p (\Pi_s(h_k))^2\,ds\right)\ . \tag{4.1}$$

Proof: Applying Itô's formula and Lemma 3.1 to the process Z_t, we get

$$\begin{aligned}\rho_t\,1 &= \overline{E}[Z_t\,/\,\mathcal{Y}_t]\\ &= 1 + \int_0^t \sum_{k=1}^p \overline{E}\Big[Z_s\,h_k(s,x_s)\,/\,\mathcal{Y}_s\Big]\,dy^k_s\\ &= 1 + \int_0^t \sum_{k=1}^p \Pi_s(h_k)\,\rho_s\,1\,dy^k_s\ ,\end{aligned}$$

which, according to Itô's formula, is the same exponential process than the one in equality (4.1). ∎

This allows us to prove the following result:

Theorem 4.2. *For all t in $[0,T]$ and every function ψ in $\mathfrak{D}^2$, the filter $\Pi_t(\psi)$ is the solution of the stochastic differential equation*

$$\begin{aligned}\Pi_t(\psi) &= \Pi_0(\psi) + \int_0^t \Pi_s(L\psi)\,ds \\ &\quad + \int_0^t \sum_{k=1}^p \Big(\Pi_s(L_k\psi) - \Pi_s(h_k)\,\Pi_s(\psi)\Big)\Big(dy_s^k - \Pi_s(h_k)\,ds\Big)\,.\end{aligned} \tag{4.2}$$

Proof: We successively apply Itô's formula to the processes $(\rho_t\,1)^{-1}$ and $\rho_t\,\psi(\rho_t\,1)^{-1}$ to find

$$d((\rho_t)^{-1}) = (\rho_t\,1)^{-1}\Big(-\sum_{k=1}^p \Pi_t(h_k)\,dy_t^k + \sum_{k=1}^p (\Pi_t(h_k))^2\,dt\Big)$$

and

$$\begin{aligned}d(\Pi_t(\psi)) &= \frac{1}{\rho_t\,1}\Big(\rho_t(L\psi)\,dt + \sum_{k=1}^p \rho_t(L_k\psi)\,dy_t^k\Big) \\ &\quad + \frac{\rho_t\,\psi}{\rho_t\,1}\Big(-\sum_{k=1}^p \Pi_t(h_k)\,dy_t^k + \sum_{k=1}^p (\Pi_t(h_k))^2\,dt\Big) \\ &\quad + \frac{1}{\rho_t\,1}\Big(-\sum_{k=1}^p \Pi_t(h_k)\,\rho_t(L_k\psi)\,dt\Big) \\ &= \Pi_t(L\psi)\,dt + \sum_{k=1}^p \Pi_t(L_k\psi)\,dy_t^k \\ &\quad + \sum_{k=1}^p \Pi_t\,\psi\Big(-\Pi_t(h_k)\,dy_t^k + (\Pi_t(h_k))^2\,dt\Big) \\ &\quad - \sum_{k=1}^p \Pi_t(h_k)\,\Pi_t(L_k\psi)\,dt\,.\ \blacksquare\end{aligned}$$

5 – The robust form of the Zakai equation

In this section, we derive the robust form of the Zakai equation (3.1) obtained previously. This allows to turn up the resolution of an Itô type stochastic differential equation to an ordinary partial differential equation parameterized by the paths of the observation process. Since, in the case of a multidimensional observation process, this method is however only adapted to the case of a non-correlated filtering problem, we shall suppose in this section that $g \equiv 0$, so we consider the system process-observation pair $(x_t, y_t) \in (L^2(\gamma) \times \mathbb{R}^p)$ solution of the stochastic differential system

$$
(5.1) \qquad \begin{cases} x_t^i = x_0^i + \displaystyle\int_0^t b_i(s, x_s)\, ds + \int_0^t \sum_{j \in \mathbb{Z}} \sigma_j^i(s, x_s)\, dw_s^j\,, \qquad i \in \mathbb{Z}\,, \\ y_t = \displaystyle\int_0^t h(s, x_s)\, ds + v_t\ . \end{cases}
$$

Remark. In the case of a one-dimensional observation process, M. Davis [4] respectively J.M. Bismut and D. Michel [1] have derived a robust form for the Zakai equation for a correlated nonlinear filtering system with bounded coefficients, whereas P. Florchinger [10] proved a similar result for a correlated nonlinear filtering system with unbounded observation coefficients.

Moreover, we define

Definition 5.1. For every t in $[0, T]$ and every function ψ in $\mathfrak{D}^2$, set

$$
(5.2) \qquad \nu_t\, \psi = \overline{E}\Big[\psi(t, x_t)\, U_t \,/\, \mathcal{Y}_t\Big]\ ,
$$

where U_t is the stochastic process defined by

$$
(5.3) \qquad U_t = \exp\bigg(-\int_0^t \sum_{k=1}^p y_s^k\, dh_k(s, x_s) - \frac{1}{2}\int_0^t \sum_{k=1}^p (h_k(s, x_s))^2\, ds\bigg)\ .
$$

Besides, by an integration by parts in the stochastic integral appearing in the formula (2.1), we get

$$
Z_t = \exp\bigg(\langle h(t, x_t),\, y_t\rangle - \int_0^t \sum_{k=1}^p y_s^k\, dh_k(s, x_s) - \frac{1}{2}\int_0^t \sum_{k=1}^p (h_k(s, x_s))^2\, ds\bigg)\ ,
$$

which allows us to deduce that for any function ψ in $\mathfrak{D}^2$, we have

$$
(5.4) \qquad \nu_t\, \psi = \rho_t\Big(\psi\, \exp\Big(-\langle h(t, x_t),\, y_t\rangle\Big)\Big)\ .
$$

With this, we can prove the following theorem:

Theorem 5.2. *For every function ψ in $\mathfrak{D}^2$, $\nu_t(\psi)$ solves the ordinary partial differential equation*

$$\left\{\begin{array}{l} \dfrac{d}{dt}\nu_t\,\psi = \nu_t\,L_{y_t}\psi, \\ \nu_0\,\psi = E[\psi(0,x_0)]\ , \end{array}\right. \tag{5.5}$$

where L_{y_t} is the second order partial differential operator, parameterized by the paths of the process y, defined for any function ψ in $\mathfrak{D}^2$ by

$$\begin{aligned} L_{y_t}\psi(t,x) &= L\psi(t,x) - \sum_{i,j\in\mathbb{Z}}\sum_{k=1}^{p}\gamma_j\,(\sigma_j^i(t,x))^2\,\nabla_i h_k(t,x)\,\nabla_i\psi(t,x)\,y_t^k \\ &\quad - \Big(\frac{1}{2}\sum_{k=1}^{p}(h_k(t,x))^2 - \sum_{k=1}^{p}Lh_k(t,x)\,y_t^k \\ &\quad + \frac{1}{2}\sum_{i,j\in\mathbb{Z}}\sum_{k=1}^{p}\gamma_j\Big(\sigma_j^i(t,x)\,\nabla_i h_k(t,x)\,y_t^k\Big)^2\Big)\,\psi(t,x)\ . \end{aligned}$$

Proof: By Itô's formula,

$$dh_k(t,x) = Lh_k(t,x_t)\,dt + \sum_{i,j\in\mathbb{Z}}\sigma_j^i(t,x_t)\,\nabla_i h_k(t,x_t)\,dw_t^j\ .$$

Consequently,

$$\begin{aligned} U_t = \exp\Big(&-\int_0^t\sum_{k=1}^{p}y_s^k\,Lh_k(s,x_s)\,ds - \frac{1}{2}\int_0^t\sum_{k=1}^{p}(h_k(s,x_s))^2\,ds \\ &-\int_0^t\sum_{i,j\in\mathbb{Z}}\sum_{k=1}^{p}y_s^k\,\sigma_j^i(s,x_s)\,\nabla_i h_k(s,x_s)\,dw_s^j\Big)\ . \end{aligned}$$

Further applications of Itô's formula give

$$\begin{aligned} dU_t = &-\sum_{k=1}^{p}U_t\,Lh_k(t,x_t)\,y_t^k\,dt - \frac{1}{2}\sum_{k=1}^{p}U_t\,(h_k(t,x_t))^2\,dt \\ &-\sum_{i,j\in\mathbb{Z}}\sum_{k=1}^{p}U_t\,\sigma_j^i(t,x_t)\,\nabla_i h_k(t,x_t)\,y_t^k\,dw_t^j \\ &+\frac{1}{2}\sum_{i,j\in\mathbb{Z}}\sum_{k=1}^{p}U_t\,\gamma_j\,\Big(\sigma_j^i(t,x_t)\,\nabla_i h_k(t,x_t)\,y_t^k\Big)^2\,dt \end{aligned}$$

and

$$d\psi(t,x_t) = L\psi(t,x_t)\,dt + \sum_{i,j\in\mathbb{Z}} \nabla_i\psi(t,x_t)\,\sigma^i_j(t,x_t)\,dw^j_t \ .$$

Hence,

$$\begin{aligned} d(\psi(t,x_t)\,U_t) &= L\psi(t,x)\,U_t\,dt + \sum_{i,j\in\mathbb{Z}} \sigma^i_j(t,x_t)\,\nabla_i\psi(t,x_t)\,U_t\,dw^j_t \\ &\quad - \sum_{k=1}^p \psi(t,x_t)\,U_t\,y^k_t\,Lh_k(t,x_t)\,dt - \frac{1}{2}\sum_{k=1}^p \psi(t,x_t)\,U_t\,(h_k(t,x_t))^2\,dt \\ &\quad - \sum_{i,j\in\mathbb{Z}}\sum_{k=1}^p \psi(t,x_t)\,U_t\,\sigma^i_j(t,x_t)\,\nabla_i h_k(t,x_t)\,y^k_t\,dw^j_t \\ &\quad + \frac{1}{2}\sum_{i,j\in\mathbb{Z}}\sum_{k=1}^p \gamma_j\,\psi(t,x_t)\,U_t\left(\sigma^i_j(t,\,x_t\,\nabla_i h_k(t,x_t))\,y^k_t\right)^2 dt \\ &\quad - \sum_{i,j\in\mathbb{Z}}\sum_{k=1}^p \gamma_j\,U_t\,(\sigma^i_j(t,x_t))^2\,\nabla_i h_k(t,x_t)\,\nabla_i\psi(t,x_t)\,y^k_t\,dt \ . \end{aligned}$$

The conclusion is then straightforward by means of Definition 5.1 and Lemma 3.1. ∎

REFERENCES

[1] Bismut, J.M. and Michel, D. – Diffusions conditionelles. Part I, *Funct. Anal.,* 44 (1981), 174–211. Part II, *J. Funct. Anal.,* 45 (1982), 274–292.

[2] Cattiaux, P., Roelly, S. and Zessin, H. – Une approche Gibbsienne des diffusions Browniennes infini-dimensionnelles, *Probab. Th. Rel. Fields,* 104 (1996), 147–179.

[3] Clark, J. – *The design of a robust approximation to the stochastic differential equations of nonlinear filtering*, In "Communication Systems and Random Process Theory" (J. Swirzynski, Ed.), Sijthoff and Nordhoof, 1978.

[4] Davis, M. – *Pathwise nonlinear filtering*, In "Stochastic Systems, the Mathematics of Filtering and Identification and Applications" (Hazenwinkel and Willems, Eds.), Reidel, Dordrecht, 1981.

[5] Davis, M. and Marcus, S.I. – *An introduction to nonlinear filtering*, In "Stochastic Systems the Mathematics of Filtering and Identification and Applications" (Hazenwinkel and Willems, Eds.), Reidel, Dordrecht, 1981.

[6] Doss, H. and Royer, G. – Processus de diffusion associé aux mesures de Gibbs, *Z. Warsch. Verw. Geb.,* 46 (1978), 125–158.

[7] Elliott, R.J. – Stochastic calculus and applications, *Applications of Mathematics,* Vol. 18 (1982), Springer-Verlag, Berlin, Heidelberg, New York.

[8] Ferreyra, G.S. – *Smoothness of the unnormalized conditional measures of stochastic nonlinear filtering*, In "Proceedings of the 23rd IEEE Conference on Decision and Control", Las Vegas, 1984.

[9] Florchinger, P. – Malliavin calculus with time depending coefficients and application to nonlinear filtering, *Probab. Th. Rel. Fields,* 86 (1990), 203–223.
[10] Florchinger, P. – Zakai equation of nonlinear filtering with unbounded coefficients. The case of dependent noises, *System & Control Letters,* 21 (1993), 413–422.
[11] Florchinger, P. – *Zakai equation of nonlinear filtering in infinite dimension*, In "Proceedings of the 30th IEEE Conference on Decision and Control" (1991), 2754–2755, Brighton, England.
[12] Florchinger, P. and Schiltz, J. – *Smoothness of the density for the filter under infinite dimensional noise and unbounded observation coefficients*, In "Proceedings of the 2nd Portuguese Conference on Automatic Control", (1996), 115–118, Porto.
[13] Fujisaki, M., Kallianpur, G. and Kunita, H. – Stochastic differential equations for the non-linear filtering problem, *Osaka J. Math.,* 9 (1972), 19–40.
[14] Holley, R. and Stroock, D. – Diffusions on an infinite dimensional torus, *J. Funct. Anal.,* 42 (1981), 29–63.
[15] Hopkins, W. – *Nonlinear Filtering of Non Degenerate Diffusions with Unbounded Coefficients*, Thesis, University of Maryland at College Park, 1982.
[16] Kallianpur, G. – *Stochastic Filtering Theory*, Springer-Verlag: Berlin, Heidelberg, New York.
[17] Krylov, N.V. – On L_p-theory of stochastic partial differential equations in the whole space, *SIAM J. Math. Anal.,* 27(2) (1996), 313–340.
[18] Michel, D. – Régularité des lois conditionnelles en théorie du filtrage non linéaire et calcul des variations stochastiques, *J. Funct. Anal.,* 41 (1981), 8–36.
[19] Millet, A., Nualart, D, and Sanz, M. – Time reversal for infinite dimensional diffusions, *Probab. Theor. Rel. Fields,* 82 (1989), 315–347.
[20] Pardoux, E. – Stochastic partial differential equations and filtering of diffusion processes, *Stochastics,* 3 (1979), 127–167.
[21] Pardoux, E. – *Filtrage non linéaire et équations aux dérivées partielles stochastiques associées*, In "Ecole d'été de Probabilités de Saint-Flour", Lect. Notes Math., vol. 1464 (1989), 69–163, Springer-Verlag: Berlin, Heidelberg, New York.
[22] Schiltz, J. – Malliavin calculus with time depending coefficients applied to a class of stochastic differential equations, *Stochastic Analysis and Applications,* 16(6) (1998), 1073–1100.
[23] Shiga, T. and Shimizu, A. – Infinite dimensional stochastic differential equations and their applications, *J. Math. Kyoto Univ.,* 20 (1980), 395–416.
[24] Zakai, M. – On the optimal filtering of diffusion processes, *Z. Wahrscheinlichkstheor. Verw. Gebiete,* 11 (1969), 230–243.

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