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RANDOM EVOLUTIONS PROCESSES INDUCED BY DISCRETE TIME MARKOV CHAINS

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Abstract: Research on the random evolution of a family of semigroups induced by a finite-state, continuous-time, stationary Markov chain was begun by Griego and Hersh in 1969. Subsequently limit theorems and applications for random evolutions have appeared in many places and the theory of random evolutions has been extended and generalized in many directions. In this paper, we extend the theory of random evolutions to discrete time Markov chains. Also, we use the idea of a reversed Markov chain to exhibit subtle connections between forward and backward random evolutions. Aside from their probabilistic significance, the results contribute to the general theory of discrete semigroups.

1 – Introduction

A random evolution describes a situation in which a Markov process controls the development of another process, the other process being described by operators on a Banach space. A connection between random evolutions and products of random matrices is made by Cohen in [3]. This connection is then used in predicting the long-run growth rate of a single-type, continuously changing population in a randomly varying environment using only a sample of the continuous-time random evolution taken at time $t = 1$. In [4 and 5], he describes in more detail how a population in a random environment can be modeled by a continuous-time random evolution observed at discrete points in time and why eigenvalue inequalities arise naturally. The discrete random evolution considered

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by Korolyuk and Swishchuk [17] can be considered as an imbedded random evolution, as can the ones considered by Cohen [3–5], in analogy with the notation of an imbedded Markov chain. Conditions necessary for a discrete time Markov chain to be imbeddable in a continuous time Markov chain with the same state space were formulated by Kingman in [16].

Griego and Hersh [7 and 8] were the first to find that certain abstract Cauchy problems are related to random evolutions. In [11] we gave an alternative formulation of random evolutions that furnished both the backward and forward equations for random evolutions. Forward random evolutions provide a probabilistic approach to the study of a different class of Cauchy problems to those of the backward random evolutions of Griego–Hersh. The theory of random evolutions was extended to the case where the inducing Markov chain is non-stationary in [12], there an explicit connection between random evolutions and the famous Phillips Perturbation series is developed. Earlier in this journal [14], the author studied random evolutions with underlying continuous-time, countable state space Markov chains.

In this paper we consider random evolutions with underlying discrete-time Markov chains. The proofs utilize the sample path methods of [8 and 15]. The sample path approach gives additional insight as to how the random evolution structure relates to the mechanisms of the Markov chain.

Surveys of the early literature on random evolutions are given in the papers of Cohen [5] and Hersh [9]. Applications and limit theorems for random evolutions are given in the book of Ethier–Kurtz [6]. In his book [10], Iordache studies the role played by random evolutions and other dynamical systems in chemical engineering. Abstract mathematical models of evolutionary stochastic systems in random media, namely, random evolutions are studied in detail by Korolyuk and Swishchuk in the monograph [17]. There the random evolutions studied are induced by semi-Markov processes, for the most part. The reader is referred to Ethier–Kurtz [6] and Korolyuk–Swishchuk [17] for the necessary facts about semigroups and to Adke and Manjunath [1] and Chung [2] for information about Markov chains.

2 – We use the following notation and assumptions

$$Z = \{0, 1, 2, ...\} \ .$$

$\{X(n);\ n \in Z\}$ is a stationary Markov chain taking values in $\{1, 2, ..., N\}$ and having transition matrix P, where

$$P = \langle p_{\alpha\beta} \rangle\,, \quad 1 \le \alpha,\ \beta \le N \quad \text{with}$$
$$p_{\alpha\beta} \ge 0 \ \text{ for } 1 \le \alpha,\ \beta \le N \quad \text{and} \quad \sum_{\beta=1}^{N} p_{\alpha\beta} = 1\ .$$

$P_j[\cdot]$ denotes the probability measure define on the sample paths $X(\cdot)$, given that $X(0) = j$; $E_j[\cdot]$ denotes integration with respect to $P_j[\cdot]$.

$$\tilde{f} = \langle f_j \rangle_{1 \le j \le N} \ \in \ \tilde{B} = B^N$$

where B is a fixed Banach space. We equip $\tilde{B}$ with any appropriate norm so that $\|\tilde{f}\| \to 0$ as $\|f_i\| \to 0$ for each i. Let $\{T_i,\ 1 \le i \le N\}$ be bounded linear operators defined on B. We define the *backward random evolution* $\{R(n)\colon\ n \in Z\}$ by

$$R(n) = T_{X(0)}\, T_{X(1)} \cdots T_{X(n)}\,, \quad n \ge 1\ ,$$

with $R(0) = I$, the identity operator. For $n \in Z$ define $\tilde{R}(n)$ on $\tilde{B} = B^N$, specified componentwise, by

$$(\tilde{R}(n)\,\tilde{f})_j = E_j[R(n)\, f_{X(n)}]\ .$$

Theorem 1. **(i)** *$\{\tilde{R}(n);\ n \in Z\}$ is a discrete semigroup of bounded, linear operators on $\tilde{B}$, and*

(ii) *$\tilde{u}(n) = \tilde{R}(n)\,\tilde{f}$ solves the initial value problem*

$$\begin{aligned} u_j(n+1) &= \sum_{k=1}^{N} p_{jk}\, T_j\, u_k(n)\ , \\ \tilde{u}(0) &= \tilde{f} \in \tilde{B}\ . \end{aligned} \tag{2.1}$$

Proof of (i): (See Griego–Hersh [8, page 410]).

It is easy to see that $\tilde{R}(n)$ is a bounded linear operator. Thus, we need to check the semigroup property. It suffices to show that for each i,

$$\left(\tilde{R}(n+m)\,\tilde{f}\right)_i = \left(\tilde{R}(n)\,\tilde{R}(m)\,\tilde{f}\right)_i\ .$$

Now,

$$\left(\tilde{R}(n+m)\,\tilde{f}\right)_i = E_i\Big[R(n+m)\,f_{X(n+m)}\Big]$$

$$= E_i\Big[E_i\Big[R(n+m)\,f_{X(n+m)}\,|\,\mathcal{F}_n\Big]\Big]$$

(where $\mathcal{F}_n$ is the σ-algebra generated by the random variables $X(k)$, $0 \le k \le n$)

$$= E_i\Big[E_i\Big[R(n)\,R(m)\circ\theta_n\,f_{X(m)}\circ\theta_n\,|\,\mathcal{F}_n\Big]\Big]$$

(θ_n is the shift operator defined by the requirement that $X(k,\theta_n w) = X(k+n,\, w)$ for every $k \in Z$)

$$= E_i\Big[R(n)\,E_i\Big[R(m)\circ\theta_n\,f_{X(m)}\circ\theta_n\,|\,\mathcal{F}_n\Big]\Big]$$

$$= E_i\Big[R(n)\,E_{X(n)}\Big[R(m)\,f_{X(m)}\Big]\Big]$$

(by the Markov property of X)

$$= E_i\Big[R(n)\,(\tilde{R}(m)\,\tilde{f})_{X(n)}\Big] = (\tilde{R}(n)\,\tilde{R}(m)\,\tilde{f})_i\ . \blacksquare$$

Proof of (ii):

$$u_j(n+1) = E_j\Big[R(n+1)\,f_{X(n+1)}\Big]$$

$$= \sum_{k=1}^{N} E_j\Big[R(n+1)\,f_{X(n+1)};\ X(1) = k\Big]$$

$$= \sum_{k=1}^{N} T_j\,p_{jk}\,E_k[R(n)\,f_{X(n)}]$$

(by the Markov property of X

$$= \sum_{k=1}^{N} p_{jk}\,T_j\,u_k(n)\ . \blacksquare$$

Now, consider the forward random evolution

$$S(0) = I\ ,$$
$$S(n) = T_{X(n)}\,T_{X(n-1)}\cdots T_{X(0)}\,, \qquad n \ge 1\ .$$

Define $\tilde{S}(n)$ on $\tilde{B}$ by

$$(\tilde{S}(n)\,\tilde{f})_j = \sum_{k=1}^{N} E_k\Big[S(n)\,f_k\,;\ X(n) = j\Big]\ .$$

Theorem 2. (**i**) *$\{\tilde{S}(n);\ n \in Z\}$ is a discrete semigroup of bounded linear operators in $\tilde{B}$, and*

(**ii**) *$\tilde{w}(n) = \tilde{S}(n)\,\tilde{f}$ solves the initial value problem*

$$w_j(n+1) = \sum_{i=1}^{N} p_{ij}\, T_j\, w_i(n)\,, \qquad \tilde{w}(0) = \tilde{f}\ . \tag{2.2}$$

Proof of (ii):

$$\begin{aligned} w_j(n+1) &= \sum_{R=1}^{N} E_k\Big[S(n+1) f_k\,;\ X(n+1) = j\Big] \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} E_k\Big[T_{X(n+1)}\, T_{x(n)} \cdots T_{X(0)}\, f_k\,;\ X(n) = i,\ X(n+1) = j\Big] \\ &= \sum_{i=1}^{N} p_{ij}\, T_j \sum_{k=1}^{N} E_k\Big[T_{X(n)} \cdots T_{X(0)}\, f_k\,;\ X(n) = i\Big] \\ &= \sum_{i=1}^{N} p_{ij}\, T_j\, w_i(n)\ . \ \blacksquare \end{aligned}$$

The proof of part (i) is like that of part (i) of Theorem 1 with changes suggested by the proof of Theorem 1 in [15].

Let $A = \operatorname{diag}(T_1, T_2, ..., T_N)$ then by (2.1) and (2.2) we have, in matrix form,

$$\begin{aligned} \tilde{u}(n+1) &= A\,P\,\tilde{u}(n) \quad \text{and} \\ \tilde{w}(n+1) &= A\,P^T\,\tilde{w}(n)\ . \ \blacksquare \end{aligned}$$

3 – In order to compare the backward and forward random evolutions further, we now assume that the transition matrix P of X has one as a simple eigenvalue. Thus, P has a unique left eigenvector $p = \langle p_\alpha \rangle$, $\sum_{\alpha=1}^{N} p_\alpha = 1$. Such a Markov chain has an associated reversed chain $\hat{X}$ with transition matrix $\hat{P} = \langle \hat{p}_{\alpha\beta} \rangle$, $1 \le \alpha,\ \beta \le N$, where $\hat{p}_{\alpha\beta} = \frac{p_\beta}{p_\alpha}\, p_{\beta\alpha}$.

Now, define the backward forward random evolutions induced by $\hat{X}$, $\hat{R}(n)$ and $\hat{S}(n)$ respectively. As in Theorems 1 and 2, the expectation operators, $\tilde{\hat{R}}(n)$ and $\tilde{\hat{S}}(n)$ are discrete semigroups of bounded linear operators on $\tilde{B}$, and

(i) $\tilde{u}(n) = \tilde{\hat{R}}(n)\tilde{f}$ solves the initial value problem

$$(3.1) \qquad \begin{aligned} u_j(n+1) &= \sum_{k=1}^{N} \hat{p}_{jk}\, T_j\, u_k(n) \\ \tilde{u}(0) &= \tilde{f}\ . \end{aligned}$$

(ii) $\tilde{w}(n) = \tilde{\hat{S}}(n)\tilde{f}$ solves the initial value problem

$$(3.2) \qquad \begin{aligned} w_j(n+1) &= \sum_{i=1}^{N} \hat{p}_{ij}\, T_j\, w_i(n) \\ \tilde{w}(0) &= \tilde{f}\ . \end{aligned}$$

With some manipulation of the matrices defined above we obtain the following comparison theorem.

Theorem 3. *Let* $\prod = \mathrm{diag}(p_1, p_2, ..., p_N)$.
Then equations (2.1) and (2.2) may be written in matrix form as

$$\tilde{u}(n+1) = A\,P\,\tilde{u}(n)$$

$$\text{and} \quad \tilde{w}(n+1) = A\,P^T\,\tilde{w}(n)\,, \quad \text{respectively}\,.$$

Similarly, (3.1) and (3.2) in matrix form are

$$\tilde{u}(n+1) = A\Big[\prod^{-1} P^T \prod\Big]\,\tilde{u}(n)$$

$$\text{and} \quad \tilde{w}(n+1) = A\Big[\prod P^T \prod^{-1}\Big]\,\tilde{w}(n)\,, \quad \text{respectively}\,. \blacksquare$$

Since $\prod$ is a diagonal matrix, we have shown (see an analogous result for reversible countable state space Markov chains in [14])) that the forward evolution for the reversed chain is essentially the same as the backward evolution for the original chain and the backward evolution for the reversed chain is essentially the same as the forward evolution for the original chain.

4 – The "jump" operator case can be handled in a similar manner. We suppose B, $\tilde{B}$, A, and X are given as in Section 2 and let $\{C_{\alpha\beta}\}$, $1 \le \alpha, \beta \le N$, be bounded linear operators defined on the Banach space B. We define the jump *backward random evolution* $\{R(n);\, n \in Z\}$ by

$$R(n) = T_{X(0)}\, C_{X(0)X(1)}\, T_{X(1)} \cdots C_{X(n-1)X(n)}\, T_{X(n)}\,, \qquad n \ge 1\ ,$$

with $R(0) = I$. The expectation semigroup $\{\tilde{R}(n);\ n \in Z\}$ is defined on $\tilde{B}$ by $\{\tilde{R}(n)\tilde{f})_j = Ej[R(n) f_{X(n)}]$. With this notation the following theorem is proved using the methods of Section 2.

Theorem 4. *(i) $\{\tilde{R}(n)\,;\ n \in Z\}$ is a discrete semigroup of bounded linear operators on $\tilde{B}$, and*

(ii) *$\tilde{u}(n) = \tilde{R}(n)\tilde{f}$ solves the initial value problem*

$$
\begin{aligned}
u_j(n+1) &= \sum_{k=1}^{N} T_j\, p_{jk}\, C_{jk}\, u_k(n)\ , \\
\tilde{u}(0) &= \tilde{f} \in \tilde{B}\ .
\end{aligned}
\tag{4.1}
$$

Now, define a jump forward random evolution $\{S(n)\,;\ n \in Z\}$ by reversing the order of the operators in $R(n)$ above. Analogous to the case of Section 2, we define

$$
(\tilde{S}(n)\, \tilde{f})_j = \sum_{k=1}^{N} E_k \Big[S(n)\, f_k\, ;\ X(n) = j \Big]\ .
$$

Using the method of Section 2, we get that $\tilde{S}(n)$ is a discrete semigroup of bounded linear operators in $\tilde{B}$ and

Theorem 5. *$\tilde{W}(n) = \tilde{S}(n)\tilde{f}$ solves the initial value problem*

$$
\begin{aligned}
w_j(n+1) &= \sum_{i=1}^{N} T_j\, p_{ij}\, C_{ij}\, w_i(n)\ , \\
\tilde{w}(0) &= \tilde{f} \in \tilde{B}\ .
\end{aligned}
\tag{4.2}
$$

5 – The system of equations (2.1) taken with the system of equations (2.2) form a formally adjoint system. The relation between (2.1) and (2.2) shows up more clearly in the case of non-stationary transition probability matrix. Also, in the theory of continuous-time random evolutions, the non-stationary case is both interesting and applicable (see [12] and [17]). Thus, in this section we shall extend the theory of backward and forward random evolutions to the nonstationary case.

Suppose $X(n)$ is a nonstationary Markov chain on $\{1, 2, ..., N\}$ with transition matrix

$$
P(m) = \langle p_{\alpha\beta}(m) \rangle \qquad 1 \le \alpha,\ \beta \le N\ ,
$$

such that $p_{\alpha\beta}(m) = P[X(m+1) = \beta \,|\, X(m) = i]$ for $m \in Z^+$. Let $P_{\alpha,m}$ be the probability measure defined on sample paths for X under the condition $X(m) = \alpha$. $E_{\alpha,m}$ denotes integration with respect to $P_{\alpha,m}$.

A *backward random evolution* $\{R(n,m)\,;\ 0 \le n \le m\}$ is defined by $R(n,m) = T_{X(n)}\, T_{X(n+1)} \cdots T_{X(m)}$ where $R(n,n) = I$. For $0 \le n \le m$ define the operator $\tilde{R}(n,m)$ on $\tilde{B}$ by

$$\left(\tilde{R}(n,m)\,\tilde{f}\right)_j = E_{j,n}\left[R(n,m)\, f_{X(m)}\right] .$$

A *forward random evolution* $\{S(n,m)\,;\ 0 \le n \le m\}$ is defined by $S(n,m) = T_{X(m)}\, T_{X(m-1)} \cdots T_{X(n)}$, where $S(m,m) = I$. For $0 \le n \le m$ define the operator $\tilde{S}(n,m)$ on $\tilde{B}$ by

$$\left(\tilde{S}(n,m)\,\tilde{f}\right)_j = \sum_{k=1}^{N} E_{k,n}\left[S(n,m) f_k\,;\ X(m) = k\right] .$$

The following two theorems are nonstationary analogues of Theorems 1 and 2.

Theorem 6. (**i**) *$\{\tilde{R}(n,m)\,;\ 0 \le n \le m\}$ is a two-parametric family of bounded linear operators satisfying the convolution equation (semigroup property)*

$$\tilde{R}(n,m) = \tilde{R}(n,l)\,\tilde{R}(l,m)\,, \qquad 0 \le n \le l \le m\ .$$

(**ii**) *$\tilde{u}(n,m) = \tilde{R}(n,m)\tilde{f}$ solves the initial value problem*

$$u_j(n,m) = \sum_{k=1}^{N} p_{jk}(n)\, T_j\, u_k(n+1,\, m), \quad \tilde{u}(n,n) = \tilde{f}\ . \tag{5.1}$$

Theorem 7. (**i**) *$\{\tilde{S}(n,m)\,;\ 0 \le n \le m\}$ is a two-parametric family of bounded linear operators satisfying the convolution equation*

$$\tilde{S}(n,m) = \tilde{S}(l,m)\,\tilde{S}(n,l)\,, \qquad 0 \le n \le l \le m\ .$$

(**ii**) *$\tilde{w}(n,m) = \tilde{S}(n,m)\,\tilde{f}$ solves the initial value problem*

$$\begin{aligned} w_j(n,m) &= \sum_{i=1}^{N} p_{ij}(m-1)\, T_j\, w_i(n,m-1)\ , \\ \tilde{w}(m,m) &= \tilde{f}\ . \end{aligned} \tag{5.2}$$

(5.1) exhibits the backward random evolution semigroup, $\tilde{R}(n,m)$, inherent connection with the first jump of the Markov chain and (5.2) clearly shows the inherent connection of the forward random evolution with the last jump of X.

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