PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 4 – 1997

SOME REMARKS ON TARDIFF'S FIXED POINT THEOREM ON MENGER SPACES

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1 – Introduction

Let D_+ be the family of all distribution functions $F: \mathbb{R} \to [0,1]$ such that $F(0) = 0$, and H_0 be the element of D_+ which is defined by

$$H_0 = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0 . \end{cases}$$

A t-norm T is a binary operation on $[0,1]$ which is associative, commutative, has 1 as identity, and is non-decreasing in each place. We say that T' is stronger than T'' and we write $T' \ge T''$ if $T'(a,b) \ge T''(a,b)$, $\forall\, a,b \in [0,1]$.

Definition 1.1. Let X be a set, $\mathcal{F}\colon X^2 \to D_+$ a mapping ($\mathcal{F}(x,y)$ will be denoted F_{xy}) and $T\colon [0,1]\times[0,1] \to [0,1]$ a t-norm. The triple $(X, \mathcal{F}, T)$ is called a *Menger space* iff it satisfies the following properties:

(**PM0**) If $x \ne y$ then $F_{xy} \ne H_0$;
(**PM1**) If $x = y$ then $F_{xy} = H_0$;
(**PM2**) $F_{xy} = F_{yx}$, $\forall\, x,y \in X$;
(**M**) $F_{xy}(u+v) \ge T(F_{xz}(u), F_{zy}(v))$, $\forall\, x,y,z \in X$, $\forall\, u,v \in \mathbb{R}$.

Let $f\colon [0,1] \to [0,\infty]$ be a continuous function which is strictly decreasing and vanishes at 1.

Received: March 11, 1996; *Revised*: October 3, 1996.
AMS Subject Classification: 60E54.
Keywords and Phrases: Menger spaces, Fixed point theorems.

Definition 1.2 ([6]). The pair $(X,\mathcal{F})$ which has the properties (PM0)–(PM2) is called a *probabilistic f-metric structure* iff

$$\forall\, t>0 \;\; \exists\, s>0 \quad \text{such that} \quad \Big[f\circ F_{xz}(s)<s,\; f\circ F_{zy}(s)<s\Big] \;\Rightarrow\; f\circ F_{xy}(t)<t\ .$$

Remark 1.3. If $(X,\mathcal{F})$ is a probabilistic f-metric structure then the family $\mathcal{W_F}^f:=\{W_\epsilon^f\}_{\epsilon\in(0,f(0))}$, where $W_\epsilon^f:=\{(x,y)|\ F_{xy}(\epsilon)>f^{-1}(\epsilon)\}$, is a uniformity base which generates a uniformity on X called $\mathcal{U_F}$ [6, p.46, Th. 1.3.39].

We define the t-norm generated by f by:

$$T_f(a,b)=f^{(-1)}\Big(f(a)+f(b)\Big)$$

where $f^{(-1)}$ is the quasi-inverse of f, namely

$$f^{(-1)}(x)=\begin{cases} f^{-1}(x), & x\le f(0),\\ 0, & x>f(0)\ .\end{cases}$$

It is well known and easy to see that $f\circ f^{(-1)}(x)\le x$, $\forall\, x\in[0,\infty]$ and $f^{(-1)}\circ f(a)=a$, $\forall\, a\in[0,1]$.

In the next section of this note we'll construct generalized metrics on Menger spaces, related to some ideas which have appeared in [11] and [4], and using some properties of the probabilistic f-metric structures.

In the last section, using this generalized metrics, we'll obtain a fixed point theorem on complete Menger spaces and we'll give some consequences. We'll give also, a fixed point alternative in complete Menger spaces.

The notations and the notions not given here are standard and follow [1], [8].

2 – A generalized metric on probabilistic f-metric structures

Let $f\colon\ [0,1]\to[0,1]$ a continuous and strictly decreasing function, such that $f(1)=0$.

Lemma 2.1. *We consider a Menger space $(X,\mathcal{F},T)$, where $T\ge T_f$. For each $k>0$ let us define*

$$d_k(x,y):=\sup_{s>0} s^k\int_s^\infty \frac{f\circ F_{xy}(t)}{t}\,dt$$

and

$$\rho_k(x,y) := \Big(d_k(x,y)\Big)^{\frac{1}{k+1}} .$$

Then ρ_k is a generalized metric on X.

Proof: It is clear that ρ_k is symmetric and $\rho_k(x,x) = 0$.

If $\rho_k(x,y) = 0$, then for each $s > 0$, $\displaystyle\int_s^\infty \frac{f \circ F_{xy}(t)}{t}\,dt = 0$ which implies $\displaystyle\int_s^t \frac{f \circ F_{xy}(u)}{u} du = 0,\ \forall\, t > s > 0.$

Since $\dfrac{f \circ F_{xy}(u)}{u} \geq \dfrac{f \circ F_{xy}(t)}{t} \geq 0$ for each $u \in (s,t)$, then

$$0 = \int_s^t \frac{f \circ F_{xy}(u)}{u}\,du \geq \frac{f \circ F_{xy}(t)}{t}\,(t-s)\,, \quad \forall\, t > s > 0\,,$$

which implies $f \circ F_{xy}(t) = 0$, $\forall\, t > 0$. Since f is a stictly decreasing function and $f(1) = 0$ then $F_{xy}(t) = 1$, $\forall\, t > 0$, that is $x = y$.

Because $(X, \mathcal{F}, T)$ is a Menger space and $T \geq T_f$, we have

$$F_{xy}(u+v) \geq T\Big(F_{xz}(u), F_{zy}(v)\Big) \geq T_f\Big(F_{xz}(u), F_{zy}(v)\Big), \quad \forall\, x,y,z \in X,\ \ \forall\, u,v \in \mathbb{R}.$$

Let us take $u = \alpha t$ and $v = \beta t$, where $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$. Then

$$F_{xy}(t) \geq f^{(-1)}\Big(f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t)\Big)\,,$$
$$\forall\, x,y,z \in X,\ \ \forall\, t > 0,\ \ \forall\, \alpha,\beta \in (0,1),\ \ \alpha + \beta = 1\,,$$

and so

$$f \circ F_{xy}(t) \leq (f \circ f^{(-1)})\,\Big(f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t)\Big) \leq f \circ F_{xz}(\alpha t) + f \circ F_{zy}(\beta t)\,,$$
$$\forall\, x,y,z \in X,\ \ \forall\, t > 0,\ \ \forall\, \alpha,\beta \in (0,1),\ \ \alpha + \beta = 1\,.$$

We divide the both members of inequality by t, integrate from s to ms and multiply with s^k, where $s > 0$, $m > 1$, $k > 0$. We obtain

$$s^k \int_s^{ms} \frac{f \circ F_{xy}(t)}{t}\,dt \leq s^k \int_s^{ms} \frac{f \circ F_{xz}(\alpha t)}{t}\,dt + s^k \int_s^{ms} \frac{f \circ F_{zy}(\beta t)}{t}\,dt, \quad \forall\, m > 1,\ \ \forall\, s > 0.$$

We take $\alpha t = u$, respectively $\beta t = v$ in the first, respectively, the second term of the right side of the previous inequality and it follows that:

$$\begin{aligned} s^k \int_s^{ms} \frac{f\circ F_{xy}(t)}{t}\, dt &\le \frac{1}{\alpha^k}(\alpha s)^k \int_{\alpha s}^{m\alpha s} \frac{f\circ F_{xz}(u)}{u}\, du + \frac{1}{\beta^k}(\beta s)^k \int_{\beta s}^{m\beta s} \frac{f\circ F_{zy}(v)}{v}\, dv \\ &\le \frac{1}{\alpha^k}(\alpha s)^k \int_{\alpha s}^{\infty} \frac{f\circ F_{xz}(u)}{u}\, du + \frac{1}{\beta^k}(\beta s)^k \int_{\beta s}^{\infty} \frac{f\circ F_{zy}(v)}{v}\, dv \\ &\le \frac{1}{\alpha^k} \sup_{s>0}(\alpha s)^k \int_{\alpha s}^{\infty} \frac{f\circ F_{xz}(u)}{u}\, du + \frac{1}{\beta^k} \sup_{s>0}(\beta s)^k \int_{\beta s}^{\infty} \frac{f\circ F_{zy}(v)}{v}\, dv, \\ &\qquad \forall\, m > 1, \quad \forall\, s > 0\ . \end{aligned}$$

By making $m \to \infty$ and taking $\sup\limits_{s>0}$ in the left side of the previous inequality and by observing that

$$\sup_{s>0}(\alpha s)^k \int_{\alpha s}^{\infty} \frac{f\circ F_{xz}(u)}{u}\, du = \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{xz}(t)}{t}\, dt$$

and

$$\sup_{s>0}(\beta s)^k \int_{\beta s}^{\infty} \frac{f\circ F_{zy}(v)}{v}\, dv = \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{zy}(t)}{t}\, dt\ ,$$

we obtain that

$$\begin{aligned} \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{xy}(t)}{t}\, dt &\le \frac{1}{\alpha^k} \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{xz}(u)}{u}\, du \\ &\quad + \frac{1}{\beta^k} \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{zy}(v)}{v}\, dv\ . \end{aligned} \tag{2.1}$$

Let us denote

$$\begin{cases} a = \sup\limits_{s>0} s^k \displaystyle\int_s^{\infty} \frac{f\circ F_{xy}(t)}{t}\, dt\ , \\ b = \dfrac{1}{\alpha^k} \sup\limits_{s>0} s^k \displaystyle\int_s^{\infty} \frac{f\circ F_{xz}(t)}{t}\, dt\ , \\ c = \dfrac{1}{\beta^k} \sup\limits_{s>0} s^k \displaystyle\int_s^{\infty} \frac{f\circ F_{zy}(t)}{t}\, dt\ . \end{cases}$$

If $b = \infty$ or/and $c = \infty$ it follows that $\rho_k(x,z) = b^{\frac{1}{k+1}} = \infty$ or/and $\rho_k(z,y) = c^{\frac{1}{k+1}} = \infty$ and it is obvious that $\rho_k(x,y) \le \infty = \rho_k(x,z) + \rho_k(z,y)$.

We suppose that $b < \infty$ and $c < \infty$. The inequality (2.1) becomes:

$$a \le \frac{b}{\alpha^k} + \frac{c}{\beta^k} = \frac{b}{\alpha^k} + \frac{c}{(1-\alpha)^k}\,, \qquad \forall\, \alpha \in (0,1)\ ,$$

which implies $a \le \inf\limits_{0<\alpha<1} \Big(\frac{b}{\alpha^k} + \frac{c}{(1-\alpha)^k}\Big)$, $\forall\, \alpha \in (0,1)$.

We define the function $g\colon\ (0,1) \to \mathbb{R}_+$, $g(\alpha) = \dfrac{b}{\alpha^k} + \dfrac{c}{(1-\alpha)^k}$. We observe that g has a minimum in $\alpha_0 = \dfrac{b^{\frac{1}{k+1}}}{b^{\frac{1}{k+1}} + c^{\frac{1}{k+1}}}$ $(g'(\alpha_0) = 0)$.

Therefore

$$a \le \frac{b}{\alpha_0^k} + \frac{c}{(1-\alpha_0)^k} = (b^{\frac{1}{k+1}} + c^{\frac{1}{k+1}})^{k+1}$$

and it is clear that

$$\rho_k(x,y) = a^{\frac{1}{k+1}} \le b^{\frac{1}{k+1}} + c^{\frac{1}{k+1}} = \rho_k(x,z) + \rho_k(z,y)\ . \ \blacksquare$$

Lemma 2.2. *Let $(X,\mathcal{F},T)$ be a Menger space with $T \ge T_f$. Then $\mathcal{U}_{\mathcal{F}} \subset \mathcal{U}_{\rho_k}$.*

Proof: It can be shown that $\sup\limits_{a<1} T(a,a) \ge \sup\limits_{a<1} T_f(a,a) = 1$ and, using [6, p.41, Th. 1.3.22] we obtain that $(X,\mathcal{F})$ is a probabilistic f-metric structure. By using Remark 1.3 it suffices to show that

$$\text{(2.2)} \qquad \forall\, \epsilon \in (0, f(0)),\ \ \exists\, \delta(\epsilon)\colon\quad \rho_k(x,y) < \delta \ \Rightarrow\ F_{xy}(\epsilon) > f^{-1}(\epsilon)\ .$$

We observe that

$$\begin{aligned}
\rho_k(x,y) < \delta \iff & \sup_{s>0} s^k \int_s^{\infty} \frac{f \circ F_{xy}(t)}{t}\, dt < \delta^{k+1} \\
\iff & \forall\, s > 0,\ \ s^k \int_s^{\infty} \frac{f \circ F_{xy}(t)}{t}\, dt < \delta^{k+1} \\
\Longrightarrow & \forall\, m > 1,\ \forall\, s > 0,\ \ s^k \int_s^{ms} \frac{f \circ F_{xy}(t)}{t}\, dt < \delta^{k+1}\ .
\end{aligned}$$

We take s fixed, $s = \dfrac{\epsilon}{2}$ and $m = 2$. It follows

$$\Big(\frac{\epsilon}{2}\Big)^k \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{f \circ F_{xy}(t)}{t}\, dt < \delta^{k+1}\ .$$

But $t \le \epsilon \Rightarrow F_{xy}(t) \le F_{xy}(\epsilon) \Rightarrow f\circ F_{xy}(t) \ge f\circ F_{xy}(\epsilon) \Rightarrow \dfrac{f\circ F_{xy}(t)}{t} \ge \dfrac{f\circ F_{xy}(\epsilon)}{\epsilon}$.
Therefore

$$\left(\frac{\epsilon}{2}\right)^k \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{f\circ F_{xy}(\epsilon)}{\epsilon}\,dt \le \left(\frac{\epsilon}{2}\right)^k \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{f\circ F_{xy}(t)}{t}\,dt < \delta^{k+1} ,$$

which implies $\left(\frac{\epsilon}{2}\right)^{k+1} \dfrac{f\circ F_{xy}(\epsilon)}{\epsilon} < \delta^{k+1}$. If we choose $\delta = \dfrac{\epsilon}{2}$ we have $f\circ F_{xy}(\epsilon) < \epsilon$, which shows that the relation (2.2) is satisfied for $\delta(\epsilon) = \dfrac{\epsilon}{2}$. ∎

Lemma 2.3. *If $(X,\mathcal{F},T)$ is a complete Menger space under $T \ge T_f$, then (X,ρ_k) is complete.*

Proof: We suppose that (x_n) is a ρ_k-Cauchy sequence, that is,

$$\forall\,\epsilon > 0,\ \exists\, n_0(\epsilon):\quad \forall\, n \ge n_0(\epsilon),\ \forall\, p \ge 0 \ \Rightarrow\ \rho(x_n, x_{n+p}) < \epsilon\ . \tag{2.3}$$

From Lemma 2.2 we have that (x_n) is a $\mathcal{U}_{\mathcal{F}}$-Cauchy sequence. Since $(X,\mathcal{F},T)$ is a complete Menger space, we obtain that (x_n) is a $\mathcal{U}_{\mathcal{F}}$-convergent sequence, that is

$$\exists\, x_0 \in X \ \text{ such that } \forall\,\epsilon > 0,\ \exists\, n_1(\epsilon):\quad \forall\, n \ge n_1(\epsilon) \ \Rightarrow\ F_{x_n x_0}(\epsilon) > f^{-1}(\epsilon)\ .$$

It remains to show that (x_n) is a ρ_k-convergent sequence. From (2.3) we obtain that

$$\epsilon \ge \varliminf_{p\to\infty} \rho(x_n, x_{n+p}) = \varliminf_{p\to\infty} \sup_{s>0} s^k \int_s^{\infty} \frac{f\circ F_{x_n x_{n+p}}(t)}{t}\,dt \ge$$
$$\ge \varliminf_{p\to\infty} s^k \int_s^{\infty} \frac{f\circ F_{x_n x_{n+p}}(t)}{t}\,dt\,, \qquad \forall\, n \ge n_0(\epsilon),\ \ \forall\, s > 0\ .$$

By using the Fatou's lemma and the continuity of f we obtain:

$$\epsilon \ge \varliminf_{p\to\infty} s^k \int_s^{\infty} \frac{f\circ F_{x_n x_{n+p}}(t)}{t}\,dt \ge s^k \int_s^{\infty} \varliminf_{p\to\infty} \frac{f\circ F_{x_n x_{n+p}}(t)}{t}\,dt =$$
$$= s^k \int_s^{\infty} \frac{1}{t}\, f\Big(\varliminf_{p\to\infty} F_{x_n x_{n+p}}(t)\Big)\,dt\,, \qquad \forall\, n \ge n_0(\epsilon),\ \ \forall\, s > 0\ .$$

It can be proved that $\lim\limits_{p\to\infty} F_{x_n x_{n+p}}(t) = F_{x_n x_0}(t)$ (actually we 'll use only the fact that $\underline{\lim}_{p\to\infty} F_{x_n x_{n+p}}(t) \geq F_{x_n x_0}(t)$) and the previous relation becomes

$$\epsilon \geq s^k \int_s^\infty \frac{f \circ F_{x_n x_0}(t)}{t}\, dt\,, \qquad \forall\, n \geq n_0(\epsilon), \quad \forall\, s > 0\;,$$

which implies

$$\rho_k(x_n, x_0) = \sup_{s>0} s^k \int_s^\infty \frac{f \circ F_{x_n x_0}(t)}{t}\, dt \leq \epsilon\,, \qquad \forall\, n \geq n_0(\epsilon)\;.$$

Thus the lemma is proved. ∎

3 – A fixed point theorem and some consequences

It is well-known that a mapping $A\colon\, X \to X$ (where $(X, \mathcal{F})$ is a PM-space) is called *s-contraction* if there exists $L \in (0,1)$ such that $F_{AxAy}(Lt) \geq F_{xy}(t)$ for all $t \in \mathbb{R}$, for all $x, y \in X$.

Lemma 3.1. *If $(X, \mathcal{F})$ is a probabilistic f-metric structure and A is an s-contraction then A is, for each $k > 0$, a strict contraction in (X, ρ_k).*

Proof: Since $F_{AxAy}(Lt) \geq F_{xy}(t)$ for some $L \in (0,1)$, and every real t then we have

$$s^k \int_s^\infty \frac{f \circ F_{AxAy}(Lt)}{t}\, dt \leq s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t}\, dt\;.$$

If we make $Lt = u$ in the left side, then we obtain

$$\begin{aligned}\frac{1}{L^k}(sL)^k \int_{sL}^\infty \frac{f \circ F_{AxAy}(u))}{u}\, du &\leq s^k \int_s^\infty \frac{f \circ F_{xy}(t))}{t}\, dt \\ &\leq \sup_{s>0} s^k \int_s^\infty \frac{f \circ F_{xy}(t))}{t}\, dt = d_k(x,y)\,, \qquad \forall\, s > 0\;.\end{aligned}$$

Therefore, if we take $\sup\limits_{s>0}$ in the first member of the above inequality, then we obtain that $\dfrac{1}{L^k}\, d_k(Ax, Ay) \leq d_k(x,y)$ and it is clear that

$$\rho_k(Ax, Ay) \leq L_1\, \rho_k(x,y) \quad \text{ where } \; L_1 = L^{\frac{k}{k+1}} \in (0,1) \tag{3.1}$$

and the lemma is proved. ∎

Now, we can prove our main result:

Theorem 3.2. *Let $(X,\mathcal{F},T)$ be a complete Menger space with $T \geq T_f$. If there exists some $k > 0$ such that for every pair $(x,y) \in X$ one has*

$$\sup_{s>0} s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t}\,dt < \infty \ , \tag{3.2}$$

then every s-contraction on X has a unique fixed point.

Proof: The relation (3.2) shows that ρ_k is a metric. From Lemma 3.1 we obtain that A is a strict contraction in (X,ρ_k). Let $x \in X$ be an arbitrary point. From (3.1) we have that $(A^i x)$ is a $\mathcal{U}_{\rho_k}$-Cauchy sequence. By using the Lemma 2.3, we observe that $(A^i x)$ is a ρ_k-convergent sequence to x_0. It is easy to see that x_0 is the unique fixed point of A. ∎

Corollary 3.3 (cf. [10]). *Let $(X,\mathcal{F},T)$ be a complete Menger space under $T \geq T_f$, where $f(0) < \infty$ and suppose that for each pair $(x,y) \in X^2$ there exists t_{xy} for which $F_{xy}(t_{xy}) = 1$. Then every s-contraction on X has a unique fixed point.*

Proof: Since for $s \leq t_{xy}$ we have

$$\begin{aligned} 0 \leq s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t}\,dt &= s^k \int_s^{t_{xy}} \frac{f \circ F_{xy}(t)}{t}\,dt \leq \\ &\leq s^k \int_s^{t_{xy}} \frac{f \circ F_{xy}(s)}{t}\,dt \leq s^k\, f(0)\Big(\ln(t_{xy}) - \ln(s)\Big) \end{aligned}$$

and for $s > t_{xy}$ we have $s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t}\,dt = 0$ then (3.2) holds and we can apply the theorem. ∎

Corollary 3.4 ([6]). *Let $(X,\mathcal{F},T)$ be a complete Menger space with $T \geq T_1$ such that for some $k > 0$ and every pair $(x,y) \in X$ one has*

$$\sup_{s>0} s^k \int_s^\infty \frac{1 - F_{xy}(t)}{t}\,dt < \infty \ . \tag{3.3}$$

Then every s-contraction on X has a unique fixed point.

Proof: We take $f(t) = f_1(t) = 1 - t$ and we apply the theorem. ∎

Corollary 3.5. *Let $(X,\mathcal{F},T)$ be a complete Menger space under $T \geq T_1$ and suppose that there exists $k>0$ such that every F_{xy} has a finite k-moment. Then every s-contraction on X has a unique fixed point.*

Proof: It is well-known that $(\mu_k)_{xy}^k = \int\limits_0^\infty t^{k-1}(1-F_{xy}(t))\,dt < \infty$. Therefore

$$s^k \int\limits_s^\infty \frac{1-F_{xy}(t)}{t}\,dt \leq \int\limits_s^\infty t^k\,\frac{1-F_{xy}(t)}{t}\,dt = \int\limits_s^\infty t^{k-1}(1-F_{xy}(t))\,dt \leq (\mu_k)_{xy}^k < \infty$$

and the corollary follows. ∎

Remark 3.6. For $k=1$ it can be obtained a known result (see [11, Corollary 2.2]).

Generally from the fixed point alternative ([3]) we obtain the following

Theorem 3.7. *Let $(X,\mathcal{F},T)$ be a complete Menger space under $T \geq T_f$ and A an s-contraction. Then for each $x \in X$ either,*

- **i)** *there is some $k>0$ such that $(A^i x)$ is ρ_k-convergent to the unique fixed point of A, or*
- **ii)** *for all $k>0$, for all $n \in \mathbb{N}$ and for all $M>0$ there exists $s := s(k,n,M)$ such that*

$$s^k \int\limits_s^\infty \frac{f \circ F_{A^n x A^{n+1} x}(t)}{t}\,dt > M\ .$$

Proof: We suppose that ii) is not true:

$$\exists\, k>0,\ \exists\, n_0>0,\ \exists\, M>0,\ \forall\, s>0 \quad \text{such that}\quad s^k \int\limits_s^\infty \frac{f \circ F_{A^{n_0} x A^{n_0+1} x}(t)}{t}\,dt \leq M\ .$$

So, we have for some $k>0$, $\rho_k(A^{n_0}x, A^{n_0+1}x) < \infty$. It follows that

$$\forall\, p>0, \quad \rho_k(A^{n_0}x, A^{n_0+p}x) \leq \sum_{i=0}^{p-1} \rho_k(A^{n_0+p}x, A^{n_0+p+1}x) \leq$$

$$\leq (1+L_1+L_2+\ldots+L_1^{p-1})\,\rho_k(A^{n_0}x, A^{n_0+1}x) = \frac{1-L_1^p}{1-L_1}\,\rho_k(A^{n_0}x, A^{n_0+1}x) \leq$$

$$\leq \frac{\rho_k(A^{n_0}x, A^{n_0+1}x)}{1-L_1} < \infty\ ,$$

where $L_1 := L^{\frac{k}{k+1}} < 1$. Therefore, the sequence of successive approximations, $(A^i x)$ is a ρ_k-Cauchy sequence. From Lemma 2.3 we obtain that $(A^i x)$ is ρ_k-convergent and it is easy to see that the limit of the sequence $(A^i x)$ is the unique fixed point of A. ∎

REFERENCES

[1] Constantin, Gh. and Istrăţescu, I. – *Elements of Probabilistic Analysis with Applications*, Ed. Acad. and Kluwer Academic Publishers, 1989.

[2] Hadžić, O. – *Fixed Point Theory in Probabilistic Metric Spaces*, University of Novi Sad, 1995.

[3] Margolis, B. and Diaz, J.B. – A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.,* 74 (1968), 305–309.

[4] Radu, V. – *A Remark on Contractions in Menger Spaces*, Seminarul de Teoria Probabilităţilorşi Aplicaţii(=STPA), Universitatea din Timişoara, Nr.64, 1983.

[5] Radu, V. – *Some fixed Point Theorems in Probabilistic Metric Spaces*, in "Lectures Notes in Math.", Vol.1223, pp.125–133, Springer-Verlag, New York, 1987.

[6] Radu, V. – *Lectures on probabilistic analysis*, in "Surveys, Lectures Notes and Monographs", Series on Probability, Statistics and Applied Mathematics, 2, Universitatea de Vest din Timişoara, (Seminarul de Teoria Probabilităţilorşi Aplicaţii), 1994.

[7] Schweizer, B., Sherwood, H. and Tardiff, R. – Comparing two contraction notions on probabilistic metric spaces, *Stochastica,* 12 (1988).

[8] Schweizer, B. and Sklar, A. – *Probabilistic Metric Spaces*, North-Holland, New York, 1983.

[9] Sehgal, V.M. and Bharucha-Reid, A.T. – Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory,* 6 (1972), 97–102.

[10] Sherwood, H. – Complete probabilistic metric spaces, *Z. Wahrsch. Verw. Geb.,* 20 (1971), 117–128.

[11] Tardiff, R.M. – Contraction maps on probabilistic metric spaces, *Journal Math. Anal. Appl.,* (1990), 517–523.

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