PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 1 – 1997

SPACE CURVES AND THEIR DUALS

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In the real projective plane P^2, the duality between lines and points induces a map δ from the set of smooth ($\mathbf{C}^\infty$) immersions $f\colon \mathbb{R} \to P^2$ to the set of all smooth maps $g\colon \mathbb{R} \to P^2$. Thus $\delta(f) = g$, where for all $s \in \mathbb{R}$, $g(s)$ is the polar of the tangent line to f at $f(s)$. In order that g itself be an immersion, it is necessary to restrict f to have nowhere zero geodesic curvature. The map δ is then an involution on the set of such immersions.

In this paper, we examine these ideas in the slightly broader setting of smooth immersions $f\colon \mathbb{R} \to E^3$ in Euclidean 3-space. In particular, suppose that M and N are smooth surfaces in E^3 such that, for any immersion $f\colon \mathbb{R} \to E^3$, $f(\mathbb{R}) \subset M$ implies $f_*(\mathbb{R}) \subset N$, and vice-versa, where f_* is defined in §1. Then M and N are either both spheres with centre 0 or both cones with apex 0. If M is the unit sphere S^2 or is the circular cone of apex angle $\pi/2$ then $M = N$. Accordingly, we concentrate attention on these cases.

1 – The dual of a space curve

Let $f\colon \mathbb{R} \to E^3$ be a smooth immersion. Then we can define a unit tangent vector field t along $f(\mathbb{R})$ by $t(s) = f'(s)/\|f'(s)\|$, $s \in \mathbb{R}$. The *dual* $\delta(f) = f_*\colon \mathbb{R} \to E^3$ of f is then given by

$$f_* = f \wedge t\ .$$

Of course, although f_*, is smooth, it need not be an immersion. Thus $f'_* = f' \wedge t + f \wedge t' = f \wedge t'$, so f_* is an immersion iff, for all $s \in \mathbb{R}$, $f(s)$ and $t'(s)$ are linearly independent.

Received: October 8, 1995; *Revised*: June 22, 1996.

For such an f,

$$\begin{aligned} f_{**} &= (f_*)_* = f_* \wedge (f_*'/\|f_*'\|) \\ &= \frac{1}{\|f \wedge t'\|}\,(f \wedge t) \wedge (f \wedge t') \\ &= \frac{1}{\|f \wedge t'\|}\,(f \wedge t.t')f - (f \wedge t.f)\,t' \\ &= -\cos\theta\, f\ , \end{aligned}$$

where θ is the angle between $f \wedge t'$ and t, $0 \leq \theta \leq \pi$. This shows that for at least some immersions f, δ has an involutory character.

We now examine the case of immersions $f\colon \mathbb{R} \to S^2$ where $\theta = 0$ or π.

2 – Curves on S^2

Let $f : \mathbb{R} \to S^2$ be a smooth immersion into the unit sphere. Then f_* is a smooth immersion iff the geodesic curvature $\kappa_g(s)$ of f at s is nonzero, for all $s \in \mathbb{R}$, since $\nu(s)\,\kappa_g(s) = f(s).(t(s) \wedge t'(s)) \neq 0$ iff $t'(s)$ is not perpendicular to S^2 at $f(s)$, with $\nu(s) = \|f'(s)\|$.

Suppose, then, that G denotes the set of all smooth immersions $f\colon \mathbb{R} \to S^2$ for which κ_g is nowhere zero. Then G is the disjoint union of G_+ and G_- where $f \in G_+$ or G_- according as $\kappa_g > 0$ or $\kappa_g < 0$. Trivially, the antipodal involution $\alpha\colon G \to G$, given by $\alpha(f)(s) = -f(s)$, interchanges G_+ and G_-.

Proposition 1. *For all $f \in G$, $f_{**} = f$ if $f \in G_+$ and $f_{**} = -f$ if $f \in G_-$.*

Proof: We have shown in §1 above that $f_{**} = (-\cos\theta)\,f$, where θ is the angle between $f \wedge t'$ and t. Since $\kappa_g = \frac{1}{\nu}\,f.(t \wedge t') = -\frac{1}{\nu}(f \wedge t').t$ and $|\kappa_g| = \frac{1}{\nu}\,\|f \wedge t'\|$, where $\nu = \|f'\|$ is the velocity function as above, the result follows. ∎

From $f_* = f \wedge t$ and $|\kappa_g| = \frac{1}{\nu}\,\|f \wedge t'\|$, it follows immediately that $\|f_*'\| = \nu|\kappa_g|$.

Corollary 1. *There is a well-defined map $\delta\colon G \to G$ given by $\delta(f) = f_*$.*

Proof: We want to show that $f_* \in G$ for all $f \in G$. Since $f \in G$ implies $f \wedge t'$ is nowhere zero, we know that f_* is a smooth immersion. Also $\|f_*\| = \|f \wedge t\| = 1$, since $f.t = 0$ and $\|f\| = \|t\| = 1$. By Proposition 1, $f_{**} = \pm f$, so f_{**} is a smooth immersion. Hence $f_* \in G$. ∎

Corollary 2. $\delta(G) = G_+$.

Proof: If $f \in G_+$, then $-f \in G_-$ and if $f \in G_-$, $-f \in G_+$. Also, for any $f \in G$, $(-f)_* = f_*$. Suppose that for some $f \in G_+$, $f_* \in G_-$. Then $(f_*)_{**} = -f_* \in G_+$, by Proposition 1. But $(f_*)_{**} = (f_{**})_* = f_* \in G_-$, by hypothesis. So $f_* = -f_*$, which is a contradiction. It follows that $f_* \in G_+$, if $f \in G_+$. Likewise, if $f \in G_-$, then $-f \in G_+$ and $f_* = (-f)_* \in G_+$. ∎

Corollary 3. *$\delta|G_+$ is a fixed-point free involution.*

Although $\delta|G_+$ has no fixed elements, it does map each circle of radius $\sqrt{2}/2$ to itself, and each circle of radius r_1, $0 < r_1 < 1$ to the parallel circle of radius r_2 in the same hemisphere, where $r_1^2 + r_2^2 = 1$.

3 – Multiple points and homotopy

Let us now concentrate on smooth closed curves on S^2. Thus we confine attention to smooth immersions $f\colon \mathbb{R} \to S^2$ that are periodic. Denote by C the set of all such curves that are *nondegenerate* in the sense of Little [1]. That is to say, $f \in C$ iff it is periodic and $f \in G$. Denote by C_+ and C_- the sets of periodic elements of G_+ and G_-. Now regard C as a subset of the space S of $\mathbf{C}^2$ periodic nondegenerate immersions $f\colon \mathbb{R} \to S^2$, with the $\mathbf{C}^2$ topology. Then Little showed that, with obvious notation, each of S_+ and S_- has exactly three path components. Equivalently, there are exactly three nondegenerate regular homotopy classes on S_+ and S_-. These six classes are represented by curves of the form indicated in Figure 1 for plane projection from a hemisphere of S^2.

Let C_+^i denote the subsets of C_+ consisting of curves in the class of types i, $i = 1, 2, 3$.

Proposition 2. *If $f \in C_+^i$, then $f_* \in C_+^i$, $i = 1, 2, 3$.*

Proof: This follows from work of Little [1], as we now explain. Let $f \in C_+$ and suppose that $s, u \in \mathbb{R}$ with $s \neq u$. Then $f_*(s) = f_*(u)$ iff $f(s) \wedge t(s) = f(u) \wedge t(u)$. Thus $f_*(s) = f_*(u)$ iff the great circle that is tangent to f at $f(s)$ and oriented in the direction of $t(s)$ is also tangent to f at $f(u)$ in the direction of $t(u)$.

We may suppose without loss of generality that f is self-transverse (modulo periodicity) and that it has only doubly tangent great circles of the above type. That is, we may suppose that both f and f_* are self-transverse.

If f has 0, 1 or 2 double points, then f_* has 0, 1 or 4 such double points, as indicated in Figure 2. A procedure explained by Little then shows that $f_* \in C^1_+$, C^2_+ or C^3_+, respectively and the proposition follows, since δ is a homeomorphism of $C_+ \subset S_+$ to itself.

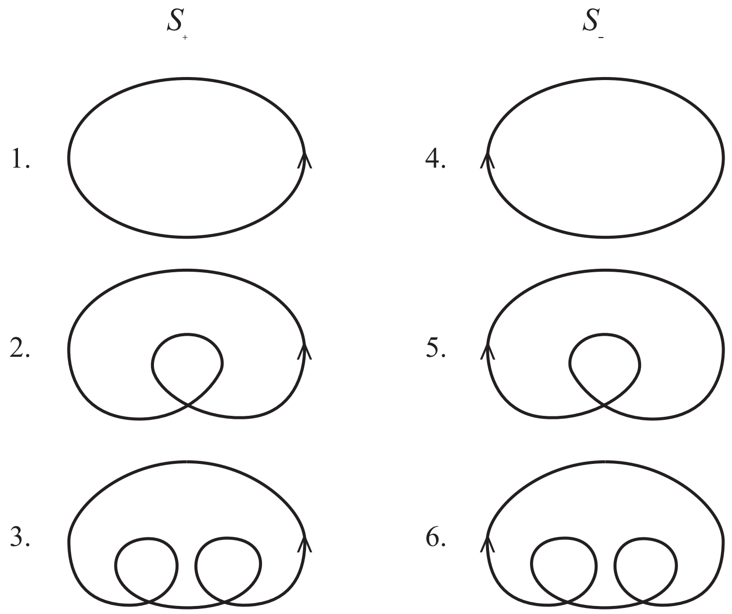

Fig. 1 – Nondegenerate regular homotopy classes of closed curves on S^2.

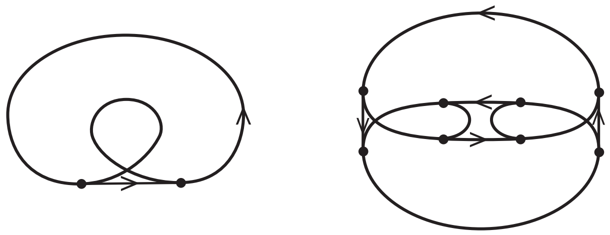

Fig. 2 – Multiple points and oriented double tangents.

Similar arguments apply to C^i_-, where we find that $f \in C^i_-$ implies that $f_* \in C^i_+$. ∎

4 – Duality on a cone

The results obtained above depend to some extent on the fact that the origin O has a privileged position in relation to S^2. Another surface where O is a centre of symmetry is a right circular cone C with apex O. For convenience, let the axis of C be the z-axis in E^3. To make things work better, we also suppose that the apex angle of C is $\frac{1}{2}\pi$. The surface C is given, therefore, by the equation

$$x^2 + y^2 = z^2\,, \qquad z \neq 0\ .$$

Let $f\colon \mathbb{R} \to$ C be a smooth immersion. Then there are smooth functions z and θ such that, for all $s \in \mathbb{R}$,

$$f(s) = \Big(z(s)\cos\theta(s),\, z(s)\sin\theta(s),\, z(s)\Big)\ ,$$

$z(s) \neq 0$. It follows that

$$f' = \Big(z'\cos\theta - z\,\theta'\sin\theta,\, z'\sin\theta + z\,\theta'\cos\theta,\, z'\Big)\ ,$$

so

$$\|f'\|^2 = 2z'^2 + z^2\,\theta'^2\ ,$$

and

$$f_* = f \wedge t = \frac{1}{\|f'\|}\,z^2\,\theta'(-\cos\theta, -\sin\theta, 1)$$

is well-defined as a smooth map $f_*\colon \mathbb{R} \to$ C, provided that θ' is nowhere zero.

Moreover, $f'_* = f \wedge t'$ implies that $f'_* = 0$ at $s \in \mathbb{R}$ iff $f(s)$ and $t'(s)$ are linearly dependent. Since we shall require that θ' is nowhere zero, f is transverse to the generators of C and hence the normal curvature of f is nowhere zero. We conclude that $t'(s)$ is nowhere zero, so $f'_*(s) \neq 0$ for all $s \in \mathbb{R}$. Hence f_* is a smooth immersion, transverse to the generators of C.

We have now shown that there is a well-defined map $\gamma\colon K \to K$ of the set K of smooth immersions of $\mathbb{R}$ into C, transverse to its generators, into itself, given by $\gamma(f) = f_*$.

Now C has two components or sheets C_+ and C_- given by $z > 0$ and $z < 0$ respectively. So K may be partitioned into four disjoint subsets K_{pq}, where $p = \pm 1$ according as $z > 0$ or $z < 0$ and $q = \pm 1$ according as $\theta' > 0$ or $\theta' < 0$, for any $f \in K_{pq}$.

The following proposition is easy to establish.

Proposition 3. *$\gamma(K_{++} \cup K_{-+}) \subset K_{++}$, and $\gamma(K_{--} \cup K_{+-}) \subset K_{--}$.*

We do not know whether either inclusion is strict.

The map γ cannot be an involution, even on, say K_{++}, as we now show.

Proposition 4. *For all $f \in K$ and all $s \in \mathbb{R}$, $\|f_*(s)\| \le \|f(s)\|$ with equality iff $z'(s) = 0$.*

Proof: Since $f_*(s) = f(s) \wedge t(s)$,

$$\|f_*(s)\| = \|f(s)\|\,\|t(s)\| \sin\phi(s) = \|f(s)\| \sin\phi(s) \ ,$$

where $\phi(s)$ is the angle between $t(s)$ and $f(s)$. ∎

Proposition 4 shows that if $f \in K$ is such that z has a critical point at $s \in \mathbb{R}$, then with the obvious notation, $z_*(s) = z(s)$. If s is not a critical point of z, however, then $|z_*(s)| < |z(s)|$.

So if γ is a closed curve on C_+ then the range of values of $z(s)$, $s \in \mathbb{R}$, is a compact interval $[a,b]$, where $a < b$ except when γ is a 'circle of latitude'. For such γ, the range of $z(s)$ for the n-th iteration γ^n of γ, is $[a_n, b]$, where $a_{n+1} < a_n < a$, for sufficiently large n. We do not know whether $\alpha = \lim_{n\to\infty} a_n$ must be 0 or whether it can be positive.

ACKNOWLEDGEMENT – The first-named author gratefully acknowledges partial financial support from Academia das Ciências, Lisboa, and The Royal Society of London.

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