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HOMOGENIZATION OF ELLIPTIC EQUATIONS WITH QUADRATIC GROWTH IN PERIODICALLY PERFORATED DOMAINS: THE CASE OF UNBOUNDED SOLUTIONS

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Abstract: This paper is devoted to the homogenization of the following non linear problem

$$\begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)Du_\varepsilon\right)+H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)=f & \text{in } \Omega_\varepsilon, \\ \left(A\left(\dfrac{x}{\varepsilon}\right)Du_\varepsilon\right)\underline{\mu}=0 & \text{on } \partial T_\varepsilon, \\ u_\varepsilon=0 & \text{on } \partial\Omega, \\ u_\varepsilon\in H^1(\Omega_\varepsilon),\quad H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)\in L^1(\Omega_\varepsilon),\quad H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)u_\varepsilon\in L^1(\Omega_\varepsilon)\ , \end{cases}$$

where $\Omega_\varepsilon=\Omega-T_\varepsilon$ is obtained by removing from a bounded open set Ω of $\mathbb{R}^n$ a closed set T_ε of ε-periodic balls of size ε, $H(y,s,\xi)$ is $]0,1[^n$-periodic in y, has the same sign as s and has a quadratic growth with respect to ξ, and f belongs to $L^2(\Omega)$. (The corresponding problem with bounded solutions has been treated by P. Donato, A. Gaudiello and L. Sgambati in [11]).

We prove that the linear part gives the homogenized matrix of the linear part and the nonlinear one changes into $H^0(u,Du)$, where H^0 is defined by

$$H^0(s,\xi)=\int_{]0,1[^n-\overline{T}}H(y,s,C(y)\xi)\,dy\qquad\forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n\ ,$$

with $C(\frac{x}{\varepsilon})$ the corrector matrices of the linear problem and T the reference hole.

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0 – Introduction

This paper is devoted to the study of the asymptotic behaviour, as ε tends to zero, of the solutions u_ε of

$$
(0.1)\quad \begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)Du_\varepsilon\right)+H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)=f & \text{in } \Omega_\varepsilon,\\[2mm] \left(A\left(\dfrac{x}{\varepsilon}\right)Du_\varepsilon\right)\underline{\mu}=0 & \text{on } \partial T_\varepsilon,\\[2mm] u_\varepsilon=0 & \text{on } \partial\Omega,\\[2mm] u_\varepsilon\in H^1(\Omega_\varepsilon),\quad H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)\in L^1(\Omega_\varepsilon),\quad H\left(\dfrac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon\right)u_\varepsilon\in L^1(\Omega_\varepsilon)\ , \end{cases}
$$

where Ω is a bounded open set of $\mathbb{R}^n$, $\Omega_\varepsilon = \Omega - T_\varepsilon$ is a domain obtained by removing from Ω a closed set T_ε of ε-periodic balls of size ε, $A(y)$ is a $]0,1[^n$-periodic bounded definite positive matrix, $H(y,s,\xi)$ is a Caratheodory function defined on $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $]0,1[^n$-periodic in y, with the same sign as s (sign hypothesis) and with quadratic growth with respect to ξ (see assumptions (1.7)–(1.10)), f belongs to $L^2(\Omega)$ and $\underline{\mu}$ denotes the unitary external normal vector with respect to Ω_ε.

The following main result is proved in Theorem 1.3 for a suitable class of extension-operators $\{P_\varepsilon\}_\varepsilon$, up to a subsequence of $\{\varepsilon\}$,

$$
P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } \ H_0^1(\Omega)
$$

as ε tends to zero, and

$$
\begin{cases} -\operatorname{div}(A^0Du)+H^0(u,Du)=\theta f & \text{in } \ \Omega\ ,\\ u\in H_0^1(\Omega),\quad H^0(u,Du)\in L^1(\Omega),\quad H^0(u,Du)u\in L^1(\Omega)\ , \end{cases}
$$

where $-\operatorname{div}(A^0Du)$ is the homogenized operator of the linear part and H^0 is defined by

$$
H^0(s,\xi)=\int_{]0,1[^n-\overline{T}} H(y,s,C(y)\xi)\,dy \qquad \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n\ ,
$$

with $C(\frac{\cdot}{\varepsilon})$ the corrector matrices of the linear problem, $T\subset\subset\,]0,1[^n$ the reference hole and $\theta=|]0,1[^n-T|$.

If the limit problem has a unique solution, then we need not pass to a subsequence of $\{\varepsilon\}$.

The general notion of H-convergence and correctors have been introduced by F. Murat and L. Tartar in [13]. We refer also to [7] and [8] for the case of the perforated domains.

The extension-operators have been introduced by D. Cioranescu and J. Saint Jean Paulin in [9].

This paper is a sequel of [11]. In [11] the homogenization of a problem analogous to (0.1) is studied, but in the case where u_ε is bounded while no sign condition is imposed on H. In the present paper we emphasize the main changes and we refer to [11] for the remaining common parts.

The homogenization in a fixed domain of the nonlinear problem with quadratic growth in the gradient was treated by A. Bensoussan, L. Boccardo, A. Dall'Aglio and F. Murat in [2] both in the bounded case and in the unbounded case. Also the homogenization of the nonlinear problem with subquadratic growth in the gradient has been studied by L. Boccardo and T. Del Vecchio in [4] for a fixed domain and by P. Donato and L. Sgambati in [12] for periodically perforated domains.

Here we adapt some ideas introduced for a fixed domain in [2] and partially developed for a perforated domain in [11]. The proof of the main result is based on verifying that the correctors for the nonlinear problem are the same as for corresponding linear problem.

We refer to [1], [3], [10] and [14] for a detailed bibliography on the homogenization theory.

Contents :

1 – Position of the problem and the main result

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$, $Y =]0,1[^n$ the reference cell, $T \subset\subset Y$ an open subset with smooth boundary ∂T and ε a parameter taking values in a decreasing positive sequence which tends to zero.

Assume that for every ε there exists a subset K_ε of Z^n such that:

$$\Omega \cap \bigcup_{k \in Z^n} \varepsilon(k+\overline{T}) = \bigcup_{k \in K_\varepsilon} \varepsilon(k+\overline{T}) \ .$$

Then, for every ε, we define the perforated domain Ω_ε by:

$$T_\varepsilon = \bigcup_{k \in K_\varepsilon} \varepsilon(k + \overline{T}), \qquad \Omega_\varepsilon = \Omega - T_\varepsilon$$

and we introduce the space

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon) \colon v|_{\partial\Omega} = 0 \right\}$$

equipped with the H^1 norm.

In the following we denote by

- χ_E the characteristic function of a subset E of $\mathbb{R}^n$,
- $|E|$ the Lebesgue measure of a Lebesgue-measurable subset E of $\mathbb{R}^n$,
- $\widetilde{v}$ or $v^\sim$ the zero extension on Ω of every (vector) function v defined on Ω_ε,
- $\underline{\mu}$ the unitary external normal vector with respect to $Y - T$ or Ω_ε according to the situation.

Moreover we recall that

$$\chi_{\Omega_\varepsilon} \rightharpoonup \theta = |Y - T| \quad \text{in} \ \ L^\infty(\Omega) \ \text{weak}^* \tag{1.1}$$

as ε tends to zero.

We now introduce a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension-operators such that for every ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)) \ , \\ (P_\varepsilon v)|_{\Omega_\varepsilon} = v \qquad \forall\, v \in V_\varepsilon \ , \\ \|D(P_\varepsilon v)\|_{(L^2(\Omega))^n} \leq c \|Dv\|_{(L^2(\Omega_\varepsilon))^n} \qquad \forall\, v \in V_\varepsilon \ , \\ \qquad \text{where } c \text{ is a constant independent of } \varepsilon \end{cases} \tag{1.2}$$

and, for every g in $H^{-1}(\Omega)$, as in [12] we define $P_\varepsilon^* g$ in V_ε' as follows:

$$P_\varepsilon^* g \colon v \in V_\varepsilon \to \langle g, P_\varepsilon v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \in \mathbb{R}^n \ . \tag{1.3}$$

Remark 1.1. The existence of a sequence $\{P_\varepsilon\}_\varepsilon$ satisfying (1.2) is proved in [9]. Moreover (1.2) provides the Poincaré and Sobolev inequalities in V_ε with a constant independent of ε.

Let $A(y) = (a_{ij}(y))_{ij}$ be an $n \times n$ matrix-valued function defined on $\mathbb{R}^n$ such that

$$\tag{1.4} \begin{cases} A \in (L^\infty(\mathbb{R}^n))^{n^2} \ , \\ A \ \ Y\text{-periodic} \ , \\ \exists\, \alpha > 0\colon \displaystyle\sum_{i,j=1}^{n} a_{ij}(y)\, \lambda_i\, \lambda_j \geq \alpha |\lambda|^2 \,, \quad y \ \text{ a.e. in } \mathbb{R}^n, \ \ \forall\, \lambda \in \mathbb{R}^n \ , \end{cases}$$

and let us denote for every ε

$$\tag{1.5} A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. in } \ \mathbb{R}^n \ .$$

Moreover let us give a Caratheodory function H defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ such that for y a.e. in $\mathbb{R}^n$, for every s and $\overline{s}$ in $\mathbb{R}$ and for every ξ and $\overline{\xi}$ in $\mathbb{R}^n$

$$\tag{1.6} |H(y, s, \xi)| \leq b_2(|s|)\,(1 + |\xi|^2) \ ,$$

$$\tag{1.7} |H(y, s, \xi) - H(y, s, \overline{\xi})| \leq b_1(|s|)\,(1 + |\xi| + |\overline{\xi}|)\, |\xi - \overline{\xi}| \ ,$$

$$\tag{1.8} |H(y, s, \xi) - H(y, \overline{s}, \xi)| \leq b_2(|s - \overline{s}|)\,(1 + |\xi|^2) \ ,$$

$$\tag{1.9} H(x, s, \xi)\, s \geq 0 \ ,$$

$$\tag{1.10} H(\cdot, s, \xi) \ \ Y\text{-periodic} \ ,$$

where b_1 and b_2 are continuous increasing functions with $b_1(0) \geq 0$ and $b_2(0) = 0$.

Remark 1.2. Observe that (1.6) follows from (1.8) and (1.9) by virtue of the continuity of $H(x, \cdot, \xi)$ in $\mathbb{R}$.

For every ε, set

$$\tag{1.11} H_\varepsilon(x, s, \xi) = H\left(\frac{x}{\varepsilon}, s, \xi\right), \quad x \ \text{ a.e. in } \Omega_\varepsilon, \ \ \forall\, (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \ .$$

In this paper we study the asymptotic behaviour, as ε tends to zero, of the solutions u_ε of the following problem:

$$\tag{1.12} \begin{cases} \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon\, Du_\varepsilon\, Dv\, dx + \int_{\Omega_\varepsilon} H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, v\, dx = \int_{\Omega_\varepsilon} f\, v\, dx \\ \hfill \forall\, v \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon) \ , \\ u_\varepsilon \in V_\varepsilon, \ \ H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \in L^1(\Omega_\varepsilon), \ \ H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, u_\varepsilon \in L^1(\Omega_\varepsilon) \ , \end{cases}$$

with f in $L^2(\Omega)$.

Following the same outline as [4] and [5], it is easy to prove that problem (1.12) admits a solution u_ε. Moreover [6] shows that every solution of (1.12) can be used as test function in (1.12). Then, taking $v = u_\varepsilon$ in (1.12) and using (1.4) and (1.9), we obtain the following a priori estimates:

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c_1 \ , \tag{1.13}$$

$$\int_{\Omega_\varepsilon} H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, u_\varepsilon \, dx \leq c_1 \ , \tag{1.14}$$

where c_1 is a positive constant independent of ε.

As a consequence of (1.2) and (1.13) we deduce that, for some subsequence (still denoted $\{\varepsilon\}$),

$$\begin{cases} P_\varepsilon u_\varepsilon \rightharpoonup u & \text{weakly in } \ H_0^1(\Omega) \ , \\ P_\varepsilon u_\varepsilon \to u & \text{strongly in } \ L^2(\Omega) \ , \\ P_\varepsilon u_\varepsilon \to u & \text{a.e. in } \ \Omega \ , \end{cases} \tag{1.15}$$

as ε tends to zero.

To describe the problem satisfied by u, we use the $n \times n$ matrix-valued function $A^0 = (a^0_{ij})_{ij}$ and $C(y) = (c_{ij}(y))_{ij}$ defined respectively by

$$a^0_{ij} = \int_{Y-\overline{T}} a_{ij}(y) - \sum_{k=1}^{n} a_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \, dy \tag{1.16}$$

and

$$c_{ij}(y) = \delta_{ij} - \frac{\partial \chi_j}{\partial y_i} \qquad \text{a.e. in } \ Y - \overline{T} \ , \tag{1.17}$$

in terms of the solution χ_j, $j = 1, ..., n$, of

$$\begin{cases} -\operatorname{div}\Big(A(y)\, D(y_i - \chi_j)\Big) = 0 & \text{in } \ Y - \overline{T} \ , \\ \Big(A(y)\, D(y_i - \chi_j)\Big) \cdot \underline{\mu} = 0 & \text{on } \ \partial T \ , \\ \chi_j \quad Y\text{-periodic} \ , \end{cases} \tag{1.18}$$

where $A(y) = (a_{ij}(y))_{ij}$ is given in (1.4).

Then we set

$$H^0(s,\xi) = \int_{Y-\overline{T}} H(y, s, C(y)\xi)\, dy \qquad \forall\, (s,\xi) \in \mathbb{R} \times \mathbb{R}^n \ , \tag{1.19}$$

where H is the Caratheodory function satisfying (1.7)–(1.10).

Now we can state the main result of this paper.

Theorem 1.3. *Let $\{A^\varepsilon\}_\varepsilon$ be the sequence of $n\times n$ matrix-valued functions defined by (1.5), $\{H_\varepsilon\}_\varepsilon$ be the sequence of Caratheodory functions defined by (1.11) under assumptions (1.7)–(1.10) and $\{u_\varepsilon\}_\varepsilon$ a sequence of solutions of (1.12).*

Let $\{P_\varepsilon\}_\varepsilon$ be a sequence of linear extension-operators satisfying (1.2), A^0 be the $n\times n$ matrix defined in (1.16), H^0 the function given in (1.19) and $\theta = |Y-T|$.

Then there exists a subsequence (still denoted $\{\varepsilon\}$) and a function u in $H^1_0(\Omega)$ such that, as ε tends to zero,

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega)\ . \tag{1.20}$$

The function u satisfies the following:

$$\begin{cases} \displaystyle\int_\Omega A^0\, Du\, Dv\, dx + \int_\Omega H^0(u,Du)\, v\, dx = \int_\Omega \theta\, f\, v\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, v\in H^1_0(\Omega)\cap L^\infty(\Omega)\ , \\ u\in H^1_0(\Omega), \quad H^0(u,Du)\in L^1(\Omega), \quad H^0(u,Du)\, u\in L^1(\Omega)\ , \\ \displaystyle\int_\Omega A^0\, Du\, Du\, dx + \int_\Omega H^0(u,Du)\, u\, dx = \int_\Omega \theta\, f\, u\, dx\ . \end{cases} \tag{1.21}$$

Moreover, as ε tends to zero, the following convergences hold:

$$-\operatorname{div}\Big(A^\varepsilon(x)\,(Du_\varepsilon)^\sim\Big) \to -\operatorname{div}(A^0\, Du) \quad \text{strongly in } H^{-1}(\Omega)\ , \tag{1.22}$$

$$H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)^\sim \rightharpoonup H^0(u,Du) \quad \text{weakly in } L^1(\Omega)\ . \tag{1.23}$$

The proof of Theorem 1.3 follows essentially the same outline as that of Theorem 1.3 in [11]. Consequently we shall give a detailed proof of the main changes and refer to [11] for the remaining common parts.

Remark 1.4. The function

$$H(y,s,\xi) = \Big(h(y) + |\xi|^2\Big)\, g(s)\ , \qquad y \text{ a.e. in } \mathbb{R}^n,\quad \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$$

with h a positive bounded measurable Y-periodic function and g an increasing Lipschitz continuous function on $\mathbb{R}$, such that $g(0)=0$ (for example $g(s) = \operatorname{arctg}(s)$), satisfies assumptions (1.6)–(1.10). In this case

$$H^0(s,\xi) = g(s)\int_{Y-\overline{T}} \Big(h(y) + |C(y)\,\xi|^2\Big)\, dy \qquad \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n\ .$$

2 – A corrector result

In this Section we give a corrector result for problem (1.12).

Let $C(y)$ be defined by (1.17). Then, for every ε, we set

$$C^\varepsilon(x) = C\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. in } \Omega_\varepsilon . \tag{2.1}$$

First we recall a result proved in [11] (Proposition 2.1 and Theorem 2.3).

Proposition 2.1 [11]. *Let $\{A^\varepsilon\}_\varepsilon$ be defined in (1.5), $\{P_\varepsilon\}_\varepsilon$ satisfy (1.2), $\{P^*_\varepsilon g\}_\varepsilon$ be defined in (1.3), for g in $H^{-1}(\Omega)$, and, for every ε, let w_ε be the unique solution of the following linear problem:*

$$\begin{cases} -\operatorname{div}\Big(A^\varepsilon(x)\, Dw_\varepsilon\Big) = P^*_\varepsilon g & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial\Omega, \\ \Big(A^\varepsilon(x)\, Dw_\varepsilon\Big)\cdot \underline{\mu} = 0 & \text{on } \partial T_\varepsilon \; . \end{cases}$$

Moreover let A^0 and $\{C^\varepsilon\}_\varepsilon$ be defined in (1.16) and (2.1) respectively.

Then, as ε tends to zero, the following holds:

$$P_\varepsilon w_\varepsilon \rightharpoonup w \quad \text{weakly in } H^1_0(\Omega) \; ,$$

$$\begin{cases} \limsup\limits_{\varepsilon\to 0} \|Dw_\varepsilon - C^\varepsilon\phi\|_{(L^2(\Omega_\varepsilon))^n} \le c\|Dw-\phi\|_{(L^2(\Omega))^n} \quad \forall\,\phi\in (C^\infty_0(\Omega))^n, \\ \qquad \text{with } c \text{ a constant independent of } \phi \; , \end{cases}$$

where w is the unique solution of:

$$\begin{cases} -\operatorname{div}(A^0\, Dw) = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \; . \end{cases}$$

Moreover, if g is in $W^{-1,\infty}(\Omega)$, then

$$A^\varepsilon (Dw_\varepsilon)^\sim (Dw_\varepsilon)^\sim \rightharpoonup A^0\, Dw\, Dw \quad \text{weakly in } L^1(\Omega) \; ,$$

as ε tends to zero.

In the sequel we make use of the following known lemma:

Lemma 2.2. *As ε tends to zero, let $\{g_\varepsilon\}_\varepsilon$, be a sequence of functions which converges weakly in $L^1(\Omega)$ to a function g_0 and let $\{t_\varepsilon\}_\varepsilon$ be a sequence of equibounded and measurable functions which converges almost everywhere in Ω to a function t_0. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega g_\varepsilon\, t_\varepsilon\, dx = \int_\Omega g_0\, t_0\, dx \ .$$

From now on, $\{\varepsilon\}$ denotes a subsequence for which (1.15) holds.

Now we state the corrector result for problem (1.12).

Proposition 2.3. *Under assumptions (1.4), (1.5), (1.8), (1.9), (1.10) and (1.11), let $\{u_\varepsilon\}_\varepsilon$ be a subsequence of solutions of problem (1.12) for which convergence (1.15) holds, u be the function defined in (1.15) and $\{C^\varepsilon\}_\varepsilon$ be defined in (2.1).*

Then

$$\begin{cases} \limsup\limits_{\varepsilon\to 0} \|Du_\varepsilon - C^\varepsilon\phi\|_{(L^2(\Omega_\varepsilon))^n} \le c\|Du-\phi\|_{(L^2(\Omega))^n} \qquad \forall\,\phi\in(C_0^\infty(\Omega))^n, \\ \qquad\qquad \text{where } c \text{ is a constant independent of } \phi \ . \end{cases}$$

Proposition 2.3 is an immediate consequence of Proposition 2.1 and of the following result.

Theorem 2.4. *Under assumptions (1.4), (1.5), (1.8), (1.9), (1.10) and (1.11), let $\{u_\varepsilon\}_\varepsilon$ be a subsequence of solutions of problem (1.12) for which convergence (1.15) holds and let $\{v_\varepsilon\}_\varepsilon$ be the sequence of the solutions of the following linear problem:*

$$\begin{cases} -\operatorname{div}\Big(A^\varepsilon(x)\, Dv_\varepsilon\Big) = P_\varepsilon^*\Big(-\operatorname{div}(A^0\, Du)\Big) & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \\ \Big(A^\varepsilon(x)\, Dv_\varepsilon\Big)\cdot \underline{\mu} = 0 & \text{on } \partial T_\varepsilon \ , \end{cases}$$

with $\{A_\varepsilon\}_\varepsilon$ defined in (1.5), $\{P_\varepsilon\}_\varepsilon$ satisfying (1.2), A^0 given in (1.16) and u defined in (1.15).

Then

$$\lim_{\varepsilon\to 0}\|D(u_\varepsilon - v_\varepsilon)\|_{(L^2(\Omega_\varepsilon))^n} = 0 \ .$$

Proof: The proof of this theorem follows the outline of the proof of Proposition 4.1 in [2], modified as follows due to the presence of holes.

Let $\{u_\lambda\}_{\lambda\in\mathbb{N}}$ be a sequence in $C_0^\infty(\Omega)$ such that

$$u_\lambda \to u \quad \text{strongly in } \ H_0^1(\Omega) \quad \text{as } \ \lambda \to +\infty \tag{2.2}$$

and let, for every λ in $\mathbb{N}$, $\{v_{\lambda,\varepsilon}\}_\varepsilon$ be the sequence of the solution of

$$\begin{cases} -\operatorname{div}\Big(A^\varepsilon(x)\, Dv_{\lambda,\varepsilon}\Big) = P_\varepsilon^*\Big(-\operatorname{div}(A_0\, Du_\lambda)\Big) & \text{in } \Omega_\varepsilon, \\ v_{\lambda,\varepsilon} = 0 & \text{on } \partial\Omega, \\ \Big(A^\varepsilon(x)\, Dv_{\lambda,\varepsilon}\Big)\, \underline{\mu} = 0 & \text{on } \partial T_\varepsilon \ . \end{cases} \tag{2.3}$$

Then Proposition 2.1 implies that for every λ in $\mathbb{N}$,

$$P_\varepsilon v_{\lambda,\varepsilon} \rightharpoonup u_\lambda \quad \text{weakly in } \ H_0^1(\Omega) \ \text{ as } \varepsilon \to 0 \ . \tag{2.4}$$

Moreover it is easy to prove that (see proof of (3.6) in [11]), for every λ in $\mathbb{N}$ and ε,

$$\begin{cases} \|Dv_{\lambda,\varepsilon} - Dv_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n} \le c\|Du_\lambda - Du\|_{(L^2(\Omega))^n}, \\ \quad \text{where } c \text{ is a constant independent of } \lambda \text{ and } \varepsilon \ . \end{cases} \tag{2.5}$$

Set

$$\varphi_\mu\colon\ s \in \mathbb{R} \to s\, e^{\mu s^2} \in \mathbb{R}$$

with μ a positive parameter (depending on k) to be defined and for every k in $\mathbb{R}^+$

$$T_k\colon\ s \in \mathbb{R} \to \max\Big\{-k, \min\{s,k\}\Big\} \in [-k,k] \ .$$

Then the function $\varphi_\mu(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon}))$ belongs to $V_\varepsilon \cap L^\infty(\Omega_\varepsilon)$. Consequently, if we choose $\varphi_\mu(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon}))$ as test function in (1.12), we obtain, for every λ in $\mathbb{N}$ and ε,

$$\begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon\, Du_\varepsilon\, D\Big(\varphi_\mu(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon}))\Big)\, dx \, + \\ &\qquad + \int_{\Omega_\varepsilon} H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\mu\Big(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon})\Big)\, dx = \\ &= \int_{\Omega_\varepsilon} f\, \varphi_\mu\Big(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon})\Big)\, dx \ , \end{aligned}$$

i.e.

$$\tag{2.6} \begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon\, D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx\, + \\ &\quad + \int_{\Omega_\varepsilon} A^\varepsilon\, Dv_{\lambda,\varepsilon}\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx \\ &\quad - \int_{\Omega_\varepsilon} A^\varepsilon\, D\Big(G_k(v_{\lambda,\varepsilon})\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx \\ &\quad + \int_{\Omega_\varepsilon} A^\varepsilon D\Big(G_k(u_\varepsilon)\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx \\ &\quad + \int_{\Omega_\varepsilon} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\, \varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, dx = \\ &\qquad\qquad = \int_{\Omega_\varepsilon} f\, \varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, dx\ , \end{aligned}$$

where G_k is the function defined by

$$G_k\colon\ s \in \mathbb{R}\ \to\ s - T_k(s) \in \mathbb{R}\ .$$

Now we give an estimate from below for the left hand-side of (2.6).

Since φ'_μ is a positive function, by virtue of (1.4) and for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$ we obtain the following estimate for the first term in the left hand-side of (2.6):

$$\tag{2.7} \begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon\, D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx = \\ &= \int_{\Omega_\varepsilon} A^\varepsilon\, D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, \varphi'_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, dx \\ &\geq \alpha \int_{\Omega_\varepsilon} \Big|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big|^2\, \varphi'_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, dx\ . \end{aligned}$$

If we choose $\varphi_\mu(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon}))$ as test function in (2.3), for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$ we can rewrite the second term in the left hand-side of (2.6) in the following way:

$$\tag{2.8} \begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon\, Dv_{\lambda,\varepsilon}\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\, dx = \\ &\qquad\qquad = \int_{\Omega} A^0\, Du_\lambda\, D\Big(P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\Big)\, dx\ . \end{aligned}$$

By the definition of G_k, the following equality for the third term in the left hand-side of (2.6) holds for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$:

$$\int_{\Omega_\varepsilon} A^\varepsilon D\Big(G_k(v_{\lambda,\varepsilon})\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\,dx = \\ = \int_{\Omega_\varepsilon} A^\varepsilon Dv_{\lambda,\varepsilon}\, Du_\varepsilon\, \varphi'_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, \chi_{\{|v_{\lambda,\varepsilon}|>k\}}\, \chi_{\{|u_\varepsilon|<k\}}\,dx\ .$$

Thus, by virtue of (1.2), (1.4) and (1.13), we deduce that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\text{(2.9)}\quad \left|\int_{\Omega_\varepsilon} A^\varepsilon D\Big(G_k(v_{\lambda,\varepsilon})\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\,dx\right| \le \\ \le c(\mu,k)\, \|Du_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n} \Big(\int_{\Omega_\varepsilon} |Dv_{\lambda,\varepsilon}|^2\, \chi_{\{|v_{\lambda,\varepsilon}|>k\}}\, \chi_{\{|u_\varepsilon|<k\}}\,dx\Big)^{\frac12} \\ \le c(\mu,k)\, c_1\, \frac{1}{\alpha}\Big(\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\, (Dv_{\lambda,\varepsilon})^\sim \chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|>k\}}\, \chi_{\{|P_\varepsilon u_\varepsilon|<k\}}\,dx\Big)^{\frac12},$$

where $c(\mu,k)$ is a positive constant dependent only on μ and k.

The fourth term in the left hand-side of (2.6) can be treated similarly. Then for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$, it results

$$\text{(2.10)}\quad \left|\int_{\Omega_\varepsilon} A^\varepsilon D\Big(G_k(u_\varepsilon)\Big)\, D\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\,dx\right| \le \\ \le c(\mu,k)\, \frac{c_1}{\alpha}\Big(\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\, (Dv_{\lambda,\varepsilon})^\sim\, \chi_{\{|P_\varepsilon u_\varepsilon|>k\}}\, \chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|<k\}}\,dx\Big)^{\frac12}\ .$$

Regarding the last term in the left hand-side of (2.6) we remark that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\text{(2.11)}\quad \int_{\Omega_\varepsilon} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\, \varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\,dx = \\ = \int_{\{|u_\varepsilon|\le k\}} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\, \varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\,dx \\ + \int_{\{|u_\varepsilon|>k\}} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, e^{\mu(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon}))^2}\,dx\ .$$

On the other hand, (1.9) implies that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\text{(2.12)}\quad \int_{\{|u_\varepsilon|>k\}} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\, e^{\mu(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon}))^2}\,dx \ge 0\ .$$

Moreover from (1.2), (1.4) and (1.6) it follows that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\begin{aligned}
(2.13)\qquad & \left|\int_{\{|u_\varepsilon|\le k\}} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\,\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\,dx\right| \le \\
&\le b_2(k)\int_{\{|u_\varepsilon|\le k\}}\Big(1+|D(T_k(u_\varepsilon))|^2\Big)\left|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|dx \\
&\le 2b_2(k)\int_{\{|u_\varepsilon|\le k\}}\left|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|^2\left|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|dx \\
&\quad + b_2(k)\int_{\{|u_\varepsilon|\le k\}}\Big(1+2|Dv_{\lambda,\varepsilon}|^2\Big)\left|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|dx \\
&\le 2b_2(k)\int_{\Omega_\varepsilon}\left|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|^2\left|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|dx \\
&\quad + b_2(k)\int_\Omega \chi_{\Omega_\varepsilon}\left|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\right|dx \\
&\quad + 2b_2(k)\,\frac{1}{\alpha}\int_\Omega A^\varepsilon(Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\left|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\right|dx\ .
\end{aligned}$$

Then from (2.11), (2.12) and (2.13) we deduce that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\begin{aligned}
(2.14)\qquad & \int_{\Omega_\varepsilon} H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\,\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\,dx \ge \\
&\ge -2b_2(k)\int_{\Omega_\varepsilon}\left|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|^2\left|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|dx \\
&\quad - b_2(k)\int_\Omega \chi_{\Omega_\varepsilon}\left|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\right|dx \\
&\quad - 2b_2(k)\,\frac{1}{\alpha}\int_\Omega A^\varepsilon(Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\left|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\right|dx\ .
\end{aligned}$$

Now, combining (2.6) with (2.7), (2.8), (2.9), (2.10) and (2.14) we obtain, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$\begin{aligned}
&\alpha\int_{\Omega_\varepsilon}\left|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\right|^2\varphi_\mu'\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\,dx + \\
&\qquad + \int_\Omega A^0\,Du_\lambda\,D\Big(P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\Big)\,dx -
\end{aligned}$$

$$
\begin{aligned}
&- c(\mu,k)\,\frac{c_1}{\alpha}\Big(\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|>k\}}\,\chi_{\{|P_\varepsilon u_\varepsilon|<k\}}\Big)^{\frac{1}{2}}\\
&- c(\mu,k)\,\frac{c_1}{\alpha}\Big(\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\chi_{\{|P_\varepsilon u_\varepsilon|>k\}}\,\chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|<k\}}\Big)^{\frac{1}{2}}\\
&- 2b_2(k)\int_{\Omega_\varepsilon}\Big|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big|^2\,\Big|\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big|\,dx\\
&- b_2(k)\int_\Omega \chi_{\Omega_\varepsilon}\Big|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\Big|\,dx\\
&- 2b_2(k)\,\frac{1}{\alpha}\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\Big|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\Big|\,dx \le\\
&\qquad\qquad\le \int_\Omega f\,\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\,\chi_{\Omega_\varepsilon}\,dx\ ,
\end{aligned}
$$

from which, choosing μ (depending on k) such that

$$
\alpha\,\varphi'_\mu(s) - 2b_2(k)\,|\varphi_\mu(s)| \ge \frac{\alpha}{2} \qquad \forall\, s\in\mathbb{R}\ ,
$$

it follows that, for every ε, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$
\begin{aligned}
(2.15)\quad &\Big\|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \le\\
&- \frac{2}{\alpha}\int_\Omega A^0\,Du_\lambda\,D\Big(P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\Big)\,dx\\
&+ \frac{2c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A_\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|>k\}}\,\chi_{\{|P_\varepsilon u_\varepsilon|<k\}}\Big)^{\frac{1}{2}}\\
&+ \frac{2c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\chi_{\{|P_\varepsilon u_\varepsilon|>k\}}\,\chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|<k\}}\Big)^{\frac{1}{2}}\\
&+ \frac{2b_2(k)}{\alpha}\int_\Omega \chi_{\Omega_\varepsilon}\Big|P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\Big|\,dx\\
&+ \frac{4b_2(k)}{\alpha^2}\int_\Omega A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim\,\Big|\varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big)\Big|\,dx\\
&+ \frac{2}{\alpha}\int_\Omega f\,P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big)\,\chi_{\Omega_\varepsilon}\,dx\ ,
\end{aligned}
$$

where, now, $c(k)$ is a constant dependent only on k.

In order to pass to the limit, as ε tends to zero, in (2.15), we make some remarks.

Making use of (1.1), (1.2), (1.15) and (2.4) it is easy to prove that (cf. (3.17) in [9]), passing possibly to a subsequence of $\{\varepsilon\}$, for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$,

$$(2.16)\qquad \begin{cases} P_\varepsilon\Big(\varphi_\mu\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big) \to \varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big), \\ \varphi_\mu\Big(T_k(P_\varepsilon u_\varepsilon)-T_k(P_\varepsilon v_{\lambda,\varepsilon})\Big) \to \varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big), \\ \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega\ , \end{cases}$$

as ε tends to zero.

Moreover, from Proposition 2.1, it follows that, for λ in $\mathbb{N}$

$$(2.17)\qquad A^\varepsilon (Dv_{\lambda,\varepsilon})^\sim\,(Dv_{\lambda,\varepsilon})^\sim \rightharpoonup A^0\,Du_\lambda\,Du_\lambda \quad \text{weakly in } \ L^1(\Omega)\ ,$$

as ε tends to zero.

Furthermore, if we set

$$A=\Big\{k\in\mathbb{R}^+ :\ \exists\,\lambda\colon\ \Big|\{|u_\lambda|=k\}\Big|\neq 0\Big\}\cup\Big\{k\in\mathbb{R}^+ :\ \Big|\{|u|=k\}\Big|\neq 0\Big\}\ ,$$

from (1.15) and (2.4) we deduce that, passing possibly to a subsequence of $\{\varepsilon\}$, for every λ in $\mathbb{N}$ and for k in $\mathbb{R}^+ - A$,

$$(2.18)\qquad \begin{cases} \chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|>k\}}\,\chi_{\{|P_\varepsilon u_\varepsilon|<k\}} \ \to\ \chi_{\{|u_\lambda|>k\}}\,\chi_{\{|u|<k\}} \\ \chi_{\{|P_\varepsilon u_\varepsilon|>k\}}\,\chi_{\{|P_\varepsilon v_{\lambda,\varepsilon}|<k\}} \ \to\ \chi_{\{|u|>k\}}\,\chi_{\{|u_\lambda|<k\}} \end{cases} \quad \text{a.e. in } \ \Omega$$

as ε tends to zero.

Observe that, since λ takes values in a sequence and u_λ and u are in $L^1(\Omega)$, it results that $|A|=0$. Consequently (2.18) holds for a.e. k in $\mathbb{R}^+$.

Then passing to the limit, as ε tends to zero, in (2.15) and making use of (1.1), (2.16), (2.17), (2.18) and Lemma 2.2, we obtain, for every λ in $\mathbb{N}$ and for a.e. k in $\mathbb{R}^+$,

$$(2.19)\qquad \limsup_{\varepsilon\to 0}\Big\|D\Big(T_k(u_\varepsilon)-T_k(v_{\lambda,\varepsilon})\Big)\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \le$$

$$\le -\frac{2}{\alpha}\int_\Omega A^0\,Du_\lambda\,D\Big(\varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big)\Big)\,dx$$

$$+\frac{2c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A^0\,Du_\lambda\,Du_\lambda\,\chi_{\{|u_\lambda|>k\}}\,\chi_{\{|u|<k\}}\,dx\Big)^{\frac12}+$$

$$+\frac{2c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A^0\,Du_\lambda\,Du_\lambda\,\chi_{\{|u|>k\}}\,\chi_{\{|u_\lambda|<k\}}\,dx\Big)^{\frac{1}{2}}$$
$$+\frac{2b_2(k)}{\alpha}\int_\Omega \theta\Big|\varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big)\Big|\,dx$$
$$+\frac{4b_2(k)}{\alpha^2}\int_\Omega A^0\,Du_\lambda\,Du_\lambda\,\Big|\varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big)\Big|\,dx$$
$$+\frac{2}{\alpha}\int_\Omega f\,\varphi_\mu\Big(T_k(u)-T_k(u_\lambda)\Big)\,\theta\,dx\ ,$$

where $c(k)$ is a constant dependent on k only.

Now we prove that for every λ in $\mathbb{N}$ and k in $\mathbb{R}^+$

$$\limsup_{\varepsilon\to 0}\Big\|D(G_k(u_\varepsilon))\Big\|^2_{(L^2(\Omega_\varepsilon))^n}\le\frac{1}{\alpha}\int_\Omega f\,\theta\,G_k(u)\,dx \tag{2.20}$$

and

$$\limsup_{\varepsilon\to 0}\Big\|D(G_k(v_{\lambda,\varepsilon}))\Big\|^2_{(L^2(\Omega_\varepsilon))^n}\le\frac{1}{\alpha}\int_\Omega A^0\,Du_\lambda\,G_k(u_\lambda)\,dx\ . \tag{2.21}$$

Fix k and choose $v=T_n(G_k(u_\varepsilon))$, $n\in\mathbb{N}$, as test function in (1.12). Then from (1.2) and (1.9) it follows that

$$\int_{\Omega_\varepsilon} A^\varepsilon\,Du_\varepsilon\,D\Big(T_n(G_k(u_\varepsilon))\Big)\,dx\le\int_\Omega f\,T_n(G_k(P_\varepsilon u_\varepsilon))\,\chi_{\Omega_\varepsilon}\,dx\ . \tag{2.22}$$

Passing to the limit, as $n\to+\infty$, in (2.22), by virtue of definition of G_k and (1.4) it results

$$\begin{aligned}\alpha\int_{\Omega_\varepsilon}\Big|D(G_k(u_\varepsilon))\Big|^2\,dx&\le\int_{\Omega_\varepsilon} A^\varepsilon\,D(G_k(u_\varepsilon))\,D(G_k(u_\varepsilon))\,dx\\&=\int_{\Omega_\varepsilon} A^\varepsilon\,Du_\varepsilon\,D(G_k(u_\varepsilon))\,dx\\&\le\int_\Omega f(G_k(P_\varepsilon u_\varepsilon))\,\chi_{\Omega_\varepsilon}\,dx\ .\end{aligned} \tag{2.23}$$

Then passing to the limit, as ε tends to zero, in (2.23), by (1.1) and (1.15) we obtain (2.20).

To prove (2.21) fix k and choose $G_k(v_{\lambda,\varepsilon})$ as test function in (2.3). Then, by definition of G_k and (1.4) it follows

$$\begin{aligned}\alpha\int_{\Omega_\varepsilon}\Big|D(G_k(v_{\lambda,\varepsilon}))\Big|^2\,dx&\le\int_{\Omega_\varepsilon} A^\varepsilon\,D(G_k(v_{\lambda,\varepsilon}))\,D(G_k(v_{\lambda,\varepsilon}))\,dx\\&=\int_{\Omega_\varepsilon} A^\varepsilon\,Dv_{\lambda,\varepsilon}\,D(G_k(v_{\lambda,\varepsilon}))\,dx\\&=\int_\Omega A^0\,Du_\lambda\,D\Big(P_\varepsilon(G_k(v_{\lambda,\varepsilon}))\Big)\,dx\end{aligned}$$

from which passing to the limit, as ε tends to zero, by virtue of (1.2) and (2.4) we obtain (2.21).

Now combining (2.5) with (2.19), (2.20) and (2.21) it follows that for every λ in $\mathbb{N}$ and for a.e. k in $\mathbb{R}^+$

$$
\begin{aligned}
(2.24)\qquad & \limsup_{\varepsilon\to 0} \|Du_\varepsilon - Dv_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^n} \le \\
&\le 2\limsup_{\varepsilon\to 0}\Big\|Dv_\varepsilon - Dv_{\lambda,\varepsilon}\Big\|^2_{(L^2(\Omega_\varepsilon))^n} + 2\limsup_{\varepsilon\to 0}\Big\|Dv_{\lambda,\varepsilon} - Du_\varepsilon\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \\
&\le 2c\Big\|Du_\lambda - Du\Big\|^2_{(L^2(\Omega))^n} + 8\limsup_{\varepsilon\to 0}\Big\|D\Big(T_k(u_\varepsilon) - T_k(v_{\lambda,\varepsilon})\Big)\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \\
&\quad + 8\limsup_{\varepsilon\to 0}\Big\|D(G_k(u_\varepsilon))\Big\|^2_{(L^2(\Omega_\varepsilon))^n} + 4\limsup_{\varepsilon\to 0}\Big\|D\Big(G_k(v_{\lambda,\varepsilon})\Big)\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \\
&\le 2c\Big\|Du_\lambda - Du\Big\|^2_{(L^2(\Omega))^n} - \frac{16}{\alpha}\int_\Omega A^0\, Du_\lambda\, D\Big(\varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big)\Big)\, dx \\
&\quad + \frac{16c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A^0\, Du_\lambda\, Du_\lambda\, \chi_{\{|u_\lambda|>k\}}\, \chi_{\{|u|<k\}}\, dx\Big)^{\frac12} \\
&\quad + \frac{16c(k)\,c_1}{\alpha^2}\Big(\int_\Omega A^0\, Du_\lambda\, Du_\lambda\, \chi_{\{|u|>k\}}\, \chi_{\{|u_\lambda|<k\}}\, dx\Big)^{\frac12} \\
&\quad + \frac{16b_2(k)}{\alpha}\int_\Omega \theta\Big|\varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big)\Big|\, dx \\
&\quad + \frac{32b_2(k)}{\alpha^2}\int_\Omega A^0\, Du_\lambda\, Du_\lambda\, \Big|\varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big)\Big|\, dx \\
&\quad + \frac{16}{\alpha}\int_\Omega f\, \varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big)\,\theta\, dx + \frac{8}{\alpha}\int_\Omega f\,\theta\, G_k(u)\, dx \\
&\quad + \frac{4}{\alpha}\int_\Omega A^0\, Du_\lambda\, D(G_k(u_\lambda))\, dx\ ,
\end{aligned}
$$

where $c(k)$ is a constant dependent only on k.

Observe, now, that, by virtue of (2.2), for a.e. k in $(0,+\infty)$

$$
(2.25)\qquad
\begin{cases}
\varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big) \rightharpoonup 0 & \text{weakly in } H^1_0(\Omega), \\
\Big|\varphi_\mu\Big(T_k(u) - T_k(u_\lambda)\Big)\Big| \rightharpoonup 0 & \text{weakly* in } L^\infty(\Omega), \\
\chi_{\{|u_\lambda|>k\}}\,\chi_{\{|u|<k\}} \rightharpoonup 0 & \text{weakly* in } L^\infty(\Omega), \\
\chi_{\{|u|>k\}}\,\chi_{\{|u_\lambda|<k\}} \rightharpoonup 0 & \text{weakly* in } L^\infty(\Omega), \\
G_k(u_\lambda) \rightharpoonup G_k(u) & \text{weakly in } H^1_0(\Omega)\ ,
\end{cases}
$$

as λ tends to infinity.

Then passing to the limit, as λ tends to infinity, in (2.24) and making use of (2.2) and (2.25) we have that for a.e. k in $\mathbb{R}^+$,

$$\begin{aligned}(2.26)\qquad \limsup_{\varepsilon\to 0}\Big\|Du_\varepsilon - Dv_\varepsilon\Big\|^2_{(L^2(\Omega_\varepsilon))^n} \le \\ \le \frac{8}{\alpha}\int_\Omega f\,\theta\, G_k(u)\,dx + \frac{4}{\alpha}\int_\Omega A^0\,Du\,D(G_k(u))\,dx\ .\end{aligned}$$

Finally passing to the limit, as k tends to infinity, in (2.26) we obtain the desired result. ∎

The following result is an immediate consequence of Proposition 2.6 in [9].

Proposition 2.5. *Let $\{H_\varepsilon\}_\varepsilon$ be the sequence of Caratheodory functions defined by (1.11) under assumptions (1.7)–(1.10), H^0 be the function given in (1.19) and let $\{C^\varepsilon\}_\varepsilon$ be defined by (2.1) under assumption (1.4).*

Then H^0 satisfies (1.7)–(1.9) (up to a multiplicative constant β) and

$$H_\varepsilon(x, T_k\varphi_\varepsilon, C^\varepsilon\phi)^\sim \rightharpoonup H^0(T_k\varphi_0,\phi) \quad \text{weakly in } L^1(\Omega)$$

as ε tends to zero, for k in $\mathbb{R}^+$, for ϕ in $(C_0^\infty(\Omega))^n$ and for sequence $\{\varphi_\varepsilon\}_\varepsilon$ of measurable functions on Ω such that

$$\varphi_\varepsilon \to \varphi_0 \quad \text{a.e. in } \Omega\ ,$$

where

$$T_k\colon\ s\in\mathbb{R}\to\max\Big\{-k,\min\{s,k\}\Big\}\ .$$

3 – Proof of Theorem 1.3

Let $\{\varepsilon\}$ be a subsequence for which (1.15) holds and let u and H^0 be defined by (1.15) and (1.19) respectively.

From (1.15) and Proposition 2.5 it follows that, for k in $\mathbb{R}^+$

$$(3.1)\qquad H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon\phi\Big)^\sim \rightharpoonup H^0(T_k u,\phi) \quad \text{weakly in } L^1(\Omega)$$

as ε tends to zero, for ϕ in $(C_0^\infty(\Omega))^n$.

Let us prove that, for k in $\mathbb{R}^+$,

$$(3.2)\qquad H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon\Big)^\sim \rightharpoonup H^0(T_k u, Du) \quad \text{weakly in } L^1(\Omega)$$

as ε tends to zero.

Let $\{\phi_h\}_{h\in\mathbb{N}}$ be a sequence in $(C_0^\infty(\Omega))^n$ such that

$$\phi_h \to Du \quad \text{strongly in} \quad (L^2(\Omega))^n \ . \tag{3.3}$$

Fix k in $\mathbb{R}^+$. Then (1.7) and Proposition 2.5 imply that, for φ in $L^\infty(\Omega)$,

$$\begin{aligned}
&\left|\int_\Omega \Big(H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon\Big)^\sim - H^0\Big(T_k(u), Du\Big)\Big)\varphi\, dx\right| \leq \\
&\quad \leq \int_{\Omega_\varepsilon}\Big|H_\varepsilon(x, T_k u_\varepsilon, Du_\varepsilon) - H_\varepsilon(x, T_k u_\varepsilon, C^\varepsilon \phi_h)\Big|\,|\varphi|\, dx \\
&\quad\quad + \left|\int_\Omega\Big(H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon\phi_h\Big)^\sim - H^0(T_k u, \phi_h)\Big)\varphi\, dx\right| \\
&\quad\quad + \int_\Omega \Big|H^0(T_k u, \phi_h) - H^0(T_k u, Du)\Big|\,|\varphi|\, dx \\
&\quad \leq b_1(k)\int_{\Omega_\varepsilon}\Big(1 + |Du_\varepsilon| + |C^\varepsilon\phi_h|\Big)\,|Du_\varepsilon - C^\varepsilon\phi_h|\,|\varphi|\, dx \\
&\quad\quad + \left|\int_\Omega\Big(H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon\phi_h\Big)^\sim - H^0(T_k u, \phi_h)\Big)\varphi\, dx\right| \\
&\quad\quad + \beta\, b_1(k)\int_\Omega \Big(1 + |\phi_h| + |Du|\Big)\,|\phi_h - Du|\,|\varphi|\, dx \ ,
\end{aligned} \tag{3.4}$$

for every ε and h. Moreover from (1.13) and Proposition 2.3 it follows that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Big(1 + |Du_\varepsilon| + |C^\varepsilon\phi_h|\Big)\,|Du_\varepsilon - C^\varepsilon\phi_h|\,|\varphi|\, dx \leq \\
&\quad \leq \limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Big(1 + 2|Du_\varepsilon| + |C^\varepsilon\phi_h - Du_\varepsilon|\Big)\,|Du_\varepsilon - C^\varepsilon\phi_h|\,|\varphi|\, dx \\
&\quad \leq c\limsup_{\varepsilon\to 0}\Big(\Big\|Du_\varepsilon - C^\varepsilon\phi_h\Big\|_{(L^2(\Omega_\varepsilon))^n} + \Big\|Du_\varepsilon - C^\varepsilon\phi_h\Big\|^2_{(L^2(\Omega_\varepsilon))^n}\Big) \\
&\quad \leq c\Big(\Big\|Du - \phi_h\Big\|_{(L^2(\Omega))^n} + \Big\|Du - \phi_h\Big\|^2_{(L^2(\Omega))^n}\Big),
\end{aligned} \tag{3.5}$$

where c is a constant independent of h.

Then from (1.15), Proposition 2.5, (3.3), (3.4) and (3.5) we obtain (3.2).

Now we prove that

$$H^0(u, Du) \in L^1(\Omega)\,, \qquad H^0(u, Du)\, u \in L^1(\Omega) \ . \tag{3.6}$$

By virtue of (1.9) and (1.14) it results that, for every ε and for every k in $\mathbb{R}^+$,

$$(3.7)\qquad \int_\Omega H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon\Big)^\sim T_k(P_\varepsilon u_\varepsilon)\, \chi_{\{|P_\varepsilon u_\varepsilon|<k\}}\, dx =$$

$$= \int_{\{|P_\varepsilon u_\varepsilon|<k\}} H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)^\sim P_\varepsilon u_\varepsilon\, dx$$

$$\leq \int_\Omega H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)^\sim P_\varepsilon u_\varepsilon\, dx \leq c_1 \ .$$

On the other hand, from (1.15) it follows that, passing possibly to a subsequence of $\{\varepsilon\}$, for a.e. k in $\mathbb{R}^+$,

$$(3.8)\qquad \chi_{\{|P_\varepsilon u_\varepsilon|<k\}} \to \chi_{\{|u|<k\}} \quad \text{a.e. in } \ \Omega \ ,$$

$$(3.9)\qquad T_k(P_\varepsilon u_\varepsilon) \to T_k(u) \quad \text{a.e. in } \ \Omega \ ,$$

as ε tends to zero.

Then passing to the limit, as ε tends to zero, in (3.7) and making use of (3.2), (3.8), (3.9) and Lemma 2.2 we have that, for a.e. k in $\mathbb{R}^+$,

$$(3.10)\qquad \int_\Omega H^0(u, Du)\, u\, \chi_{\{|u|<k\}}\, dx = \int_\Omega H^0(T_k u, Du)\, T_k u\, \chi_{\{|u|<k\}}\, dx \leq c_1 \ .$$

Passing to the limit in (3.10) as k goes to infinity, by virtue of sign property for H^0 and of Beppo Levi's Theorem, we deduce that

$$(3.11)\qquad H^0(u, Du)\, u \in L^1(\Omega)\ , \qquad \int_\Omega H^0(u, Du)\, u\, dx \leq c_1 \ .$$

Moreover, for k fixed we have

$$\Big|H^0(u, Du)\Big| = \Big|H^0(u, Du)\, \chi_{\{|u|<k\}} + \frac{1}{u}\, H^0(u, Du)\, u\, \chi_{\{|u|\geq k\}}\Big|$$

$$\leq \Big|H^0(T_k u, Du)\, \chi_{\{|u|<k\}}\Big| + \frac{1}{k}\, H^0(u, Du)\, u \ ,$$

which, by virtue of (3.2) and (3.11), implies

$$H^0(u, Du) \in L^1(\Omega) \ .$$

So we have obtained (3.6). Now we prove (1.23).

At first let us observe that for ε and for ϕ in $L^\infty(\Omega)$,

$$(3.12)\qquad \int_\Omega H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)^\sim \phi\, dx =$$

$$= \int_{\{|P_\varepsilon u_\varepsilon|<k\}} H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon\Big)^\sim \phi\, dx + \int_{\{|P_\varepsilon u_\varepsilon|\geq k\}} H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)^\sim \phi\, dx \ ,$$

$$\int_\Omega H^0(u,Du)\,\phi\,dx = \int_{\{|u|<k\}} H^0(T_k u, Du)\,\phi\,dx + \int_{\{|u|\geq k\}} H^0(u,Du)\,\phi\,dx\ . \tag{3.13}$$

On the other hand from (3.2), (3.8) and Lemma 2.2, we deduce that, for a.e. k in $\mathbb{R}^+$,

$$\lim_{\varepsilon\to 0}\int_{\{|P_\varepsilon u_\varepsilon|<k\}} H_\varepsilon\Big(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon\Big)^{\sim}\phi\,dx = \int_{\{|u|<k\}} H^0(T_k u, Du)\,\phi\,dx\ . \tag{3.14}$$

Moreover (3.6) implies that

$$\lim_{k\to+\infty}\int_{\{|u|\geq k\}} H^0(u,Du)\,\phi\,dx = 0 \tag{3.15}$$

whereas (1.9) and (1.14) imply that, for every ε and k,

$$\int_{\{|P_\varepsilon u_\varepsilon|\geq k\}} \Big|H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)^{\sim}\Big|\,dx \leq \frac{c_1}{k}\ . \tag{3.16}$$

Then combining (3.12) with (3.13), (3.14), (3.15) and (3.16) we obtain (1.23). To prove (1.22) observe that, for every ε,

$$\begin{aligned} -\operatorname{div}\Big(A^\varepsilon(Du_\varepsilon)^{\sim}\Big) &= -\operatorname{div}\Big(A^\varepsilon(Du_\varepsilon - Dv_\varepsilon)^{\sim}\Big) + \Big(-\operatorname{div}(A^\varepsilon(Dv_\varepsilon))^{\sim}\Big) \\ &\quad - P_\varepsilon^*\Big(-\operatorname{div}(A^0\,Du)\Big) + P_\varepsilon^*\Big(-\operatorname{div}(A^0\,Du)\Big)\ , \end{aligned} \tag{3.17}$$

where v_ε is the solution of the auxiliary problem given in Theorem 2.4. By virtue of Theorem 2.4 the first term in the right hand-side of (3.17) converges to zero strongly in $H^{-1}(\Omega)$, as ε tends to zero. Note also that, from the definition of V_ε, the second term in the right hand-side of (3.17) is zero. Consequently, since the last term in the right hand-side of (3.17) converges to $-\operatorname{div}(A^0\,Du)$ strongly in $H^{-1}(\Omega)$ as ε tends to zero (see [11]), (1.22) holds.

Finally combining (1.22) with (1.23), (1.1) and by virtue of [6] we have that u is a solution of (1.21). ∎

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