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ON DECOMPOSABLY REGULAR OPERATORS

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Abstract: Let X be a complex Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X. $T \in \mathcal{L}(X)$ is said to be *decomposably regular* provided there is an operator S such that S is invertible in $\mathcal{L}(X)$ and $TST = T$. For each $T \in \mathcal{L}(X)$ we introduce the following subset $\rho_{gr}(T)$ of the resolvent set of T: $\mu \in \rho_{gr}(T)$ if and only if there is a neighbourhood U of μ and a holomorphic function $F\colon U \to \mathcal{L}(X)$ such that $F(\lambda)$ is invertible for all $\lambda \in U$ and $(T-\lambda)\,F(\lambda)\,(T-\lambda) = T-\lambda$ on U. In this note we determine the interior points of the class of decomposably regular operators and we prove a spectral mapping theorem for $\mathbb{C}\backslash\rho_{gr}(T)$.

1 – Terminology

X always denotes an infinite-dimensional complex Banach space, and the Banach algebra of all bounded linear operators on X is denoted by $\mathcal{L}(X)$.

If $T \in \mathcal{L}(X)$ we denote by $N(T)$ the kernel of T and by $\alpha(T)$ the dimension of $N(T)$. $T(X)$ denotes the range of T, and we define $\beta(T) = \operatorname{codim} T(X)$ $(= \dim X/T(X))$. We write $\sigma(T)$ for the spectrum of T and $\rho(T)$ for the resolvent set $\mathbb{C}\backslash\sigma(T)$.

The maximal group of invertible elements in $\mathcal{L}(X)$ is denoted by $\mathcal{G}(X)$. Let $\mathcal{R}(X)$ denote the set of all relatively regular operators in $\mathcal{L}(X)$, that is, operators T such that $TST = T$ for some $S \in \mathcal{L}(X)$. Observe that $T \in \mathcal{R}(X)$ if and only if T has complemented kernel and range ([1], p. 10). $T \in \mathcal{L}(X)$ is called *decomposably regular* if $T = TST$ for some $S \in \mathcal{G}(X)$. Let us write $\mathcal{GR}(X)$ for the class of all decomposably regular operators. In [8] decomposably regular operators are called *unit regular*. For a subset $\mathcal{M}$ of $\mathcal{L}(X)$ let $\operatorname{cl}\mathcal{M}$ and $\operatorname{int}\mathcal{M}$ denote, respectively, the closure and the interior of $\mathcal{M}$.

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Proposition 1.1.

(**1**) $\mathcal{GR}(X) = \mathcal{R}(X) \cap \operatorname{cl} \mathcal{G}(X)$.

(**2**) $T \in \mathcal{GR}(X) \Leftrightarrow T \in \mathcal{R}(X)$, *$N(T)$ and $X/T(X)$ are isomorphic.*

Proof: (**1**) [2], Theorem 1.1. (**2**) [3], Theorem 3.8.6. ∎

If X is a separable Hilbert space, then part (2) of the above proposition shows that $T \in \mathcal{GR}(X)$ if and only if $T(X)$ is closed and $\alpha(T) = \beta(T)$. $T \in \mathcal{L}(X)$ is said to be an *Atkinson operator* if $T \in \mathcal{R}(X)$ and at least one of $\alpha(T)$, $\beta(T)$ is finite. $\mathcal{A}(X)$ denotes the set of Atkinson operators. For $T \in \mathcal{A}(X)$ we define the index of T by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

We call $T \in \mathcal{L}(X)$ *Fredholm operator* if $\alpha(T)$ and $\beta(T)$ are both finite. Observe that a Fredholm operator is relatively regular ([6], Satz 74.4). For $n \in \mathbb{Z}$ let $\mathcal{F}_n(X) = \{T \in \mathcal{L}(X) \colon T \text{ is Fredholm and } \operatorname{ind}(T) = n\}$. Denote by $\mathcal{F}(X)$ the set of all Fredholm operators, thus $\mathcal{F}(X) = \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n(X)$.

It is well known that $\mathcal{A}(X)$, $\mathcal{F}(X)$ and $\mathcal{F}_n(X)$ are open subsets of $\mathcal{L}(X)$ ([6], Satz 82.4).

Proposition 1.2. $\operatorname{int} \mathcal{R}(X) = \mathcal{A}(X) = \{T \in \mathcal{L}(X) \colon T + K \in \mathcal{R}(X)$ *for all compact operators* $K\}$.

Proof: [14], Theorem 6. ∎

Remarks.

(**1**) We call $T \in \mathcal{L}(X)$ a *semi-Fredholm operator* if $T(X)$ is closed and at least one of $\alpha(T)$, $\beta(T)$ is finite. We have shown in [14] that

$$\begin{aligned}\operatorname{int}\Big\{T \in \mathcal{L}(X) \colon T(X) \text{ is closed}\Big\} &= \Big\{T \in \mathcal{L}(X) \colon T \text{ is semi-Fredholm}\Big\} = \\ &= \Big\{T \in \mathcal{L}(X) \colon T + K \text{ has closed range for each compact } K\Big\} .\end{aligned}$$

(**2**) If X is a Hilbert space then $\mathcal{A}(X) = \{T \in \mathcal{L}(X) \colon T \text{ is semi-Fredholm}\}$. In this case M. Mbekhta ([9], Théorème 2.2) has shown that $\operatorname{int} \mathcal{R}(X) = \mathcal{A}(X)$.

(**3**) If X is a Hilbert space and $T \in \operatorname{int} \mathcal{R}(X)$ then there is $\delta > 0$ and a *meromorphic* function $F \colon \{\lambda \in \mathbb{C} \colon |\lambda| < \delta\} \to \mathcal{L}(X)$ such that

$$(T - \lambda)\, F(\lambda)\, (T - \lambda) = T - \lambda \quad \text{for} \quad |\lambda| < \delta$$

(see [9], Corollaire 2.3).

2 – Interior points of some classes of relatively regular operators

Proposition 1.1 (2) shows that we have for a Fredholm operator T : $T \in \mathcal{GR}(X) \Leftrightarrow T \in \mathcal{F}_0(X)$. We can be more precise:

Theorem 2.1. $\operatorname{int} \mathcal{GR}(X) = \mathcal{F}_0(X)$.

Proof: Since $\mathcal{F}_0(X)$ is open and $\mathcal{F}_0(X) \subseteq \mathcal{GR}(X)$, we only have to show that $\operatorname{int} \mathcal{GR}(X) \subseteq \mathcal{F}_0(X)$.

Let $T \in \operatorname{int} \mathcal{GR}(X)$, then $T \in \operatorname{int} \mathcal{R}(X)$, thus $T \in \mathcal{A}(X)$ by Proposition 1.2. We have $T \in \operatorname{cl} \mathcal{G}(X)$ (Proposition 1.1(1)), thus there is a sequence (T_n) in $\mathcal{G}(X)$ such that $\|T - T_n\| \to 0$ $(n \to \infty)$. Since $T \in \mathcal{A}(X)$ and $\mathcal{A}(X)$ is open, the stability of the index ([6], Satz 82.4) shows

$$\operatorname{ind}(T) = \operatorname{ind}(T_n) \quad \text{for } n \text{ sufficiently large.}$$

Thus $\operatorname{ind}(T) = 0$. This gives $T \in \mathcal{F}_0$. ∎

Let us consider some further classes of relatively regular operators: For $n \in \mathbb{Z}$ let

$$\mathcal{F}_n\mathcal{R}(X) = \Big\{T \in \mathcal{L}(X) \colon TST = T \quad \text{for some } S \in \mathcal{F}_n(X)\Big\} .$$

Define $\mathcal{FR}(X) = \bigcup_{n \in \mathbb{Z}} \mathcal{F}_n\mathcal{R}(X)$. Thus we have

$$\mathcal{FR}(X) = \Big\{T \in \mathcal{L}(X) \colon TST = T \quad \text{for some } S \in \mathcal{F}(X)\Big\} .$$

It is shown in [10], Theorem 3, that

$$\mathcal{FR}(X) = \mathcal{R}(X) \cap \operatorname{cl} \mathcal{F}(X) .$$

Theorem 2.2.

(**1**) $\operatorname{int} \mathcal{F}_n\mathcal{R}(X) = \mathcal{F}_{-n}(X)$.

(**2**) $\operatorname{int} \mathcal{FR}(X) = \mathcal{F}(X)$.

Proof: (**1**) Take $T \in \mathcal{F}_{-n}(X)$, then T is relatively regular, hence $T = TST$ for some $S \in \mathcal{L}(X)$. [3], Theorem 6.5.5, gives $S \in \mathcal{F}(X)$ and $\operatorname{ind}(S) = -\operatorname{ind}(T) = n$, therefore $T \in \mathcal{F}_n\mathcal{R}(X)$. Thus we have $\mathcal{F}_{-n}(X) \subseteq \mathcal{F}_n\mathcal{R}(X)$. Since $\mathcal{F}_{-n}(X)$ is open, we get $\mathcal{F}_{-n}(X) \subseteq \operatorname{int} \mathcal{F}_n\mathcal{R}(X)$.

Let $T \in \operatorname{int} \mathcal{F}_n\mathcal{R}(X)$, thus $T \in \operatorname{int} \mathcal{R}(X) = \mathcal{A}(X)$ and $T \in \mathcal{FR}(X) \subseteq \operatorname{cl} \mathcal{F}(X)$. There is a sequence (T_n) in $\mathcal{F}(X)$ such that $\|T - T_n\| \to 0$ $(n \to \infty)$. Since $T \in \mathcal{A}(X)$ and $\mathcal{A}(X)$ is open, the stability of the index shows that

$$\operatorname{ind}(T) = \operatorname{ind}(T_n) \quad \text{for } n \text{ sufficiently large.}$$

Thus $\operatorname{ind}(T)$ is finite, hence $T \in \mathcal{F}(X)$. Since $T \in \mathcal{F}_n\mathcal{R}(X)$, $T = TST$ for some $S \in \mathcal{F}_n(X)$. As above, we see that $\operatorname{ind} T = -\operatorname{ind}(S) = -n$. This gives $T \in \mathcal{F}_{-n}(X)$.

(**2**) Similar. ∎

3 – A spectral mapping theorem

In [11] and [13] we introduced the following concepts for $T \in \mathcal{L}(X)$:

$$\rho_K(T) = \Big\{\lambda \in \mathbb{C}\colon\ (T-\lambda)(X) \text{ is closed, } \ N(T-\lambda) \subseteq \bigcap_{n=1}^{\infty} (T-\lambda)^n(X)\Big\}\ ,$$

$$\rho_{rr}(T) = \Big\{\lambda \in \rho_K(T)\colon\ T-\lambda \in \mathcal{R}(X)\Big\}\ ,$$

$$\sigma_K(T) = \mathbb{C}\backslash\rho_K(T) \quad \text{ and } \quad \sigma_{rr}(T) = \mathbb{C}\backslash\rho_{rr}(T)\ .$$

Write $\mathcal{H}(T)$ for the set of all complex valued functions which are analytic in some neighbourhood of $\sigma(T)$. For $f \in \mathcal{H}(T)$ let the operator $f(T) \in \mathcal{L}(X)$ be defined by the well-known analytic calculus (see [6], §99).

The following proposition lists some properties of the above defined 'essential spectra' of $T \in \mathcal{L}(X)$.

Proposition 3.1.

(**1**) $\partial\sigma(T) \subseteq \sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma(T)$.

(**2**) $\mu \in \rho_{rr}(T) \Leftrightarrow$ *there is a neighbourhood* $U(\mu)$ *of* μ *and a holomorphic function* $F\colon U(\mu) \to \mathcal{L}(X)$ *such that*

$$(T-\lambda)\,F(\lambda)\,(T-\lambda) = T-\lambda \quad \textit{for all } \ \lambda \in U(\mu)\ .$$

(**3**) $\rho_{rr}(T)$ *and* $\rho_K(T)$ *are open.*

(**4**) $f(\sigma_K(T)) = \sigma_K(f(T))$, $f(\sigma_{rr}(T)) = \sigma_{rr}(f(T))$ *for all* $f \in \mathcal{H}(T)$.

Proof: (**1**) The first inclusion is shown in [11], Satz 2. The other inclusions are clear.

(**2**) is shown in [12], Theorem 1.4 (in a more general context).

(**3**) By (2), $\rho_{rr}(T)$ is open. $\rho_K(T)$ is open by [7], Theorem 3.

(**4**) See [11] and [13]. ∎

Remark. Some of the arguments for Proposition 3.1 are also given in [4], Theorems 9.10 and in [5].

Definition 3.2. For $T \in \mathcal{L}(X)$ we define the set $\rho_{gr}(T) \subset \mathbb{C}$ by:

$\mu \in \rho_{gr}(T)$ if and only if there is a neighbourhood $U(\mu)$ of μ and a holomorphic function $F\colon U(\mu) \to \mathcal{L}(X)$ such that

$$F(\lambda) \in \mathcal{G}(X) \quad \text{and} \quad (T-\lambda)\,F(\lambda)(T-\lambda) = T-\lambda \qquad \text{for all} \quad \lambda \in U(\mu)\ .$$

$\sigma_{gr}(T) := \mathbb{C} \backslash \rho_{gr}(T)$.

An operator $T \in \mathcal{L}(X)$ for which $0 \in \rho_{gr}(T)$ is called *holomorphically decomposably regular*. The following condition is equivalent to holomorphic decomposable regularity for $T \in \mathcal{L}(X)$ (cf. [4], Theorem 9):

There are $R \in \mathcal{G}(X)$ and sequences (S_n), (T_n) in $\mathcal{G}(X)$ for which $\|S_n\| + \|T_n - R\| \to 0$ $(n \to \infty)$, $S_nT = TS_n$ and $(T-S_n)\,T_n(T-S_n) = T - S_n$.

Proposition 3.3. *Let $T \in \mathcal{L}(X)$.*

(**1**) $\mu \in \rho_{gr}(T) \Leftrightarrow \mu \in \rho_K(T)$ *and* $T - \mu \in \mathcal{GR}(X)$.

(**2**) $\rho_{gr}(T) \subseteq \rho_{rr}(T) \subseteq \rho_K(T)$, $\sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma_{gr}(T)$.

Proof: (**1**) "$\Rightarrow$": Use Definition 3.2 and Proposition 3.1 (2).

"$\Leftarrow$": Without loss of generality let us assume that $\mu = 0$. Take $S \in \mathcal{G}(X)$ such that $TST = T$ and define the function F by $F(\lambda) = (I - \lambda S)^{-1}S$ for $|\lambda| < \|S\|^{-1}$. Then we have $F(\lambda) \in \mathcal{G}(X)$ for $|\lambda| < \|S\|^{-1}$. [12], Corollary 1.5, shows that

$$(T-\lambda)\,F(\lambda)(T-\lambda) = T - \lambda \qquad \text{for} \quad |\lambda| < \|S\|^{-1}\ .$$

(**2**) Clear. ∎

The following example shows that in general $f(\sigma_{gr}(T)) \not\subseteq \sigma_{gr}(f(T))$ $(f \in \mathcal{H}(T))$.

Example 3.4: Let $T \in \mathcal{L}(X)$, $k, m \in \{1, 2, 3, ...\}$ and $\xi, \eta \in \mathbb{C}$ such that $T - \xi \in \mathcal{F}_k(X)$ and $T - \eta \in \mathcal{F}_{-m}(X)$. We shall construct operators $T - \lambda_0$ and $T - \mu_0$, each Fredholm of positive and negative index, respectively, which also satisfy $\lambda_0, \mu_0 \in \rho_K(T)$.

The *punctured neighbourhood theorem* (see [7]) shows that there exists $\delta > 0$ such that

$$T - \lambda \in \mathcal{F}_k(X)\,, \qquad \alpha(T-\lambda) \text{ is a constant for } 0 < |\lambda - \xi| < \delta$$

and

$$T - \mu \in \mathcal{F}_{-m}(X)\,, \qquad \alpha(T-\mu) \text{ is a constant for } 0 < |\mu - \eta| < \delta\;.$$

Fix λ_0 and μ_0 with $0 < |\lambda_0 - \xi| < \delta$ and $0 < |\mu_0 - \eta| < \delta$. By [7], Theorem 3 and Theorem 5, we have

$$\lambda_0, \mu_0 \in \rho_K(T)\;.$$

Define the function f by $f(\lambda) = (\lambda - \lambda_0)^m\,(\lambda - \mu_0)^k$. This gives $f(T) = (T - \lambda_0)^m\,(T - \mu_0)^k \in \mathcal{F}(X)$, and the index theorem ([6], Satz 71.3) shows that

$$\operatorname{ind}(f(T)) = m \operatorname{ind}(T - \lambda_0) + k \operatorname{ind}(T - \mu_0) = m\,k + k(-m) = 0\;,$$

thus $f(T) \in \mathcal{GR}(X)$. The spectral mapping theorem for $\sigma_K(T)$ (Proposition 3.1 (4)) gives $0 \in \rho_K(f(T))$, since $\lambda_0, \mu_0 \in \rho_K(T)$. Therefore $0 \in \rho_{gr}(f(T))$ by Proposition 3.3 (1). We have $\lambda_0 \in \sigma_{gr}(T)$, since $\operatorname{ind}(T - \lambda_0) \neq 0$, hence $0 = f(\lambda_0) \in f(\sigma_{gr}(T))$.

Example 3.4 also shows the failure of the analogue of part of Theorem 10 of [4]: There are $S, T \in \mathcal{L}(X)$ for which

$ST = TS$ is holomorphically decomposably regular but neither S nor T are.

Proposition 3.5. *Suppose*

(**a**) *$T \in \mathcal{L}(X)$ and $g \in \mathcal{H}(T)$ has only a finite number of zeros in $\sigma(T)$,*

(**b**) *$\mu_1, ..., \mu_m$ are the zeros of g in $\sigma(T)$ with respective orders $n_1, ..., n_m$ ($\mu_i \neq \mu_j$ for $i \neq j$),*

(**c**) *$(T - \mu_j)^{n_j} \in \mathcal{GR}(X)$ for $j = 1, ..., m$.*

Then $g(T) \in \mathcal{GR}(X)$.

Proof: [6], Satz 80.1, asserts that

$$\text{(1)} \qquad \begin{aligned} N\Big(\prod_{j=1}^{k}(T-\mu_j)^{n_j}\Big) &= N\Big((T-\mu_1)^{n_1}\Big) \oplus \ldots \oplus N\Big((T-\mu_k)^{n_k}\Big) \\ &\subseteq (T - \mu_{k+1})^{n_{k+1}}(X) \end{aligned}$$

for $k = 1, ..., m-1$. There are $C_1, ..., C_m \in \mathcal{G}(X)$ such that

$$(T-\mu_j)^{n_j}\, C_j (T-\mu_j)^{n_j} = (T-\mu_j)^{n_j} \qquad (j = 1, ..., m)\ .$$

Put $C = C_m\, C_{m-1} \cdots C_1$ and $R = \prod_{j=1}^m (T-\mu_j)^{n_j}$. Hence $C \in \mathcal{G}(X)$. By (1) and [14], Lemma 5, we get

$$R\,C\,R = R\ .$$

There is a function $h \in \mathcal{H}(T)$ with $h(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$ and $g(\lambda) = (\prod_{j=1}^m (\lambda-\mu_j)^{n_j})\, h(\lambda)$. This gives $g(T) = R\, h(T)$ and $h(T) \in \mathcal{G}(X)$. Put $D = C\, h(T)^{-1}$, then we derive $D \in \mathcal{G}(X)$ and

$$\begin{aligned} g(T)\, D g(T) &= h(T)\, R\, C\, h(T)^{-1}\, h(T)\, R \\ &= h(T)\, R\, C\, R = h(T)\, R = g(T)\ . \end{aligned}$$

Hence $g(T) \in \mathcal{GR}(X)$. ∎

Proposition 3.5 can also be deduced from the analogue of the other half of Theorem 10 of [4]: If (cf. Lemma 3 of [4])

$$S, T \in \mathcal{L}(X), \quad ST = TS \quad \textit{and either} \quad S = T^n \ \textit{for some } n \ \textit{or}$$
$$S'S + TT' = I \quad \textit{for some} \ S', T' \in \mathcal{L}(X)$$

then

S, T holomorphically decomposably regular $\Rightarrow$
ST holomorphically decomposably regular.

Theorem 3.6. *If $T \in \mathcal{L}(X)$ and $f \in \mathcal{H}(T)$ then*

$$\sigma_{gr}(f(T)) \subseteq f(\sigma_{gr}(T))\ .$$

Proof: We have to show that $\mathbb{C} \backslash f(\sigma_{gr}(T)) \subseteq \rho_{gr}(f(T))$. To this end take $\lambda_0 \notin f(\sigma_{gr}(T))$ and put $g(\lambda) = f(\lambda) - \lambda_0$. This gives

(2) $$0 \notin g(\sigma_{gr}(T))$$

and

(3) $$0 \notin g(\sigma_K(T)) = \sigma_K(g(T))\ .$$

Case 1: g has no zeros in $\sigma(T)$. Then $g(T) = f(T) - \lambda_0 \in \mathcal{G}(X)$, thus $\lambda_0 \in \rho(f(T)) \subseteq \rho_{gr}(f(T))$.

Case 2: g has zeros in $\sigma(T)$. (3) shows that g does not vanish in $\sigma_K(T)$. [11], Satz 3, asserts now that g has only a finite number of zeros in $\sigma(T)$. Let $\mu_1, ..., \mu_m$ be these zeros ($\mu_i \neq \mu_j$ for $i \neq j$) and $n_1, ..., n_m$ their respective orders. By (2), $\mu_1, ..., \mu_m \in \rho_{gr}(T)$, thus for each $T - \mu_j$ there is an operator $S_j \in \mathcal{G}(X)$ with $(T - \mu_j)\, S_j (T - \mu_j) = T - \mu_j$. [13], Proposition 2, gives now

$$(T - \mu_j)^{n_j}\, S_j^{n_j} (T - \mu_j)^{n_j} = (T - \mu_j)^{n_j} \qquad (j = 1, ..., m) \ .$$

Since each $S_j^{n_j}$ is invertible, it follows that

$$(T - \mu_j)^{n_j} \in \mathcal{GR}(X) \quad \text{ for } \ j = 1, ..., m \ .$$

Now use Proposition 3.5 to derive $g(T) \in \mathcal{GR}(X)$. (3) gives $0 \in \rho_K(g(T))$, thus $0 \in \rho_{gr}(g(T))$ and therefore $\lambda_0 \in \rho_{gr}(f(T))$. ∎

If the function $f \in \mathcal{H}(T)$ is injective, we can say more:

Theorem 3.7. *If $T \in \mathcal{L}(X)$ and if $f \in \mathcal{H}(T)$ is injective, then*

$$\sigma_{gr}(f(T)) = f(\sigma_{gr}(T)) \ .$$

Proof: We only have to show the inclusion "$\supseteq$". Let $\lambda_0 \in f(\sigma_{gr}(T))$, thus $\lambda_0 = f(\mu_0)$ for some $\mu_0 \in \sigma_{gr}(T)$. Put $g(\lambda) = f(\lambda) - \lambda_0$ and

$$h(\lambda) = \begin{cases} \dfrac{g(\lambda)}{\lambda - \mu_0}, & \lambda \neq \mu_0, \\ f'(\mu_0), & \lambda = \mu_0 \ . \end{cases}$$

Since $f'(\mu_0) \neq 0$, we have $h(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$ and $g(\lambda) = (\lambda - \mu_0)\, h(\lambda)$. This gives $g(T) = (T - \mu_0)\, h(T)$ and $h(T) \in \mathcal{G}(X)$.

Let us assume, to the contrary, that $\lambda_0 \in \rho_{gr}(f(T))$. Therefore $0 \in \rho_{gr}(g(T))$, hence there is an operator S in $\mathcal{G}(X)$ with $g(T)\, S g(T) = g(T)$. Thus

$$(T - \mu_0)\, h(T)\, S h(T)\, (T - \mu_0) = (T - \mu_0)\, h(T) \ .$$

It follows that

$$(T - \mu_0)\, (h(T)\, S)\, (T - \mu_0) = T - \mu_0 \ ,$$

since $h(T)$ is invertible. Hence $T - \mu_0 \in \mathcal{GR}(X)$. Furthermore, $\lambda_0 \in \rho_{gr}(f(T))$ gives $\lambda_0 \in \rho_K(f(T))$. The spectral mapping theorem for $\sigma_K(T)$ implies that $\mu_0 \in \rho_K(T)$. Therefore we have $\mu_0 \in \rho_{gr}(T)$, a contradiction. ∎

We close this paper with a proposition concerning operators in $\mathcal{FR}(X)$.

Proposition 3.8. *Let $T \in \mathcal{L}(X)$ and $g \in \mathcal{H}(T)$ satisfy the hypotheses (a) and (b) of Proposition 3.5. If*

$$(T-\mu_j)^{n_j} \in \mathcal{FR}(X) \quad \text{for } \; j=1,...,m \;,$$

then $g(T) \in \mathcal{FR}(X)$. To be more precise, if $(T-\mu_j)^{n_j} \in \mathcal{F}_{k_j}\mathcal{R}(X)$ and $k = k_1+...+k_m$, then $g(T) \in \mathcal{F}_k\mathcal{R}(X)$.

Proof: With the notation in the proof of Proposition 3.5, there are operators $C_1,...,C_m$ with $C_j \in \mathcal{F}_{k_j}(X)$, thus $D = C_m\, C_{m-1} \cdots C_1\, h(T)^{-1} \in \mathcal{F}(X)$ and, by the index theorem,

$$\mathrm{ind}(D) = k_m + ... + k_1 + \underbrace{\mathrm{ind}(h(T)^{-1})}_{=0} = k \;. \blacksquare$$

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REFERENCES

[1] Caradus, S.R. – *Generalized Inverses and Operator Theory*, Queen's papers in Pure and Applied Math. No.50, 1978.

[2] Harte, R. – Regular boundary elements, *Proc. Amer. Math. Soc.,* 99 (1987), 328–330.

[3] Harte, R. – *Invertibility and singularity for bounded linear operators*, Dekker, 1987.

[4] Harte, R. – Taylor exactness and Kato's jump, *Proc. Amer. Math. Soc.,* 119 (1993), 793–801.

[5] Harte, R. – On Kato nonsingularity, *Studia Math.,* 117 (1996), 107–114.

[6] Heuser, H. – *Funktionalanalysis*, 2nd ed., Teubner, 1986.

[7] Kato, T. – Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Anal. Math.,* 6 (1958), 261–322.

[8] Krishnan, E. and Nambooripad, K.S.S. – The semigroup of Fredholm operators, *Forum Math.,* 5 (1993), 313–368.

[9] Mbekhta, M. – Sur la structure des composantes connexes semi-Fredholm de $B(H)$, *Proc. Amer. Math. Soc.,* 116 (1992), 521–524.

[10] Rakočević, V. – A note on regular elements in Calkin algebras, *Collect. Math.,* 43 (1992), 37–42.

[11] Schmoeger, Ch. – Ein Spektralabbildungssatz, *Arch. Math.,* 55 (1990), 484–489.

[12] Schmoeger, Ch. – The punctured neighbourhood theorem in Banach algebras, *Proc. R. Ir. Academy,* 91A (1991), 205–218.

[**13**] Schmoeger, Ch. – Relatively regular operators and a spectral mapping theorem, *J. Math. Anal. Appl.,* 175 (1993), 315–320.

[**14**] Schmoeger, Ch. – On operators of Saphar type, *Portugaliae Math.,* 51 (1994), 617–628.

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