PORTUGALIAE MATHEMATICA
Vol. 53 Fasc. 4 – 1996

SOLITON TYPE ASYMPTOTIC SOLUTIONS OF THE CONSERVED PHASE FIELD SYSTEM

G.A. Omel'yanov, V.G. Danilov and E.V. Radkevich

Abstract: We consider the conserved phase field system with a small parameter in the n-dimensional case ($n \leq 3$). The soliton type solution describes the first stage of separation in an alloy, when a set of "superheated liquid" appears inside the "solid" part. The corresponding Hugoniot type condition is obtained and the asymptotic solution is justified.

1 – Introduction

The aim of this paper is to consider the metastable processes of solidification. We shall consider the conserved phase field system (the non-isothermal Cahn–Hilliard system), proposed by G. Caginalp [5]

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(\theta + \frac{l}{2}\,\varphi\Big) = k\,\Delta\theta + f(x,t)\,, \qquad (x,t) \in Q\ , \\
&-\tau_0\,\frac{\partial\varphi}{\partial t} = \xi^2\,\Delta\Big(\xi^2\,\Delta\varphi + \frac{1}{2a}\,(\varphi-\varphi^3) + \kappa_1\,\theta\Big)\ .
\end{aligned}
\tag{1}
$$

Here $Q = \Omega \times (0,T)$, $\Omega \subset R^n$ is a bounded domain with smooth (C^∞) boundary $\partial\Omega$, $n \leq 3$, $T < \infty$; Δ is the Laplace operator; θ is the normalized temperature; φ is the order function (that is, the values $\varphi = 1$ and $\varphi = -1$ outside the free boundary correspond to the pure phases); $l > 0$, $k > 0$, κ_1 are constants; $f(x,t)$ is a certain smooth functions (in [5] $f = 0$); $\tau_0 > 0$, $\xi > 0$, and $a > 0$ are parameters.

The physical meaning of τ_0, ξ, a, and of the whole model (1) for the phase transitions was discussed in the papers by G. Caginalp [5, 6] and by A. Novick–Cohen [17]. H. Alt and I. Pawlow [1] proposed general models for non-isothermal

Received: January 23, 1996; *Revised*: February 21, 1996.

phase transitions with a conserved order function. Nevertheless, the simplest model by G. Caginalp is of independent interest since it qualitatively describes the actual physical processes in a binary alloy; the solutions of system (1) can be analysed by mathematical methods.

We shall study the structure of the solution of system (1) and analyze how to pass to the limit from the microscopic description (1) to the macroscopic description. It is well-known that we can pass to the limit as $a \to 0$ and $\xi \to 0$, and that the form of the limiting problem depends on the relations between the parameters a, ξ and τ_0 (here we have the situations in which either $\tau_0 = \text{const}$ or $\tau_0 \to 0$). We restrict our consideration to the case

$$a \ll 1\,, \quad \xi \ll 1\,, \quad \xi\, a^{-1/2} = \text{const}\,, \quad \tau_0 = \text{const}\ .$$

Let us introduce a small parameter $\varepsilon \to 0$ and set

$$a = \varepsilon/2\,, \quad \xi = \sqrt{\varepsilon}\,, \quad \tau_0 = \kappa\,, \quad \kappa = \text{const} > 0\ . \tag{2}$$

For simplicity, we also assume that $k = 1$ and $l = 2$. Completing (1) with natural initial and boundary conditions, we obtain our basic mathematical model

$$\begin{aligned}
&\frac{\partial}{\partial t}(\theta + \varphi) = \Delta\theta + f(x,t)\ ,\\
&-\kappa\,\frac{\partial \varphi}{\partial t} = \Delta(\varepsilon^2\,\Delta\varphi + \varphi - \varphi^3 + \kappa_1\,\varepsilon\,\theta)\ ,\\
&\theta|_{t=0} = \theta^0(x,\varepsilon)\,, \quad \varphi|_{t=0} = \varphi^0(x,\varepsilon)\ ,\\
&\frac{\partial \theta}{\partial N}\Big|_{\Sigma} = 0\,, \quad \frac{\partial \varphi}{\partial N}\Big|_{\Sigma} = 0\,, \quad \frac{\partial}{\partial N}\Delta\varphi\Big|_{\Sigma} = 0\ .
\end{aligned} \tag{3}$$

Here N is the external normal to the boundary $\partial\Omega$ and $\Sigma = [0,T] \times \partial\Omega$.

Precisely as the solution of the Cahn–Hilliard equation (the second equation in (3) with $\theta = \text{const}$), the solution of the conserved phase field system (3) is very complicated, since its behavior varies depending on different stages of the phase separation process in binary alloys. (For example, the solution is of the oscillating type [15, 16] or of the Van der Waals tanh-type [4–6, 20–22, 25]). These stages are called stable, unstable, and metastable [15, 16]. They correspond to the cases: $\bar\varphi \geq 1$ or $\bar\varphi \leq -1$, and $-1/\sqrt{3} \leq \bar\varphi \leq 1/\sqrt{3}$, and $-1 < \bar\varphi < -1/\sqrt{3}$ or $1/\sqrt{3} < \bar\varphi < 1$ respectively. Here and below $\bar f(x) = w - \lim\limits_{\varepsilon\to 0} f(x,\varepsilon)$ denotes the weak limit in the $\mathcal{D}'$ sense. The numbers ± 1 correspond to the zero points of the equilibrium chemical potential $W'(\varphi) = \varphi - \varphi^3$. The numbers $\pm 1/\sqrt{3}$ correspond to local maxima/minima of W'. At present, the solution to problem (3) with arbitrary initial data can be analyzed in detail only by numerical methods. However, by setting some special initial data, one can select and thoroughly study a certain type of solutions.

Naturally, solutions corresponding to experimentally observed processes must be stable at each stage, i.e., theses processes must not vary much during some time. We shall study possible stable structures of solutions by using asymptotic methods. Functions of the form

$$\theta^M(x,t,\varepsilon) = \sum_{i=0}^{K_1} \varepsilon^i \, V_i(S/\varepsilon, x, t)\,, \qquad \varphi^M(x,t,\varepsilon) = \sum_{i=0}^{K_2} \varepsilon^i \, \Phi_i(S/\varepsilon, x, t) \tag{4}$$

will be called a self-similar (formal) asymptotic up to $\mathcal{O}(\varepsilon^{M+1})$ solution of system (3). Here $S = S(x,t) \in C^\infty(\bar{Q})$, $\Phi_i(\eta, x, t)$, and $V_i(\eta, x, t)$ are smooth functions uniformly bounded in $R^1_\eta \times Q$ together with their derivatives; the nonnegative numbers $K_j = K_j(M)$, $j = 1, 2$, are such that for any number $M \geq 0$ the substitution of the functions θ^M, φ^M into (3) gives a discrepancy $\mathcal{O}(\varepsilon^{M+1})$ on the right-hand side of (3), here $\mathcal{O}(\,\cdot\,)$ is a bound in the sense of $C(R^1_\eta \times \bar{Q})$.

The possible types of stable asymptotic solutions differ by the form of dependence on the fast variable $\eta = S/\varepsilon$. As is easy to see, there is a finite number of types for (3). To this end, we substitute θ^M, φ^M into the Cahn–Hilliard equation and set the coefficient at ε^{-2} equal to zero. We obtain the equation

$$\frac{\partial^2}{\partial \eta^2}\Big(|\nabla S|^2 \, \frac{\partial^2 \Phi_0}{\partial \eta^2} + \Phi_0(1 - \Phi_0^2)\Big) = 0 \,. \tag{5}$$

Integrating with respect to η, we obtain

$$|\nabla S|^2 \, \frac{\partial^2 \Phi_0}{\partial \eta^2} + \Phi_0(1 - \Phi_0^2) = C + C_1 \, \eta \,, \tag{6}$$

where the "constants" of integration $C = C(x,t)$ and $C_1 = C_1(x,t)$ are smooth functions. The assumption that the function Φ_0 is uniformly bounded implies the equality $C_1 \equiv 0$. Thus, we have obtained an autonomic second-order equation, namely, the Newton equation. For this equation we can list all possible types of solutions. Multiplying this equation by $\partial\Phi_0/\partial\eta$ and integrating with respect to η, we obtain

$$|\nabla S| \, \frac{\partial \Phi_0}{\partial \eta} = \pm\sqrt{2}\,\sqrt{E - U(\Phi_0, C)} \,, \tag{7}$$

where $E = E(x,t)$ is the second "constant" of integration, and

$$U(\Phi_0, C) = W(\Phi_0) - C\,\Phi_0\,, \qquad W(\Phi_0) = \frac{1}{2}\,\Phi_0^2 - \frac{1}{4}\,\Phi_0^4 \,.$$

Obviously, this equation has real solution if and only if $E \geq U$. Figure 1 shows the potential $U(\Phi_0, C)$ is plotted in the case $C < 0$ (for $C > 0$ the hump on the left is higher than the crest on the right and for $C = 0$ the potential $U(\Phi_0, 0)$ is an even function of Φ_0).

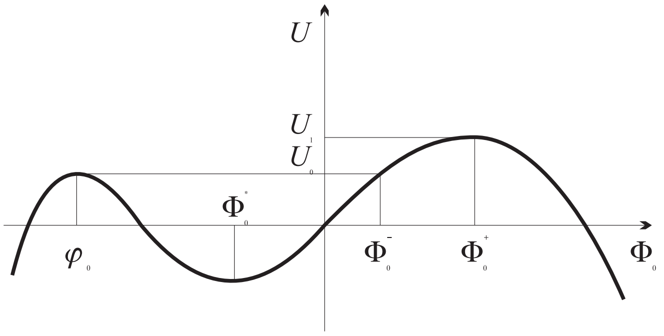

Figure 1

We denote the maximal values of the potential by U_0 and U_1, $U_0 < U_1$ ($U_0 = U(\varphi_0, C)$ and $U_1 = U(\Phi_0^+, C)$, where φ_0 and Φ_0^+ are the minimal and maximal roots of the equation $\Phi_0^3 - \Phi_0 + C = 0$, respectively). We assume that $U_2 = U(\Phi_0^*, C)$, where Φ_0^* is the point at which U takes its local minimum.

Obviously, if $E > U_0$, the equation for Φ_0 has no solutions with compact range, which means that the corresponding mechanical system has no finite motions.

If $E \in (U_2, U_0)$ (see Fig. 1), Φ_0 has a compact range such that $U(\Phi_0, C) \leq E$. Hence, to each such value of E there corresponds a periodic solution $\Phi_0(\eta, x, t)$ (see also [15–17]). Finally, for $E = U_0$ the point φ_0 at which $U = U_0$ is an isolated solution of the equation for Φ_0 and, under the condition $\frac{\partial U}{\partial \Phi_0}|_{\Phi_0 = \Phi_0^-} \neq 0$ (i.e., $C \neq 0$), the point Φ_0^- such that $U(\Phi_0^-, C) = U_0$ and $\Phi_0^- \neq \varphi_0$ is a turning point and the corresponding solution is shown in Fig. 2.

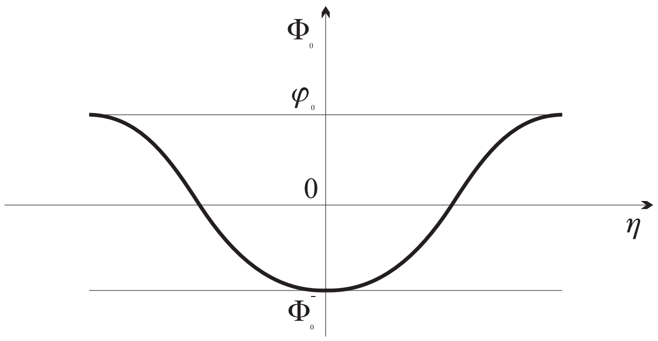

Figure 2

It is easy to see that as $|\eta| \to \infty$ this solution exponentially tends to φ_0. This solution will be called a soliton-type solution. Finally, if $C = 0$, then $U_0 = U_1$ and $\Phi_0 = \Phi_0^-$ is also an isolated solution. In this case, the solution Φ_0 is a separatrix and the explicit expression for this solution is well-known: $\Phi_0 = \tanh(\eta/\sqrt{2}\,|\nabla S|)$, i.e., Φ_0 is the Van der Waals tanh solution.

The physical setting of our problem is the following: to simplify the problem we assume that the initial concentration $\varphi^0(x) \in (1/\sqrt{3}, 1)$ for all $x \in \Omega$. The set of such points with concentration will be called "solid". At first sight, since the interval $(1/\sqrt{3}, 1)$ belongs to the domain of attraction to the point $\varphi_{\mathrm{eq}}^+ = 1$, the concentration would seem to increase and tend to φ_{eq}^+. But it is impossible to obtain the situation in which $\varphi(x,t) = \varphi_{\mathrm{eq}}^+$ at each point of Ω, since the global mass $m(\varphi)$,

$$m(\varphi) = \int_\Omega \varphi\, dx \ ,$$

conserves in time and $m(\varphi^0) < |\Omega|$. Thus, one must assume the appearance of subdomains $\Omega_t^\pm$ such that $\bar{\varphi} \in (1/\sqrt{3}, 1]$ as $x \in \Omega_t^+$ and $\bar{\varphi} \in [-1, -1/\sqrt{3})$ as $x \in \Omega_t^-$. After that, the next stage of the solidification starts at which the subdomains $\Omega_t^\pm$ transform. The soliton type asymptotic solution, constructed in the present paper, describes the first stage at which the "liquid" part Ω_t^- appears inside the "solid" part, but the volume of Ω_t^- is still small enough (note that the domains $\Omega_t^\pm$ are fulfilled by a substance that, in fact, is not solid or liquid).

The Van der Waals type asymptotic solution [21] describes the motion of $\Omega_t^\pm$ when the volumes of $\Omega_t^\pm$ are not small. Our construction allows us as well to show that the temperature remains a smooth function (in the leading term with respect to ε) during these processes. Therefore, the solidification process, described by the conserved phase field system (3), differs, in principle, from the dynamics of "solid"/"liquid", described by both the phase field model and by (1) for small relaxation time τ_0 [20].

Actually, according to the phase field model and to (1) for $\tau_0 \sim \varepsilon$, the temperature has a weak discontinuity on the free interface, whereas, according to model (3), the temperature is almost the same on the "solid" and "liquid" domains (for both $|\Omega_t^-| \ll 1$ and $|\Omega_t^-| \sim \mathrm{const}$). So we can say that the appearing set Ω_t^- is a domain of "superheated liquid". We shall consider only the case $\overline{\varphi^0} \in (1/\sqrt{3}, 1]$. Nevertheless, it is clear that our construction also allows us to describe the similar processes in the case $\overline{\varphi^0} \in [-1, -1/\sqrt{3})$.

The multidimensional Cahn–Hilliard equation with a small parameter was considered by B. Stoth [25] (in the spherical case) and by R.L. Pego [22]. The conserved phase field system (1) was considered by G. Caginalp [5, 6] and by G. Omel'yanov [20,21].

Here we use the method for constructing asymptotic solutions, which is a modification of the two-scale method for obtaining solutions with localized "fast" variation. It was developed by V. Maslov, V. Tsupin, G. Omel'yanov, V. Danilov, and K. Volosov [11–14] for constructing soliton-type and traveling wave-type asymptotic solutions. A modification of this method for phase transition problems (to the phase field system and to the conserved phase field system) was proposed in the works by E. Radkevich, V. Danilov, and G. Omel'yanov [7, 8, 18–21, 24]. The main idea of this method is to construct some analogs of the internal and external expansions assuming that they are defined in the whole range of independent variables. This implies that there are no summands polynomially increasing in the "fast" variable. Obviously, no matching is needed.

Here we have the following important points. Suppose the "fast" variation of the solution u is localized in a small neighborhood of the smooth surface Γ_t, for example, $u = A(x,t)\cosh^{-2}(S(x,t)/\varepsilon)$, $\Gamma_t = \{x \in \Omega,\ S(x,t) = 0\}$, $S, A \in C^\infty$. Then, to calculate the function S at each instant of time $t \geq 0$, it is sufficient to define only its zero surface $\{x,\ \psi(x) = t\} \equiv \{x,\ S(x,t) = 0\}$ and the first normal derivative S' on Γ_t, since in an ε-neighborhood of Γ_t the smooth function S cannot vary more than by $\mathcal{O}(\varepsilon)$ and outside an $\varepsilon^{1-\delta}$-neighborhood of Γ_t $(0 < \delta < 1)$ the solution varies slowly with precision up to $\mathcal{O}(\varepsilon^\infty)$ ($u = 0$ for the above example). Here we speak about a polynomial (with respect to the parameter ε) asymptotics, and hence, the discrepancy $\mathcal{O}(\varepsilon^\infty)$ (i.e., $\mathcal{O}(\varepsilon^M)$ for any $M > 0$) is much less than the discrepancy $\mathcal{O}(\varepsilon^{i+1})$ arising at the ith step for each fixed i.

Furthermore, in the traditional two-scale method we first construct a rapidly varying solution $u(S/\varepsilon, x, t)$ in the "extended" space $R^1 \times \Omega \times [0,T]$, assuming that $u = u(\eta, x, t)$ and that the variables η and x, t are independent. Then we calculate the trace $u(\eta,x,t)|_{\eta=S/\varepsilon}$. This approach is absolutely justified for fast oscillating solutions. However, for solutions with localized fast variation we can construct the rapidly varying components only in the subspace $R^1 \times \mathcal{T}_T^\delta$, where $\mathcal{T}_T^\delta$ is an $\varepsilon^{1-\delta}$-neighborhood of the surface $\mathcal{T}_T = \{(x,t) \in \bar{Q},\ t = \psi(x)\} = \bigcup_{t\in[0,T]} \Gamma_t$. That is, we know beforehand that our final goal is only the trace $u(\eta,x,t)|_{\eta=S/\varepsilon}$.

In its turn, since $u(\eta, x, t)$ smoothly depends on "slowly" varying variables x and t, we can calculate the solution in two stages: in the first stage we define the trace $\check{u} = u(\eta,x,t)|_{t=\psi(x)}$ of u on the section $R^1 \times \mathcal{T}_T$, in the second stage we construct a sufficiently smooth continuation of $\check{u}$ outside $R^1 \times \mathcal{T}_T$. Needless to say, this continuation must be constructed sufficiently accurate, nevertheless, there is some freedom in choosing this continuation. Namely, this freedom provides the boundedness (uniformly in $\eta \in R^1$, $x, t \in Q$) of all terms of the asymptotic expansion. In § 2 we formalize these considerations, which are obvious enough, and in detail describe the method for constructing the soliton-type asymptotic solution.

We have discussed only a formal asymptotic solution. However, in the strict sense, this does not imply that the discrepancy between the explicit solution and the asymptotic one is actually small in a certain sense. Therefore, in particular, one cannot use formal asymptotics for obtaining the limiting (as $\varepsilon \to 0$) problem. Thus, in §3 we rigorously justify the asymptotics constructed.

2 – Soliton type asymptotic solution

Let us formulate the main result of this section. By $\theta_0 = \theta_0(x,t)$, $\varphi_0 = \varphi_0(x,t) \in (1/\sqrt{3}, 1)$, $\psi = \psi(x)$, we denote the solutions of the following model problems

$$\begin{aligned} &\kappa\,\frac{\partial\varphi_0}{\partial t} = \Delta(\varphi_0^3 - \varphi_0)\,, \qquad x \in \Omega, \quad t > 0\,, \\ &\varphi_0\Big|_{t=0} = \overline{\varphi^0}(x)\,, \qquad \frac{\partial\varphi_0}{\partial N}\Big|_{\Sigma} = 0\,, \end{aligned} \tag{8}$$

$$\begin{aligned} &\frac{\partial\theta_0}{\partial t} = \Delta\theta_0 + f(x,t) - \frac{\partial\varphi_0}{\partial t}\,, \qquad x \in \Omega, \quad t > 0\,, \\ &\theta_0\Big|_{t=0} = \overline{\theta^0}(x)\,, \qquad \frac{\partial\theta_0}{\partial N}\Big|_{\Sigma} = 0\,, \end{aligned} \tag{9}$$

$$\kappa\, v_\nu = \frac{1}{3}\,\mathcal{K}_t\Big(1 + G(\check\varphi_0)\Big) + G_0\,\frac{\partial\check\varphi_0}{\partial\nu}\,, \qquad \psi|_{\Gamma_0} = 0\,. \tag{10}$$

Here $v_\nu = 1/|\nabla\psi|$ is the normal velocity of motion of the surface $\Gamma_t = \{x \in \Omega, \psi(x) = t\}$; $\mathcal{K}_t = \operatorname{div}(\nu)$ is the mean curvature of Γ_t, $\nu = \nabla\psi/|\nabla\psi|$ is the vector normal to Γ_t; $\check F = F(x,\psi(x))$ for all continuous functions $F(x,t)$; $\partial/\partial\nu = (\nu,\nabla)$;

$$G(\check\varphi_0) = \frac{I\,Q^2}{2(Q-I)}\,, \qquad G_0 = \frac{G(\check\varphi_0)}{\check\varphi_0}\,, \qquad \frac{\partial\check\varphi_0}{\partial\nu} = \Big(\nu, \nabla\varphi_0(x,\psi)\Big)\,,$$

$$Q = \sqrt{3(\check\varphi_0)^2 - 1}\,, \qquad I = \sqrt{2}\,\check\varphi_0 \ln J\,, \qquad J = (\sqrt{2}\,\check\varphi_0 + Q)/\sqrt{b}\,, \qquad b = 1 - \check\varphi_0^2\,.$$

Theorem 1. *Let $\Gamma_0 = \{x \in \Omega,\ \psi(x) = 0\}$ be a sufficiently smooth closed surface of codimension* 1. *Let sufficiently smooth solutions of problem* (8)–(10) *exist, and $\varphi_0 \in (1/\sqrt{3}, 1)$, and let $\operatorname{dist}(\Gamma_t, \partial\Omega) \geq \text{const}$ for all $t \in [0,T]$. Then,* *for any $M \geq 0$, there exists a formal asymptotic (up to $\mathcal{O}(\varepsilon^{M+1})$) solution of equations (3). The leading term of this asymptotic solution has the form*

$$\theta(x,t,\varepsilon) = \theta_0(x,t)\,, \qquad \varphi(x,t,\varepsilon) = \varphi_0(x,t) + \chi(\eta, x)\ . \tag{11}$$

Here

$$\begin{aligned} \chi &= -8\,Q^2\Big\{e^{\xi} + 8\,b\,e^{-\xi} + 8\,\check{\varphi}_0\Big\}^{-1}\,, \\ \xi &= \beta(\eta + \psi_1(x))\,, \qquad \eta = (t - \psi(x))/\varepsilon\,, \qquad \beta = Q/|\nabla\psi|\ , \end{aligned} \tag{12}$$

ψ_1 is a smooth function, the method for calculating this function is given below (see formula (31) and Lemma 2).

Remark. It is not too difficult to prove that equation (10) is a quasilinear parabolic equation, in which x_ν (along the vector ν) is a time like variable, and x_t (tangential to Γ_t) are space like variables. So, the additional condition in (10) is actually the initial condition [7, 8]. The classical solvability and uniqueness of solutions of quasilinear parabolic problems with smooth coefficients are the result of the realization of some matching conditions between the initial and boundary data [10, 23].

At first let us consider the statement of Theorem 1. Formulas (11), (12) and the solutions of problems (8)–(10) describe the motion of the soliton χ on the smooth "background" φ_0 (a soliton type solution was obtained also in [9] by numerical simulations for the one-dimensional Cahn–Hilliard equation). Obviously, the surface Γ_t is the set of maximum magnitude of $|\chi|$

$$A = \max_{x\in\Omega}|\chi| = -\chi|_{\Gamma_t} = Q^2\Big\{\frac{9}{8} + \check{\varphi}_0 - \check{\varphi}_0^2\Big\}^{-1}\ .$$

It is easy to prove that this solution exists if and only if $\check{\varphi}_0 \in (1/\sqrt{3}, 1)$. The amplitude A is a monotonically increasing function, $A'_{\check{\varphi}_0} > 0$, and trivial calculations show that $A \to 0$ and $G \to 0$ as $\check{\varphi}_0 \to 1/\sqrt{3}$. It is also clear that there exists a value $\varphi^* \in (1/\sqrt{3}, 1)$ such that $A < \check{\varphi}_0$ as $\check{\varphi}_0 \in (1/\sqrt{3}, \varphi^*)$, and $A > \check{\varphi}_0$ as $\check{\varphi}_0 \in (\varphi^*, 1)$. Thus, moving into the domain with $\check{\varphi}_0 \in (\varphi^*, 1)$, the soliton solution describes how the set $\Omega^-_{t,\varepsilon} = \{x \in \Omega,\, \check{\varphi}_0 + \chi < 0\}$ with negative concentration arises $|\Omega^-_{t,\varepsilon}| \sim \varepsilon$. Let us consider the behavior of the solution as φ_0 tends to 1. Setting $\varphi_0 = 1 - \tilde{\varphi}(x,t)\exp(-1/\delta)$, $\delta \ll 1$, we get the following relations $A = 16/9 + \mathcal{O}(e^{-1/\delta})$, $G = -1 + \mathcal{O}(\delta)$. Hence, $v_\nu \sim \delta$ and the velocity of the soliton motion decreases as $\delta \to 0$. On the other hand, the volume of the set $\Omega^-_{t,\varepsilon}$

increases, since $b \sim \exp(-1/\delta)$ and $|\Omega^-_{t,\varepsilon}| \sim |\Omega^-_{t,\varepsilon/\delta}| \sim \varepsilon/\delta$ for $\delta \ll 1$. Thus, this solution describes the appearance of a sufficiently large domain of "superheated liquid", since the concentration $\varphi \sim -7/9$ on $\Omega^-_{t,\varepsilon/\delta}$ and the temperature θ_0 is almost independent of φ_0 at these points. Nevertheless, this asymptotic solution is correct only if $|\Omega^-_{t,\varepsilon/\delta}| \to 0$ as $\varepsilon \to 0$.

We shall also see that the first corrections of the asymptotic expansions for the temperature and concentration have the form of smoothed shock waves. So,

$$w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\theta - \theta_0) = A_{1,\theta}\, H(t-\psi)\,, \qquad w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\varphi - \varphi_0 - \chi) = A_{1,\varphi}\, H(t-\psi)\ ,$$

where H is the Heaviside function, $A_{1,\theta} = 2I/(\check{\varphi}_0 v_\nu)$, $A_{1,\varphi} = 2\kappa I/(\check{\varphi}_0 Q^2 v_\nu)$ are the amplitudes of jumps on Γ_t. It is easy to calculate that $A_{1,\theta}$, $A_{1,\varphi}$ are bounded as $\check{\varphi}_0 \to 1$.

Let us prove Theorem 1 and consider the general method for constructing the asymptotic solution of problem (3) (with special initial data) up to an arbitrary precision.

First, we introduce some classes of functions, which we shall need for constructing asymptotic solutions with localized fast variation. That is,

$$\begin{aligned}
\mathcal{H} = \Big\{ & f(\eta,x,t) \in C^\infty(R^1 \times \bar{Q}),\ \exists\, f^\pm \in C^\infty(\bar{Q}), \\
& \lim_{\eta\to\pm\infty} \eta^m \frac{\partial^{r+|\alpha|+\gamma}}{\partial\eta^r\,\partial x^\alpha\,\partial t^\gamma}\Big(f(\eta,x,t) - f^\pm(x,t)\Big) = 0, \quad \forall\, m,r,\alpha,\gamma \ge 0 \Big\}\ , \\
\mathcal{S} = \Big\{ & f(\eta,x,t) \in \mathcal{H},\ f^+ = f^- = 0 \Big\}\ , \\
\mathcal{P} = \Big\{ & f(\tau,x',t) \in C^\infty(R^1_+ \times \Sigma), \\
& \lim_{\tau\to\infty} \tau^m \frac{\partial^{r+|\alpha'|+\gamma}}{\partial\tau^r\,\partial(x')^{\alpha'}\,\partial t^\gamma} f(\tau,x',t) = 0, \quad \forall\, m,r,\gamma,\alpha',\gamma \ge 0 \Big\}\ .
\end{aligned}$$

Lemma 1. *Let $S(x,t) \in C^\infty(\bar{Q})$ be such that $\partial S/\partial t|_{\Gamma_t} \ne 0$, where $\Gamma_t = \{x \in \Omega,\ S(x,t) = 0\}$, $t \ge 0$. Then, for any function $f(\eta,x,t) \in \mathcal{H}$, we get*

$$f\Big(\frac{S(x,t)}{\varepsilon}, x, t\Big) = f\Big(\beta(x)\,\frac{t-\psi(x)}{\varepsilon}, x, t\Big) + \mathcal{O}(\varepsilon)\ ,$$

where $t = \psi(x)$ is the equation of the surface $S(x,t) = 0$ and $\beta(x) = \partial S/\partial t|_{t=\psi(x)}$.

2. *Let $\mu(\eta, x, t)$, $\zeta(\eta, x, t) \in \mathcal{H}$ be such that $\mu^{\pm} = \pm 1$, $\zeta^{+} = 1$, $\zeta^{-} = 0$. Then, for any function $f \in \mathcal{H}$, we have the representations*

$$f = \frac{1}{2}(f^{+} + f^{-}) + \frac{1}{2}(f^{+} - f^{-})\,\mu(\eta, x, t) + \omega_1(\eta, x, t) \ ,$$
$$f = f^{-} + (f^{+} - f^{-})\,\zeta(\eta, x, t) + \omega_2(\eta, x, t) \ ,$$

where ω_i are functions from $\mathcal{S}$.

3. *The relations*

$$(t-\psi)^k\, f\Big(\frac{t-\psi}{\varepsilon}, x, t\Big) = \mathcal{O}(\varepsilon^k)\,, \qquad k \geq 0 \ ,$$
$$g(x,t)\, f\Big(\frac{t-\psi(x)}{\varepsilon}, x, t\Big) = g(x, \psi(x))\, f\Big(\frac{t-\psi(x)}{\varepsilon}, x, \psi(x)\Big) + \mathcal{O}(\varepsilon) \ ,$$

hold for any functions $f(\eta, x, t) \in \mathcal{S}$, $g(x,t) \in C^{\infty}(\bar{Q})$.

The proof obviously follows from the definition (see also [12]).

Let us note that the representation $S = t - \psi(x)$ does not mean that the solution must move with velocity of order $\mathcal{O}(1)$. Actually, since the function $\psi(x)$ can increase rapidly along the direction normal to the surface $\Gamma_t = \{x \in \Omega, \psi(x) = t\}$, the motion of the solution can be arbitrary slowly.

Let us begin to construct the self-similar asymptotic solution of problem (3). One can show that the leading term of the asymptotic expansion for θ must be a smooth function, since the leading term of φ is a soliton. This implies that the asymptotic solution has the following form

$$\theta(x,t,\varepsilon) = \vartheta^M(x,t,\varepsilon) + \varepsilon\, \mathcal{V}^M\Big(\frac{S(x,t)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon\Big) \ ,$$

$$\varphi(x,t,\varepsilon) = \Phi^M(x,t,\varepsilon) + \mathcal{W}^M\Big(\frac{S(x,t)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon\Big) \ , \tag{13}$$

where the functions ϑ^M, Φ^M (uniformly smooth with respect to $\varepsilon \in [0,1]$, $x, t \in Q$, the so-called "regular part" or the "background") give an analog of the external expansion,

$$\vartheta^M(x,t,\varepsilon) = \sum_{j=0}^{M} \varepsilon^j\, \theta_j(x,t)\,, \qquad \Phi^M(x,t,\varepsilon) = \sum_{j=0}^{M} \varepsilon^j\, \varphi_j(x,t) \ ,$$

the functions $\mathcal{V}^M$, $\mathcal{W}^M$ (rapidly varying near Γ_t and the external boundary) give an analog of the internal expansion,

$$\mathcal{V}^M(\eta,\tau,x,t,\varepsilon) = \sum_{j=1}^{M} \varepsilon^{j-1}\Big\{U_j(\eta,x,t) + Y_j(\tau,x',t)\Big\} ,$$

$$\mathcal{W}^M(\eta,\tau,x,t,\varepsilon) = \chi(\eta,x,t) + \sum_{j=1}^{M} \varepsilon^{j}\Big\{W_j(\eta,x,t) + Z_j(\tau,x',t)\Big\} .$$

Here x_N is the distance from a point $x \in \Omega$ to a point $x' \in \partial\Omega$ along the internal normal,

$$S,\, \theta_j,\, \varphi_j \in C^\infty(\bar{Q}) , \qquad Y_j(\tau,x',t),\, Z_j(\tau,x',t) \in \mathcal{P} ,$$
$$\chi(\eta,x,t) \in \mathcal{S} , \qquad U_j(\eta,x,t),\, W_j(\eta,x,t) \in \mathcal{H} ,$$

and

$$\text{(14)} \qquad U_j^- = 0 , \qquad W_j^- = 0 , \qquad \frac{\partial S}{\partial t}\bigg|_{\Gamma_t} \neq 0 , \qquad \Gamma_t = \Big\{x \in \Omega,\ S(x,t) = 0\Big\} .$$

By Lemma 1, without loss of generality, we can assume that $S = t - \psi(x)$, $\chi = \chi(\eta, x)$.

Furthermore, outside small neighborhoods of the soliton support and the external boundary, i.e., as $\eta \to \pm\infty$, and $\tau \to \infty$, we have $\theta_j + U_j + Y_j \asymp \theta_j + U_j^{\pm}$. Since all the functions in the latter relation are arbitrary for the moment, we can redefine θ_j (for example, by setting $\theta_j := \theta_j + U_j^+$ or $\theta_j := \theta_j + U_j^-$) and thus set one of the limiting values U_j^+ or U_j^- equal to zero. For definiteness, we set U_j^- and, similarly, $W_j^- = 0$.

Substituting (13) into equations (3), we get the relations

$$\text{(15)} \qquad \frac{\partial}{\partial t}(\vartheta^M + \Phi^M) - \Delta\vartheta^M - f = \Bigg[\frac{1}{\varepsilon}\Big\{\hat{\mathcal{L}}_2\,\mathcal{V}^M - \frac{\partial}{\partial\eta}\mathcal{W}^M\Big\} -$$
$$- \Big\{\Big(\frac{\partial}{\partial\eta} + \mathcal{L}_1\Big)\mathcal{V}^M + \frac{\partial}{\partial t}\mathcal{W}^M\Big\} + \varepsilon\Big(\Delta_x - \frac{\partial}{\partial t}\Big)\mathcal{V}^M\Bigg]\Bigg|_{\eta=S/\varepsilon,\,\tau=x_N/\varepsilon} ,$$

$$
\begin{aligned}
(16)\qquad & \Big[\frac{1}{\varepsilon^2}\hat{\mathcal{L}}_2\Big(\hat{\mathcal{L}}_2\mathcal{W}^M+\mathcal{W}^M-(\Phi^M+\mathcal{W}^M)^3\Big)+\\
&+\frac{1}{\varepsilon}\Big\{\Big(\kappa\frac{\partial}{\partial\eta}-\hat{\mathcal{L}}_2\hat{\mathcal{L}}_1\Big)\mathcal{W}^M-\hat{\mathcal{L}}_1\Big(\hat{\mathcal{L}}_2\mathcal{W}^M+\mathcal{W}^M-(\Phi^M+\mathcal{W}^M)^3\Big)\Big\}\\
&+\Big\{(\hat{\mathcal{L}}_2\Delta_x+\hat{\mathcal{L}}_1^2)\mathcal{W}^M+\kappa_1\hat{\mathcal{L}}_2\mathcal{V}^M+\kappa\frac{\partial}{\partial t}(\Phi^M+\mathcal{W}^M)\\
&+\Delta_x\Big(\hat{\mathcal{L}}_2\mathcal{W}^M+\Phi^M+\mathcal{W}^M-(\Phi^M+\mathcal{W}^M)^3\Big)\Big\}\\
&-\varepsilon\Big\{\hat{\mathcal{L}}_1(\Delta_x\mathcal{W}^M+\kappa_1\mathcal{W}^M)+\Delta_x(\hat{\mathcal{L}}_1\mathcal{W}^M-\kappa_1\vartheta^M)\Big\}+\\
&+\varepsilon^2\Delta_x\Big\{\Delta_x(\Phi^M+\mathcal{W}^M)+\kappa_1\mathcal{V}^M\Big\}\Big]\Big|_{\eta=S/\varepsilon,\,\tau=x_N/\varepsilon}=0\ .
\end{aligned}
$$

Here, as usually in the two-scale method, the expressions in square brackets are considered as functions of independent variables η, x, and t,

$$
\hat{\mathcal{L}}_2=|\nabla\psi|^2\frac{\partial^2}{\partial\eta^2}+|\nabla x_N|^2\frac{\partial^2}{\partial\tau^2}\,,\qquad \hat{\mathcal{L}}_1=\frac{\partial}{\partial\eta}\hat{\Pi}-\frac{\partial}{\partial\tau}\hat{\Pi}_b\ ,
$$

$$
\hat{\Pi}=2(\nabla\psi,\nabla_x)+\Delta\psi\,,\qquad \hat{\Pi}_b=2(\nabla x_N,\nabla_x)+\Delta x_N\ .
$$

First, let us obtain the regular terms of expansion (13). Passing to the limit as $\eta\to\pm\infty$, $\tau\to\infty$, from (15), (16) we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\vartheta^M+\Phi^M)-\Delta\vartheta^M-f=-\frac{\partial}{\partial t}\mathcal{W}^{M\pm}-\varepsilon\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathcal{V}^{M\pm}\ ,\\
(17)\qquad &\kappa\frac{\partial}{\partial t}(\Phi^M+\mathcal{W}^{M\pm})+\Delta\Big(\Phi^M+\mathcal{W}^{M\pm}-(\Phi^M+\mathcal{W}^{M\pm})^3\Big)=\\
&\qquad=\varepsilon\kappa_1\Delta\vartheta^M+\varepsilon^2\Delta\Big(\Delta(\Phi^M+\mathcal{W}^{M\pm})+\kappa_1\mathcal{V}^{M\pm}\Big)\ .
\end{aligned}
$$

Introducing the notation

$$
(18)\qquad \theta_j^\pm=\theta_j+U_j^\pm,\quad \Phi_j^\pm=\varphi_j+W_j^\pm\,,\qquad j=1,2,...,M\ ,
$$

and setting the terms of the order $\mathcal{O}(\varepsilon^j)$ equal to zero, we get equations (8), (9) for the leading terms, as well the following equations for the lower terms of the asymptotic expansion

$$
(19)\qquad
\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\theta_k^\pm=f_{k,\theta}^\pm(x,t)\,,\qquad x\in\mathcal{D}_t^\pm,\quad t>0\ ,\\
&\kappa\frac{\partial}{\partial t}\Phi_k^\pm-\Delta\Big((3\varphi_0^2-1)\,\Phi_k^\pm\Big)=f_{k,\varphi}^\pm(x,t)\ .
\end{aligned}
$$

Here $k=1,...,M$, $\mathcal{D}_t^{\pm}$ are subdomains of Ω such that

$$\mathcal{D}_t^{+}=\Big\{x\in\Omega,\ \psi(x)<t\Big\},\ \ \mathcal{D}_t^{-}=\Big\{x\in\Omega,\ \psi(x)>t\Big\},\qquad \Omega=\mathcal{D}_t^{+}\cup\mathcal{D}_t^{-}\cup\Gamma_t\ ,$$

$f_{k,\theta}^{\pm}(x,t)$, $f_{k,\varphi}^{\pm}(x,t)$ are functions of the previous terms of the expansion and their derivatives. In particular, $f_{1,\theta}^{\pm}=-\partial\Phi_1^{\pm}/\partial t$, $f_{1,\varphi}^{\pm}=\kappa_1\,\Delta\theta_0$.

Now we note that the supports of fast variations of boundary-layer functions and of functions rapidly varying on a neighborhood of Γ_t do not intersect (up to terms $\mathcal{O}(\varepsilon^{\infty})$), since $\operatorname{dist}(\partial\Omega,\Gamma_t)\geq$ const. Hence the solution in a neighborhood of Γ_t and in a neighborhood of $\partial\Omega$ is constructed differently.

Let us consider a neighborhood of Γ_t. Passing to the limit as $\tau\to\infty$, we obtain that the terms in (15), (16) belong to the space $\mathcal{S}$, since relations (17) hold. So we can use the method developed for phase transition problems by V. Danilov, G. Omel'yanov and E. Radkevich [7, 8, 20, 21, 24]: step by step we decompose the coefficients at ε^j in (15), (16), $j=-2,-1,0,...$, into the Taylor expansion at the point $t=\psi(x)$ and use the relation $\eta=(t-\psi)/\varepsilon$. Further, passing to the functions of independent variables η, x, we obtain the asymptotic solution on the surface $\mathcal{T}_t$. Finally, we define a sufficiently smooth extension of these functions on $R^1_\eta\times Q$, so that the lower terms of the asymptotic expansion exist and belong to the space $\mathcal{H}$.

Let us denote $\check{F}=F(\eta,x,t)|_{t=\psi(x)}$ and consider the terms $\mathcal{O}(\varepsilon^{-2})$ in (16):

$$\frac{\partial^2}{\partial\eta^2}\Big\{|\nabla\psi|^2\,\frac{\partial^2\check{\chi}}{\partial\eta^2}+\check{\chi}-(\check{\varphi}_0+\check{\chi})^3\Big\}=0\ .$$

After the integration, for $\check{\chi}\in\mathcal{S}$ we get

$$(20)\qquad |\nabla\psi|^2\,\frac{\partial^2\check{\chi}}{\partial\eta^2}+\check{\chi}-(\check{\varphi}_0+\check{\chi})^3=c\,,\qquad \check{\chi}\to 0\ \text{ as } \eta\to\pm\infty\ .$$

Choosing $c=-\check{\varphi}_0^3$, we get that the solution on $\mathcal{T}_t$ has the form (12), where the "constant" of integration $\psi_1=\psi_1(x)$ is an arbitrary function from $C^{\infty}(\bar{\Omega})$.

Let us extend $\check{\chi}$ (defined on the section $(x,t)\in\mathcal{T}_T=\bigcup_{t\in[0,T]}\Gamma_t$, $\eta\in R^1$) by the identity to $\chi\equiv\check{\chi}(\eta,x)$ for all $(x,t)\in\bar{Q}$, $\eta\in R^1$. Now, setting the terms $\mathcal{O}(\varepsilon^{-2+k})$ equal to zero, from (15), (16) we get the following equations

$$(21)\qquad \frac{\partial^2\check{U}_k}{\partial\eta^2}=\check{F}_k^{\theta}\,,\qquad \check{U}_k\to 0\ \text{ as } \eta\to-\infty\ ,$$

$$(22)\qquad \frac{\partial^2}{\partial\eta^2}\hat{L}\,\check{W}_k=\check{F}_k^{\varphi}\,,\qquad \check{W}_k\to 0\ \text{ as } \eta\to-\infty\ .$$

Here

$$\hat{L} = |\nabla\psi|^2 \frac{\partial^2}{\partial\eta^2} + 1 - 3(\check{\varphi}_0 + \chi)^2 \ ,$$

$\check{F}_k^{\varphi}$, $\check{F}_k^{\theta}$ are functions of $\theta_0, \varphi_0, ..., U_{k-1}, W_{k-1}$ and of their derivatives at the point $t = \psi(x)$, $k = 1, 2, ..., M$. In particular,

$$\check{F}_1^{\varphi} = \frac{\partial^2}{\partial\eta^2}\Big\{\hat{\Pi}\,\frac{\partial\chi}{\partial\eta} + 3(\check{\varphi}_0 + \chi)^2\,\varphi_1\Big\} + 3\,\frac{\partial\varphi_0}{\partial t}\Big\{2 + \eta\,\frac{\partial}{\partial\eta}\Big\}\,\frac{\partial}{\partial\eta}(\varphi_0 + \chi)^2 - \kappa\,|\nabla\psi|^{-2}\,\frac{\partial\chi}{\partial\eta}\Big|_{t=\psi} \ , \tag{23}$$

$$\check{F}_1^{\theta} = |\nabla\psi|^{-2}\,\frac{\partial\chi}{\partial\eta} \ . \tag{24}$$

It is not too difficult to prove that the following statement holds (see also [7, 12]).

Lemma 2. *The solutions* $\check{U}_k \in \mathcal{H}$, $\check{W}_k \in \mathcal{H}$ *of* (21), (22) *exist if and only if*

$$\check{F}_k^{\theta} \in \mathcal{S}\,, \qquad \check{F}_k^{\varphi} \in \mathcal{S} \ , \tag{25}$$

$$\int_{-\infty}^{\infty} \check{F}_k^{\theta}\,d\eta = 0\,, \qquad \int_{-\infty}^{\infty} \check{F}_k^{\varphi}\,d\eta = 0 \ , \tag{26}$$

$$\int_{-\infty}^{\infty} \check{f}_k^{\varphi}\,\frac{\partial\chi}{\partial\eta} d\eta = 0 \ , \tag{27}$$

where

$$\check{f}_k^{\varphi} = \int_{-\infty}^{\eta}\int_{-\infty}^{\eta'} \check{F}_k^{\varphi}(\eta'', x)\,d\eta''\,d\eta' \ .$$

Using (23), (24), it is easy to see that the conditions (25), (26) hold automatically for $k = 1$.

Further, since

$$\check{f}_1^{\varphi} = \hat{\Pi}\,\frac{\partial\chi}{\partial\eta} + 3(\varphi_1 + \xi\,\varphi_{0t})\,(2\,\varphi_0\,\chi + \chi^2)\Big|_{t=\psi} + \sqrt{2}\,\kappa\,\beta\,\ln(J^2 R) \ ,$$

$$R = \Big\{e^{\xi} + 4\hat{\varphi}_0 - 2\sqrt{2}\,Q\Big\}\,\Big\{e^{\xi} + 4\hat{\varphi}_0 + 2\sqrt{2}\,Q\Big\}^{-1} \ ,$$

where notation (12) is used, simple calculations yield the statement

Lemma 3. *For* $k = 1$ *condition* (27) *is equivalent to equation* (10).

Now, since (25)–(27) are satisfied for $k=1$, we can obtain the functions $\check{U}_1$, $\check{W}_1$:

$$\check{U}_1 = \check{U}_1^+(x)\,\zeta(\eta,x)\,, \qquad \check{W}_1 = \check{W}_1^+(x)\,\zeta(\eta,x) + \omega_1(\eta,x)\ .$$

Here

$$\zeta = \frac{\sqrt{2}}{a}\ln(J^2R) \in \mathcal{H}\,, \qquad \zeta^+ = 1, \quad \zeta^- = 0\ ,$$
$$\check{U}_1^+ = -a\,v_\nu\,, \qquad \check{W}_1^+ = \check{U}_1^+\kappa/Q^2\ ,$$
$$\omega_1(\eta,x) = \omega_{1,1}(\eta,x) + \psi_2(x)\,\chi_\eta(\eta,x) \ \in\ \mathcal{S}\ ,$$

ψ_2 is the "constant" of integration, $a = 2I/\check{\varphi}_0$.

Let us fix the functions $\theta_1(x,t)$, $\varphi_1(x,t)$ so that θ_1, φ_1 are sufficiently smooth on Q and that $\theta_1 = \theta_1^-$, $\varphi_1 = \Phi_1^-$ on $\bar{\mathcal{D}}_t^-$ for all $t \geq 0$. Let us define the functions $\theta_{1c}^-(x,t)$, $\Phi_{1c}^-(x,t)$ as $\theta_{1c}^- = \theta_1$, $\Phi_{1c}^- = \varphi_1$ for all $(x,t) \in Q$, and let θ_{1c}^+, Φ_{1c}^+ be sufficiently smooth extensions of θ_1^+, Φ_1^+ in $\bar{\mathcal{D}}_t^+ \cup \Gamma_{t,\delta}^-$, such that the heat equations (19) are satisfied for $k=1$. Here $0 < \delta \ll 1$ is an arbitrary number and $\Gamma_{t,\delta}^- \subset \mathcal{D}_t^-$ is a layer of width δ adjoined to Γ_t. Now we can define the extensions U_1, W_1

$$U_1 = u_1(x,t)\,\zeta(\eta,x)\,, \qquad W_1 = \mathrm{w}_1(x,t)\,\zeta(\eta,x) + \omega_1(\eta,x)\ , \tag{28}$$

where

$$u_1 = \theta_{1c}^+ - \theta_{1c}^-\,, \qquad \mathrm{w}_1 = \Phi_{1c}^+ - \Phi_{1c}^-\ .$$

Note that outside Γ_t we have

$$\theta_1 + U_1 = \theta_1^\pm + \Big[(\theta_{1c}^+ - \theta_{1c}^-)\,\mathcal{O}(e^{\mp\beta\eta})\Big]\,, \qquad x \in \mathcal{D}_t^\pm, \quad \eta = (t-\psi)/\varepsilon + \psi_1\ ,$$
$$\varphi_1 + W_1 = \Phi_1^\pm + \Big[(\Phi_{1c}^+ - \Phi_{1c}^-)\,\mathcal{O}(e^{\mp\beta\eta})\Big] + \omega_1\ .$$

This implies that the expressions in square brackets are of maximal value $\mathcal{O}(\varepsilon)$ in an $\varepsilon^{1-\mu}$-neighborhood of Γ_t, $0<\mu<1$, and they are exponentially small outside this neighborhood. Hence, the freedom in choosing the extensions $\theta_{1c}^\pm$, $\Phi_{1c}^\pm$ results in corrections of order $\mathcal{O}(\varepsilon)$ which automatically are taken into account when we construct the next approximations.

Thus we have the following conditions for jumps of $\theta_1^\pm$, $\Phi_1^\pm$ on Γ_t

$$[\theta_1^\pm]\Big|_{\Gamma_t} = a\,v_\nu\,, \qquad [\Phi_1^\pm]\Big|_{\Gamma_t} = \frac{\kappa\,a}{Q^2}\,v_\nu\ . \tag{29}$$

Here we use the notation $[f^\pm]|_{\Gamma_t} = f^-|_{\Gamma_t+0} - f^+|_{\Gamma_t-0}$ and note that the vector ν is directed from $\mathcal{D}_t^+$ to $\mathcal{D}_t^-$.

Let us consider equations (21), (22) in the case $k = 2$. The right-hand sides of these equations belong to $\mathcal{S}$, since equation (8) and the first equation (19) hold for $k = 1$. Further, after some calculations, we get the following statement

Lemma 4. *For $k = 2$ conditions (26) are equivalent to the equalities*

$$\left[\frac{\partial\theta_1^\pm}{\partial\nu}\right]\Big|_{\Gamma_t} = -a\,v_\nu\Big\{\Big(1+\frac{\kappa}{Q^2}\Big)\,v_\nu + \mathcal{K}_t - \frac{4}{a\,b\,Q}\,\frac{\partial\check{\varphi}_0}{\partial\nu}\Big\}\,,$$

$$\left[\frac{\partial}{\partial\nu}\Big((3\,\varphi_0^2-1)\,\Phi_1^\pm\Big)\right]\Big|_{\Gamma_t} = -12\,|\nabla\psi|\,\frac{\partial\varphi_0}{\partial t}\Big|_{\Gamma_t}\Big\{Q\mathcal{K}_t - v_\nu\,\mathrm{div}(Q\nabla\psi) + \frac{a}{2}\,\frac{\partial\check{\varphi}_0}{\partial\nu}\Big\} + v_\nu^2\,\frac{\kappa\,a}{2Q^2}(\kappa + Q^2\,\Delta\psi)\,.$$

Finally, for $k = 2$, after some trivial but cumbersome calculations condition (27) can be transformed to the following linear inhomogeneous equation for the phase correction ψ_1:

$$\kappa\,\frac{\partial\psi_1}{\partial\nu} = G_1\,\hat{\mathcal{K}}'\,\psi_1 + f^{\psi_1}(x)\,,\qquad \psi_1\Big|_{\Gamma_0} = 0\,. \tag{31}$$

Here $\hat{\mathcal{K}}'$ is the variation of the operator $\mathcal{K}_t$ from (10), the right-hand side $f^{\psi_1}(x)$ depends on the functions ψ, θ_0, φ_0, $\theta_1^\pm$, $\Phi_1^\pm$.

The following constructions are performed similarly:

1. Calculating $\check{U}_k^\pm$, $\check{W}_k^\pm$, we get conditions for jumps of the functions $\theta_k^\pm$, $\Phi_k^\pm$ on Γ_t;

2. By using formulas similar to (28), we define the extensions U_k, W_k on Q. Since $\theta_k^\pm$, $\Phi_k^\pm$ is the solution of (19), we see that conditions (25) hold;

3. Conditions (26) imply conditions for the normal derivatives of $\theta_{k-1}^\pm$, $\Phi_{k-1}^\pm$ on Γ_t;

4. Condition (27) yields an equation (similar to (31)) for the "constant" of integration ψ_k.

In fact, since

$$f\Big(\beta(\eta+\psi_1)\Big)+\varepsilon\,\beta\,f'_\eta\,\psi_2+\varepsilon^2\,\beta\,f'_\eta\,\psi_3+\ldots = f\Big(\beta(\eta+\psi_1+\varepsilon\,\psi_2+\varepsilon^2\,\psi_3+\ldots)\Big)+\mathcal{O}(\varepsilon^2)\,,$$

the functions ψ_k, $k \geq 1$, are the lower corrections to the principle phase ψ. So, these functions describe the front of the soliton wave more precisely.

We must also pose the initial conditions for the equations (3) as well as for the heat equations (8), (9), (19). Since θ_0, φ_0 are smooth functions on $\Omega\times[0,T]$

and $\theta_k^\pm$, $\Phi_k^\pm$ are smooth functions on $\mathcal{D}_t^\pm \times [0,T]$, let us define the initial data as follows

$$\overline{\theta^0} = w - \lim_{\varepsilon\to 0} \theta^0(x,\varepsilon), \quad \overline{\varphi^0} = w - \lim_{\varepsilon\to 0} \varphi^0(x,\varepsilon)\,, \qquad x \in \Omega\,,$$

$$\overline{\theta_k^0}^{\,\pm} = w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^k}\Big(\theta^0(x,\varepsilon) - \theta_0 - \sum_{j=1}^{k-1} \varepsilon^j (\theta_j + U_j + Y_j)\Big|_{t=0}\Big)\,, \qquad x \in \mathcal{D}_0^\pm\,,$$

$$\overline{\varphi_k^0}^{\,\pm} = w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^k}\Big(\varphi^0(x,\varepsilon) - \varphi_0 - \chi - \sum_{j=1}^{k-1} \varepsilon^j (\varphi_j + W_j + Z_j)\Big|_{t=0}\Big)\,, \qquad x \in \mathcal{D}_0^\pm\,.$$

These formulas yield the initial data (8), (9) and the initial data for equations (19):

$$\theta_k^\pm|_{t=0} = \overline{\theta_k^0}^{\,\pm}(x), \quad \varphi_k^\pm|_{t=0} = \overline{\varphi_k^0}^{\,\pm}(x)\,, \qquad x \in \mathcal{D}_0^\pm\,. \tag{32}$$

Furthermore, a natural form of the initial value of φ is $\varphi|_{t=0} = \varphi^0(r_0(x)/\varepsilon, x, \varepsilon)$, where r_0 is the distance function. Nevertheless, the constructed asymptotic depends on the variable $\eta = (t - \psi(x))/\varepsilon + \psi_1$ with functions ψ and ψ_1 unknown beforehand. Note that $\psi|_{\Gamma_0} = 0$ and $\psi_1|_{\Gamma_0} = 0$, the unit vector $\nabla(\psi/|\nabla\psi|)|_{\Gamma_0}$ is normal to Γ_0 and directed opposite to $\nabla r_0|_{\Gamma_0}$. Therefore, for any function $f(\eta, x) \in \mathcal{H}$ we have

$$\begin{aligned} f\Big(\frac{r_0}{\varepsilon}, x\Big) &= f\Big(\frac{-\psi + \varepsilon\,\psi_1}{|\nabla\psi|\,\varepsilon} + \frac{1}{\varepsilon}\,g_1 + g_2,\, x\Big) \\ &= f\Big(\frac{\eta}{|\nabla\psi|},\, x\Big) + \varepsilon\, f'_\eta\Big(\frac{\eta}{|\nabla\psi|},\, x\Big)\Big\{\eta\,\frac{\partial g_2}{\partial\nu}\Big|_{\Gamma_0} + \frac{1}{2}\,\eta^2\,\frac{\partial^2 g_1}{\partial \nu^2}\Big|_{\Gamma_0}\Big\} + \mathcal{O}(\varepsilon^2)\,, \end{aligned}$$

where $g_1 = r_0 + \psi/|\nabla\psi|$, $g_2 = -\psi_1/|\nabla\psi|$, $\partial/\partial\nu = |\nabla\psi|^{-1}\langle\nabla\psi, \nabla\rangle$, and $\eta = -\psi/\varepsilon + \psi_1$ for $t = 0$. We also take into account that $f'_\eta \in \mathcal{S}$, and hence, the functions $|\eta^k f'_\eta|$ are bounded in C for all $k \le M$.

Moreover, in §3 we prove that the initial perturbations $\mathcal{O}(\varepsilon^3)$ (in the sense of $L^2(\Omega)$) do not lead the soliton-type solution out of the stability domain. Therefore, we fix the initial data only up to the terms $\mathcal{O}(\varepsilon^2)$. The above constructions imply that the behavior of smooth (for the $\varepsilon > 0$) functions $\theta^0(x,\varepsilon)$, $\varphi^0(x,\varepsilon)$ may be arbitrary outside an ε-neighborhood of Γ_0, but φ^0 and the lower terms of θ^0 (w.r.t. ε) must be of a special form in this neighborhood.

Now let us consider the boundary conditions on the external boundary $\partial\Omega$ and calculate the boundary-layer functions. The soliton part of the asymptotic solution satisfies both boundary conditions in (3) up to $\mathcal{O}(\varepsilon^\infty)$. So, for φ on Σ, a discrepancy in the second boundary condition arises only from the regular part of the solution, since $\operatorname{dist}(\Gamma_t, \partial\Omega) \geq \text{const}$ and, in general,

$$\frac{\partial}{\partial N}\Delta\varphi_0\Big|_{\Sigma} \neq 0 \ .$$

Let us put $Z_j = 0$ for $j = 1, 2$ since

$$\frac{\partial}{\partial N}\Delta\varepsilon^3\, Z_3\left(\frac{x_N}{\varepsilon}, x'\right)\Big|_{x_N=0} = \mathcal{O}(1) \ .$$

Then, using the construction common for the boundary layer asymptotic solutions, we obtain the equation for Z_3:

$$\frac{\partial^2}{\partial\tau^2}Z_3 - q\, Z_3 = 0\,, \qquad Z_3 \to 0, \quad \tau \to \infty \ , \tag{32}$$

where $q = 3\,\varphi_0^2|_{\Sigma} - 1 > 0$. Obviously,

$$Z_3 = c_3(x', t)\exp(-\sqrt{q}\,\tau) \ .$$

Now we can see that the boundary condition

$$\frac{\partial^3}{\partial\tau^3}Z_3\Big|_{\tau=0} = \frac{\partial}{\partial N}\Delta\varphi_0\Big|_{\Sigma} \tag{33}$$

leads to the formula

$$c_3 = -q^{-3/2}\,\frac{\partial}{\partial N}\Delta\varphi_0\Big|_{\Sigma} \ .$$

Further, we note that the appearance of the boundary-layer function $\varepsilon^3 Z_3$ necessarily implies a correction of the Neumann condition in the term $\mathcal{O}(\varepsilon^2)$. Let, for definiteness, $\partial\Omega \cap \partial\mathcal{D}_t^- = \partial\Omega$. Then the Neumann condition for Φ_2^- has the form

$$\frac{\partial\Phi_2^-}{\partial N}\Big|_{\Sigma} = \frac{\partial Z_3}{\partial\tau}\Big|_{\tau=0} = \frac{1}{q}\,\frac{\partial}{\partial N}\Delta\varphi_0\Big|_{\Sigma} \ . \tag{34}$$

The appearance of the boundary-layer functions Z_k implies the boundary-layer terms Y_j in the θ asymptotic expansion. Since the boundary $\partial\Omega$ is fixed, for $j = 1, ..., 4$ we have $Y_j = 0$ and for Y_5 we have the equation:

$$\frac{\partial^2 Y_5}{\partial\tau^2} = \frac{\partial}{\partial t}Z_3(\tau, x', t)\,, \qquad Y_5 \to 0, \quad \tau \to \infty \ . \tag{35}$$

Therefore

$$Y_5 = -q^{-5/2}\,\frac{\partial^2}{\partial N\partial t}\Delta\varphi_0\Big|_{\Sigma}\exp(-\sqrt{q}\,\tau) \ .$$

Conversely, the appearance of Y_5 leads to a correction of the Neumann condition for the temperature in the term $\mathcal{O}(\varepsilon^4)$. Thus, the Neumann condition for θ_4^- has the form

$$\frac{\partial \theta_4^-}{\partial N}\bigg|_\Sigma = \frac{\partial Y_5}{\partial \tau}\bigg|_{\tau=0} = \frac{1}{q^2}\frac{\partial^2}{\partial N \partial t}\Delta\varphi_0\bigg|_\Sigma . \tag{36}$$

Finally, the asymptotic expansion in a small neighborhood of $\partial\Omega$ has the following form

$$\theta = \theta_0(x,t) + \sum_{j=1}^{4} \varepsilon^j \, \theta_j^-(x,t) + \sum_{j=5}^{M} \varepsilon^j \Big(\theta_j^-(x,t) + Y_j(\tau, x', t)\Big) + \mathcal{O}(\varepsilon^{M+1}) \;,$$

$$\varphi = \varphi_0(x,t) + \sum_{j=1}^{2} \varepsilon^j \, \Phi_j^-(x,t) + \sum_{j=3}^{M} \varepsilon^j \Big(\Phi_j^-(x,t) + Z_j(\tau, x', t)\Big) + \mathcal{O}(\varepsilon^{M+1}) \;.$$

Here $Z_3, Y_5 \in \mathcal{P}$ are described above, $Z_k \in \mathcal{P}$, $k \geq 4$, and $Y_j \in \mathcal{P}$, $j \geq 6$, are calculated from linear inhomogeneous problems like (32), (33), (35). In turn, we obtain the boundary conditions for problems (19):

1. Conditions (8), (9) for θ_0, φ_0;

2. Conditions

$$\frac{\partial \theta_j^-}{\partial N}\bigg|_\Sigma = 0 \,, \qquad \frac{\partial \Phi_1^-}{\partial N}\bigg|_\Sigma = 0$$

for θ_j^-, $j = 1, ..., 3$, Φ_1^-;

3. Conditions (34), (36) for Φ_2^-, θ_4^-;

4. Conditions

$$\frac{\partial \theta_j^-}{\partial N}\bigg|_\Sigma = \frac{\partial Y_{j+1}}{\partial \tau}\bigg|_{\tau=0} , \qquad \frac{\partial \Phi_k^-}{\partial N}\bigg|_\Sigma = \frac{\partial Z_{k+1}}{\partial \tau}\bigg|_{\tau=0}$$

for θ_j^-, $j \geq 5$ and Φ_k^-, $k \geq 3$.

Theorem 1 is proved. ∎

Moreover, analyzing our construction, we obtain the statement.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then for any integer $M \geq 0$ there exist the functions*

$$
\begin{aligned}
&\theta_M^{\rm as} = \theta_0 + \sum_{j=1}^{M} \varepsilon^j (\theta_j + U_j + Y_j) + \varepsilon^{M+1} (U_{M+1} + Y_{M+1}) \;, \\
(37)\qquad & \\
&\varphi_M^{\rm as} = \varphi_0 + \chi + \sum_{j=1}^{M} \varepsilon^j (\varphi_j + W_j + Z_j) + \varepsilon^{M+1} (W_{M+1} + Z_{M+1}) \;,
\end{aligned}
$$

such that

$$
(38)\qquad
\begin{aligned}
&\frac{\partial}{\partial t} (\theta_M^{\rm as} + \varphi_M^{\rm as}) - \Delta \theta_M^{\rm as} - f(x,t) = \varepsilon^M \mathcal{F}_M^\theta \;, \\
&\kappa \frac{\partial \varphi_M^{\rm as}}{\partial t} + \Delta \Big(\varepsilon^2 \Delta \varphi_M^{\rm as} + \varphi_M^{\rm as} - (\varphi_M^{\rm as})^3 + \varepsilon\, \kappa_1\, \theta_M^{\rm as} \Big) = \varepsilon^M \mathcal{F}_M^\varphi \;,
\end{aligned}
$$

$$
(39)\qquad \frac{\partial \theta_M^{\rm as}}{\partial N}\bigg|_\Sigma = \varepsilon^{M+1} F_M^\theta \,, \qquad \frac{\partial \varphi_M^{\rm as}}{\partial N}\bigg|_\Sigma = 0 \,, \qquad \frac{\partial}{\partial N} \Delta \varphi_M^{\rm as}\bigg|_\Sigma = \varepsilon^{M-1} F_M^\varphi \;,
$$

$$
(40)\qquad \theta_M^{\rm as}|_{t=0} = \overline{\theta^0} + \varepsilon (\overline{\theta_1^{0\pm}} + \tilde{\theta}) \,, \qquad \varphi_M^{\rm as}|_{t=0} = \overline{\varphi^0} + \chi\Big|_{t=0} + \varepsilon (\overline{\varphi_1^{0\pm}} + \tilde{\varphi}) \;.
$$

Here $\tilde{\theta}$, $\tilde{\varphi}$, $\mathcal{F}_M^{\varphi,\theta}$, $F_M^{\varphi,\theta}$ are (smooth for $\varepsilon > 0$) functions such that

$$
\begin{aligned}
(40')\qquad & \|\tilde{\theta}; L^2(\Omega)\| + \|\tilde{\varphi}; L^2(\Omega)\| \leq c_0 \sqrt{\varepsilon} \;, \\
& \|\mathcal{F}_M^\theta; C(\bar{Q})\| + \|\mathcal{F}_M^\varphi; C(\bar{Q})\| \leq c_1 \;, \\
(41)\qquad & \|F_M^\theta; C(\Sigma)\| + \|F_M^\varphi; C(\Sigma)\| \leq c_2 \;, \\
& \|\mathcal{F}_M^\theta; L^2(\Omega)\| + \|\mathcal{F}_M^\varphi; L^2(\Omega)\| \leq c_3 \sqrt{\varepsilon} \;,
\end{aligned}
$$

where the constants c_j are independent of ε.

3 – Justification of the soliton type asymptotic solution

In this section we shall obtain estimates for the differences between the exact θ, φ and asymptotic $\theta_M^{\rm as}$, $\varphi_M^{\rm as}$ solutions of problem (3). Let us introduce the notation $\sigma = \theta - \theta_M^{\rm as}$, $\omega = \varphi - \varphi_M^{\rm as}$ and let the initial data θ^0, φ^0 exhibit a special behavior. Then, from (3) and (38)–(40), we get the following problem for the remainders σ, ω:

$$
(42)\qquad \frac{\partial}{\partial t} (\sigma + \omega) - \Delta \sigma = -\varepsilon^M \mathcal{F}_M^\theta \;,
$$

$$\kappa \frac{\partial \omega}{\partial t} + \Delta\Big(\varepsilon^2 \Delta\omega + \omega(1 - 3\,\varphi_M^2 - 3\varphi_M\,\omega - \omega^2) + \varepsilon\,\kappa_1\,\sigma\Big) = -\varepsilon^M \mathcal{F}_M^{\varphi}\ , \tag{43}$$

$$\begin{aligned} &\frac{\partial \sigma}{\partial N}\Big|_{\Sigma} = -\varepsilon^{M+1} F_M^{\theta}, \qquad \frac{\partial \omega}{\partial N}\Big|_{\Sigma} = 0\,, \qquad \frac{\partial}{\partial N}\Delta\omega\Big|_{\Sigma} = -\varepsilon^{M-1} F_M^{\varphi}\ , \\ &\sigma|_{t=0} = -\varepsilon^{M+1/2} f_M^{\theta}\,, \qquad \omega|_{t=0} = -\varepsilon^{M+1/2} f_M^{\varphi}\ . \end{aligned} \tag{44}$$

Here $\mathcal{F}_M^{\theta,\varphi}$, $F_M^{\theta,\varphi}$ are smooth functions satisfying (41), $f_M^{\theta,\varphi}$ are functions such that

$$\|f_M^{\theta}; L^2(\Omega)\| + \|f_M^{\varphi}; L^2(\Omega)\| \le c\sqrt{\varepsilon} \tag{45}$$

with constant c independent of ε. To simplify the notation, we omit the superscript denoting asymptotic solutions.

The main result of this section is

Theorem 3. *Let there exist a sufficiently smooth solution of problem* (3) *on the time interval* $[0,T]$, *where the quantity* $T > 0$ *is independent of* ε. *Let also the assumptions of Theorem* 1 *be satisfied,* $M \ge 2$ *and there exist a constant* $\gamma > 0$ *such that* $\varphi_0 - 1/\sqrt{3} \ge \gamma$ *uniformly in* $x \in \Omega$, $t \in [0,T]$. *Then the estimates*

$$\begin{aligned} &\|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| \le c\,\varepsilon^{M+1}\ , \\ &\|\nabla\omega; L^2(Q)\| + \|\nabla\sigma; L^2(Q)\| \le c\,\varepsilon^{M+1/2}\,, \qquad \|\Delta\omega; L^2(Q)\| \le c\,\varepsilon^{M-1/2} \end{aligned} \tag{46}$$

hold with constant c *independent of* ε.

The main obstacle to the derivation of a priori estimates (46) is a rapidly varying coefficient φ_M in (43). This is typical for nonlinear equations and the following summand

$$J = \int_0^t \int_\Omega \Big(\nabla\omega, \nabla(\varphi_M^2\omega)\Big)\, dx\, dt'$$

appears on the right-hand side of the energy inequality, while on the left-hand side we have only $\|\omega; L^\infty(0,T; L^2(\Omega))\|^2$ and $\varepsilon^2\|\Delta\omega; L^2(Q)\|^2$. It is clear that trivial estimates, for example,

$$\begin{aligned} |J| &\le c\Big(\max_{x,t}|\Delta\varphi_M^2|\,\|\omega; L^2(Q)\|^2 + \|\nabla\omega; L^2(Q)\|^2\Big) \\ &\le \frac{c_1}{\varepsilon^2}\,\|\omega; L^2(Q)\|^2 + c\,\|\nabla\omega; L^2(Q)\|^2\ , \end{aligned}$$

allow us to prove that the discrepancy is bounded only for the time $T_\varepsilon \sim \varepsilon^2$. The first point of observation is that, to overcome this difficulty, we can rewrite the "bad" summand $(\nabla\omega, \nabla(\varphi_M^2\omega))$ in the form

$$-(3\varphi_0^2 - 1)\,|\nabla\omega|^2 - \varepsilon^{-2}\,\omega^2\,\Psi_1 + |\nabla\omega|^2\,\chi\,g_1 + \varepsilon^{-2}\,\omega^2\,\Psi_1(\Psi_2 + \varepsilon\,\Psi_3)\ ,$$

where nonnegative Ψ_1 belongs to $\mathcal{S}$, $\Psi_2 \in \mathcal{S}$, $\Psi_3 \in \mathcal{H}$, g_1 is a bounded in C function. Let us recall that $\varphi_0 \geq 1/\sqrt{3}+\gamma$, where $\gamma > 0$ is a fixed number. Thus, we obtain the summand $\|\sqrt{3\varphi_0^2 - 1}\,|\nabla\omega|;\, L^2(Q)\|^2 + \varepsilon^{-2}\|\sqrt{\Psi_1}\,\omega;\, L^2(Q)\|^2$ on the left-hand side of the energy inequality and the expression

$$J_1 = \int_0^t \int_\Omega \chi\, g_1\, |\nabla\omega|^2\, dx\, dt' + \frac{1}{\varepsilon^2} \int_0^t \int_\Omega \omega^2\, \Psi_1(\Psi_2 + \varepsilon\, \Psi_3)\, dx\, dt'$$

on the right-hand side. Obviously, if we again estimate the functions χg_1 and $\Psi_1(\Psi_2 + \varepsilon\Psi_3)$ by the maximum of the modulus, there is no result, but we now can use the fact that the functions χ and Ψ_1 are bounded by a constant (in C) and localized (with precision up to $\mathcal{O}(\varepsilon)$) in an ε-neighborhood of the free interface Γ_t. Here the main point is Lemma 6 about estimating integrals of the form

$$I = \int_{-\infty}^{\infty} v\left(\frac{x}{\varepsilon}\right) f(x)\, dx \ ,$$

where $v(\eta) \in \mathcal{S}$ is a known function exponentially vanishing outside the point $\eta = 0$. To estimate f we can use the norms in $L^k(R^1)$ only for $k = 1$ and $k = 2$. Lemma 6 implies that for sufficiently small ε the first summand in J_1 is bounded from above by $o(R_\omega)$, where $R_\omega = \gamma\|\nabla\omega; L^2(Q)\|^2 + \varepsilon^2\|\Delta\omega; L^2(Q)\|^2$. Since on the left-hand side of the energy inequality we have kR_ω with a constant $k > 0$ independent of ε, we see that the first summand in J_1 is no obstacle to the derivation of a priori estimate for all finite T.

Similarly, by Lemma 6, we now can prove that for sufficiently small ε the second summand in J_1 has the upper bound $k(R_\omega + 3\varepsilon^{-2}\|\omega\sqrt{\Psi_1}; L^2(0,t; L^2(\Omega))\|^2)/4$. Therefore, the second summand is no more an obstacle to the derivation of a priori estimate for all finite T.

It should be noted that, justifying a boundary-layer asymptotics for the semilinear Dirichlet problem, M. Berger and L. Fraenkel [3] proved a statement similar to Lemma 6. However, in the boundary-layer situation the fast varying asymptotics v is localized in a small neighborhood of the external boundary. Moreover, the discrepancy vanishes on the boundary. The condition $f|_{\partial\Omega} = 0$ used in [3] considerably simplifies the estimation of the integral I. In the phase transition problems the remainder does not necessarily vanish on the free boundary, so, in these problems one cannot use the estimate derived by M. Berger and L. Fraenkel.

To prove Theorem 3 we first need to obtain an auxiliary result.

Theorem 4. *Under the assumptions of Theorem 3, the following estimates*

$$
\begin{aligned}
(47)\quad & \|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| \le c\,\varepsilon^{M+1/2}\ ,\\
& \|\nabla\sigma; L^2(Q)\| + \|\nabla\omega; L^2(Q)\| \le c\,\varepsilon^{M+1/2}\,, \qquad \|\Delta\omega; L^2(Q)\| \le c\,\varepsilon^{M-1/2}\ ,
\end{aligned}
$$

hold with constant c independent of ε.

Proof: Multiplying equations (42), (43) by σ, ω respectively and integrating on Ω, we get the relations

$$
(48)\quad \frac{1}{2}\frac{d}{dt}\|\sigma\|^2 + \int_\Omega \omega_t\,\sigma\,dx + \|\nabla\sigma\|^2 = -\varepsilon^M \int_\Omega \sigma\,\mathcal{F}^\theta_M\,dx - \varepsilon^{M+1}\int_{\partial\Omega} \sigma\,F^\theta_M\,dx'\ ,
$$

$$
\begin{aligned}
(49)\quad \frac{\kappa}{2}\frac{d}{dt}\|\omega\|^2 + \varepsilon^2\,\|\Delta\omega\|^2 + 3\,\|\omega\nabla\omega\|^2 &= \int_\Omega \Big(\nabla\omega, \nabla(\omega(1-3\varphi_M^2))\Big)\,dx \,-\\
&- 3\int_\Omega \Big(\nabla\omega, \nabla(\varphi_M\omega^2)\Big)\,dx + \varepsilon\,\kappa_1 \int_\Omega (\nabla\omega,\nabla\sigma)\,dx\\
&- \varepsilon^M \int_\Omega \omega\,\mathcal{F}^\varphi_M\,dx + \varepsilon^{M+1}\int_{\partial\Omega} \omega\,F^\varphi_M\,dx' + \varepsilon^{M+2}\,\kappa_1 \int_{\partial\Omega} \omega\,F^\theta_M\,dx'\ .
\end{aligned}
$$

Here and below $\|f\|$ denotes the $L^2(\Omega)$ norm of f.

Further, multiplying (42) by ω, integrating on Ω and summing with (48), we obtain

$$
\begin{aligned}
(50)\quad \frac{1}{2}\frac{d}{dt}\Big\{\|\omega\|^2 + \|\sigma\|^2 + 2\int_\Omega \omega\,\sigma\,dx\Big\} + \|\nabla\sigma\|^2 + \int_\Omega (\nabla\omega,\nabla\sigma)\,dx &=\\
= -\varepsilon^M \int_\Omega (\omega+\sigma)\,\mathcal{F}^\theta_M\,dx - \varepsilon^{M+1}\int_{\partial\Omega} (\omega+\sigma)\,F^\theta_M\,dx'\ .
\end{aligned}
$$

Let us fix a constant $K > 1/\gamma$. Multiplying (49) by K and summing with (50), we get the equality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{(1+\kappa K)\,\|\omega\|^2 + \|\sigma\|^2 + 2\int_\Omega \omega\sigma\,dx\Big\} + \|\nabla\sigma\|^2 + \varepsilon^2 K\|\Delta\omega\|^2 + 3K\,\|\omega\,\nabla\omega\|^2 =\\
&\qquad = K\int_\Omega \Big(\nabla\omega,\ \nabla(\omega(1-3\,\varphi_M^2))\Big)\,dx - 3K\int_\Omega \Big(\nabla\omega,\ \nabla(\varphi_M\omega^2)\Big)\,dx\\
(5.1)\qquad&\qquad + (\varepsilon\,\kappa_1 K - 1)\int_\Omega (\nabla\omega,\nabla\sigma)\,dx - \varepsilon^M \int_\Omega \Big\{(\omega+\sigma)\,\mathcal{F}^\theta_M + \omega\,K\,\mathcal{F}^\varphi_M\Big\}\,dx\\
&\qquad - \varepsilon^{M+1}\int_{\partial\Omega} \Big\{(\omega+\sigma)\,F^\theta_M - \omega K(F^\varphi_M + \varepsilon\,\kappa_1\,F^\theta_M)\Big\}\,dx'\ .
\end{aligned}
$$

We shall analyze the terms in the right-hand side of (51).

Lemma 5. *Let φ_M be the asymptotical expansion* (37). *Then*

$$(52)\quad \int_\Omega \Big(\nabla\omega, \nabla(\omega(1-3\varphi_M^2))\Big)\,dx = -\int_\Omega (3\varphi_0^2-1)\,|\nabla\omega|^2\,dx - \frac{1}{\varepsilon^2}\int_\Omega \omega^2\,\Psi_1\,dx + I\;,$$

$$I = \int_\Omega |\nabla\omega|^2\,(g_1\,\chi+\varepsilon\,g_2)\,dx + \frac{1}{\varepsilon^2}\int_\Omega \omega^2\Big(\Psi_1(\Psi_2+\varepsilon\,\Psi_3)+\varepsilon^2\,\Psi_4\Big)\,dx\;,$$

where

$$(53)\qquad \Psi_1 = A_1\,\frac{\cosh^3\rho + A\cosh\rho + (1+2A)/\alpha}{(\cosh\rho+\alpha)^4} \in \mathcal{S}\;,$$

$$\rho = \xi - \frac{1}{2}\ln 8b\,,\qquad \alpha = \check{\varphi}_0\sqrt{2/b}\,,\qquad A = Q^2\sqrt{2/b}\,,\qquad A_1 = \check{\varphi}_0\,A^2/6b\;,$$

g_i, Ψ_k are smooth functions such that

$$|g_i| \le \mathrm{const}\,,\qquad \Psi_2\in\mathcal{S}\,,\qquad \Psi_3\in\mathcal{H}\,,\qquad |\Psi_4|\le \mathrm{const}\;.$$

Proof of Lemma 5: Using the expansion (37) and rewriting φ_M in the form $\varphi_M = \varphi_0+\chi+\varepsilon\,\varphi_M^*$, we get

$$(54)\qquad \int_\Omega \Big(\nabla\omega,\,\nabla(\omega(1-3\,\varphi_M^2))\Big)\,dx = -\int_\Omega (3\varphi_0^2-1)\,|\nabla\omega|^2\,dx\, + $$
$$+\int_\Omega |\nabla\omega|^2\,(g_1\,\chi+\varepsilon\,g_2)\,dx + \frac{3}{2}\int_\Omega \omega^2\,\Delta(\varphi_M^2)\,dx\;,$$

where $g_1=-3(\chi+2\varphi_0)$, $g_2=(2\varphi_0+2\chi+\varepsilon\,\varphi_M^*)\,\varphi_M^*$. Obviously, g_i are bounded functions. Further, simple calculations yield the relation

$$(55)\qquad \frac{1}{2}\,\Delta(\varphi_M^2) = \left(\frac{1}{\varepsilon^2}\,|\nabla\psi|^2\left\{\chi_\eta^2+(\varphi_0+\chi)\,\chi_{\eta\eta}\right\} + \frac{1}{\varepsilon}\,\tilde{\Psi}+\Psi_4\right)\Big|_{\eta=(t-\psi)/\varepsilon}\;.$$

Here

$$\tilde{\Psi} = -2\chi_\eta\Big(|\nabla\psi|^2\,W_{1\eta} + (\nabla\psi,\nabla\varphi_0)\Big)$$
$$+\,(\varphi_0+\chi)\,\Big(|\nabla\psi|^2\,W_{1\eta\eta} - \hat{\Pi}\,\chi_\eta\Big) + (\varphi_1+W_1)\,|\nabla\psi|^2\,\chi_{\eta\eta}\;,$$

Ψ_4 is bounded (in the C-sense) and $\hat{\Pi}$ is the operator described in (16). Let us rewrite the function χ in the form $\chi = -A\{\cosh(\rho)+\alpha\}^{-1}$, where ρ, $A>0$, $\alpha>0$ are described in (53). Now it is easy to calculate that

$$3\,|\nabla\psi|^2\left\{\chi_\eta^2+(\varphi_0+\chi)\,\chi_{\eta\eta}\right\} = \Psi_1(1-\Psi_2)\;,$$

where Ψ_1 has the form (53) and

$$\Psi_2 = \Big\{\frac{A+1}{\alpha}\cosh^2\rho + (\alpha^2+2)\cosh\rho + 2\alpha\Big\}\Big\{\cosh^3\rho + A\cosh\rho + (2A+1)/\alpha\Big\}^{-1}.$$

Obviously, $\Psi_2 \in \mathcal{S}$. Finally, let us note that the term $\tilde\Psi$ can be written in the form $\tilde\Psi = \Psi_1\,\Psi_3$, where Ψ_3 is a function from $\mathcal{H}$, since W_1 satisfies (22) and χ_η, $\chi_{\eta\eta}$ vanish like $1/\cosh\rho$ as $\eta \to \pm\infty$. This equality, (54), and (55) complete the proof of Lemma 5.

Now, using (44), (52) and integrating (51) with respect to t, we get

$$\begin{aligned}(56)\qquad &\frac{1}{2}\Big\{(1+\kappa K)\,\|\omega\|^2 + \|\sigma\|^2\Big\}(t) + \int_0^t \Big\{\|\nabla\sigma\|^2 + \varepsilon^2 K\,\|\Delta\omega\|^2 + \\ &\qquad + 3K\,\|\omega\,\nabla\omega\|^2 + K\int_\Omega (3\varphi_0^2 - 1)\,|\nabla\omega|^2\,dx + \frac{K}{\varepsilon^2}\int_\Omega \omega^2\,\Psi_1\,dx\Big\}\,dt' = \\ &= \frac{1}{2}\,\varepsilon^{2M+1}\Big\{(1+\kappa K)\,\|f^\varphi_M\|^2 + \|f^\theta_M\|^2 + 2\int_\Omega f^\varphi_M\,f^\theta_M\,dx\Big\} - \int_\Omega \omega\,\sigma\,dx \\ &+ \int_0^t \Big\{KI - 3K\int_\Omega \Big(\nabla\omega,\,\nabla(\varphi_M\,\omega^2)\Big)\,dx + (\varepsilon\,\kappa_1\,K - 1)\int_\Omega (\nabla\omega, \nabla\sigma)\,dx \\ &- \varepsilon^M \int_\Omega \Big[(\omega+\sigma)\,\mathcal{F}^\theta_M + \omega\,K\,\mathcal{F}^\varphi_M\Big]\,dx \\ &- \varepsilon^{M+1}\int_{\partial\Omega} \Big[(\omega+\sigma)\,F^\theta_M - \omega K(F^\varphi_M + \varepsilon\,\kappa_1\,F^\theta_M)\Big]\,dx'\Big\}\,dt'\ .\end{aligned}$$

It is easy to see that

$$2\Big|\int_\Omega \omega\,\sigma\,dx\Big| \le \alpha_1\,\|\omega\|^2 + \frac{1}{\alpha_1}\,\|\sigma\|^2\ ,$$

$$2\Big|\int_\Omega (\nabla\omega, \nabla\sigma)\,dx\Big| \le \alpha_2\,\|\nabla\omega\|^2 + \frac{1}{\alpha_2}\,\|\nabla\sigma\|^2\ ,$$

$$\alpha_1 = \frac{1}{2}\Big(\kappa K + \sqrt{(\kappa K)^2 + 4}\Big)\,, \qquad \alpha_2 = \frac{1}{2}\Big(\gamma K - 1 + \sqrt{(\gamma K - 1)^2 + 4}\Big)\ .$$

Let us choose the constant K large enough, so that $\gamma K - \alpha_2 \ge 1/2$. It is possible, since $\gamma K - \alpha_2$ varies from 0 to 1. Further, by the embedding theorem (see, for example, [10]) and (41), we get

$$\begin{aligned}&\varepsilon^{M+1}\Big|\int_{\partial\Omega} \Big[(\omega+\sigma)\,F^\theta_M - \omega K(F^\varphi_M + \varepsilon\,\kappa_1\,F^\theta_M)\Big]\,dx'\Big| \le \\ &\qquad \le c\,\varepsilon^{M+1}\Big(\|\omega; L^2(\partial\Omega)\| + \|\sigma; L^2(\partial\Omega)\|\Big) \le c\,\varepsilon^{2M+2} + \frac{1}{4}\,\|\omega\|_1^2 + \frac{1}{4}\,\|\sigma\|_1^2\ .\end{aligned}$$

Here and below c denotes a universal constant, $\|f\|_k$ is the $H^k(\Omega)$ norm of f, and H^k denotes the Sobolev space.

Therefore, choosing ε small enough, from (56) we obtain the following inequality

$$\begin{aligned}(57)\qquad &\frac{\alpha_1-1}{2\,\alpha_1}\Big\{\|\omega\|^2+\|\sigma\|^2\Big\}(t)+\int_0^t\Big\{\frac{1}{4}\,\|\nabla\omega\|^2+\frac{1}{4}\,\|\nabla\sigma\|^2+\\ &\qquad\qquad +\varepsilon^2 K\,\|\Delta\omega\|^2+3K\,\|\omega\nabla\omega\|^2+\frac{K}{\varepsilon^2}\int_\Omega\omega^2\,\Psi_1\,dx\Big\}\,dt'\le\\ &\le c\,\varepsilon^{2M+1}+\int_0^t\Big\{K\,|I|+c\Big(\|\omega\|^2+\|\sigma\|^2\Big)+3K\,\Big|\int_\Omega\Big(\nabla\omega,\,\nabla(\varphi_M\,\omega^2)\Big)\,dx\Big|\Big\}\,dt'\ .\end{aligned}$$

To estimate the integral I we shall need the following

Lemma 6 ([13], [18]). *For any nonnegative functions f, v,*

$$f(x)\in L^2(R^1)\cap L^1(R^1)\,,\qquad v(x)\in\mathcal{S}(R^1)\ ,$$

where $\mathcal{S}$ is the Schwartz space, there exists a constant $\varepsilon_0>0$ such that, for all $\varepsilon\in(0,\varepsilon_0]$,

$$\int_{-\infty}^{\infty}f(x)\,v\Big(\frac{x}{\varepsilon}\Big)\,dx\le\delta\,\|f;L^1(R^1)\|+c_v(\delta)\,\varepsilon^{3/2}\,\rho(\varepsilon)\,\|f;L^2(R^1)\|\ ,$$

where δ is a constant such that $\delta\ge k\,\varepsilon^{1/2-\mu}$, $\mu\in(0,1/2)$, and $k>0$ is a constant. Here $c_v(\delta)$ is a constant depending on δ and on $\|v(x);L^2(R^1)\cap L^1(R^1)\|$ such that $0<c_v(\delta)\le\mathrm{const}\,/\delta^2$, and $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Let us estimate the principal terms of I.

Lemma 7. *Let ε be small enough. Then*

$$(58)\qquad \int_\Omega|\nabla\omega|^2\,|g_1|\,\chi\,dx\le\delta_1\,\|\nabla\omega\|^2+c_\chi(\delta_1)\,\varepsilon^{9/2}\,\|\Delta\omega\|^2\ ,$$

$$(59)\qquad \frac{1}{\varepsilon^2}\int_\Omega\omega^2\,\Psi_1\,|\Psi_2|\,dx\le\frac{\delta_2}{\varepsilon^2}\int_\Omega\omega^2\,\Psi_1\,dx+c_\chi(\delta_2)\,\varepsilon^2\,\|\omega\|_1^2\ ,$$

where $\delta_i>0$ are arbitrary constants.

Proof: Denote by $\mathcal{N}_\mu$ a μ neighborhood of the interface Γ_t, where $\mu\ge 0$ is a constant independent of ε. Since $\chi=\mathcal{O}(\varepsilon^\infty)$ outside $\mathcal{N}_\mu$, we have

$$\int_\Omega|\nabla\omega|^2\,|g_1|\,\chi\,dx\le c_0\int_{\mathcal{N}_\mu}|\nabla\omega|^2\,\chi\,dx+\varepsilon^2\,c_0\int_\Omega|\nabla\omega|^2\,dx\ ,$$

where $c_0=\max\limits_{x\in\Omega}|g_1|$.

Choosing μ sufficiently small, we pass to the variables $y = (y_1, ..., y_n)$ in $\mathcal{N}_\mu$, where y_1 is the coordinate normal to $\Gamma_t = \{x \in \Omega,\ t = \psi(x)\}$. Then in $\mathcal{N}_\mu = \{y,$ $|y_1| \le \mu,\ Y_i^- \le y_i \le Y_i^+,\ i = 2, ..., n\}$ we have

$$\int_{\mathcal{N}_\mu} |\nabla\omega|^2 \chi\, dx = \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^2\, v\Big(\frac{y_1}{\varepsilon}, y, t\Big)\, J\, dy_1\, dy_i\ ,$$

where J is the Jacobian of this change of variables,

$$|\widetilde{\nabla\omega}| = |\nabla_x\omega|\Big|_{x=x(y,t)}, \qquad v(\eta, y, t) = \chi(\beta(\eta + \psi_1))\Big|_{x=x(y,t)}\ .$$

By Lemma 6 and the embedding theorem for $n = 1$, we get

$$\begin{aligned}\int_{\mathcal{N}_\mu} |\nabla\omega|^2\chi\, dx &\le \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \Big\{\delta_1 \int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^2 J\, dy_1 + c\,\varepsilon^{3/2}\rho(\varepsilon)\Big(\int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^4 J^2\, dy_1\Big)^{1/2}\Big\} dy_i \\ &\le \delta_1\, \|\nabla\omega\|^2 + c\,\varepsilon^{3/2-k_1-k_2}\,\rho(\varepsilon)\,\Big\{\varepsilon^{4k_1/3}\,\|\nabla\omega\|^2 + \varepsilon^{4k_2}\Big(\|\nabla\omega\|^2 + \|\Delta\omega\|^2\Big)\Big\} \\ &\le \delta_1\, \|\nabla\omega\|^2 + c\,\varepsilon^{1/2}\,\rho(\varepsilon)\,\Big\{\|\nabla\omega\|^2 + \varepsilon^4\,\|\Delta\omega\|^2\Big\}\ ,\end{aligned}$$

where we choose $k_2 = 1 + k_1/3$ and use that $J > 0$ is a bounded smooth function. This implies estimate (58). Here and below we omit the dependence of $c_\chi(\delta)$ on the function χ and on the constant δ. It is clear that, choosing δ, we take into account that $c_\chi(\delta) \to \infty$ as $\delta \to 0$. Similarly,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathcal{N}_\mu} \omega^2\, \Psi_1\, |\Psi_2|\, dx &\le \frac{\delta_2}{\varepsilon^2}\int_\Omega \omega^2\, \Psi_1\, dx + c\,\rho(\varepsilon)\,\varepsilon^{-1/2} \times \\ &\times \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \Big\|\tilde\omega\,\sqrt{J\,\tilde\Psi_1}\,;\ L^2(-\mu,\mu)\Big\|^{3/2} \Big\|\tilde\omega\,\sqrt{J\,\tilde\Psi_1};\ H^1(-\mu,\mu)\Big\|^{1/2} dy_i \le \\ &\le \frac{\delta_2}{\varepsilon^2}\,\Big\|\omega\,\sqrt{\Psi_1}\Big\|^2 + c\,\rho(\varepsilon)\,\Big\{\frac{1}{\varepsilon^2}\,\Big\|\omega\,\sqrt{\Psi_1}\Big\|^2 + \varepsilon^2\,\|\omega\|^2 + \varepsilon^4\,\|\nabla\omega\|^2\Big\}\ .\end{aligned}$$

Lemma 7 is proved. ∎

Further, choosing ε small enough, we have the trivial estimate

$$\varepsilon\int_\Omega |\nabla\omega|^2\, |g_2|\, dx + \frac{1}{\varepsilon}\int_\Omega \omega^2\, \Psi_1\, |\,\Psi_3 + \varepsilon\,\Psi_4\,|\, dx \le \\ \le c\,\|\omega\|^2 + \delta_3\,\Big\{\|\nabla\omega\|^2 + \frac{1}{\varepsilon^2}\int_\Omega \omega^2\, \Psi_1\, dx\Big\} \tag{60}$$

with an arbitrary constant $\delta_3 > 0$.

Let us estimate the last term in the right-hand side of (57). Using the Galliardo-Nierenberg inequality, we get that

$$
\begin{aligned}
(61) \qquad \Big|\int_\Omega \Big(\nabla\omega,\, \nabla(\varphi_M\omega^2)\Big)\, dx\Big| &\le c\int_\Omega |\omega|\,|\nabla\omega|^2\, dx + \frac{c}{\varepsilon}\int_\Omega |\nabla\omega|\,\omega^2\, dx \le \\
&\le c\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{(4+n)/4} + \frac{c}{\varepsilon}\,\|\nabla\omega\|\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{n/4} \\
&\le \delta_4\Big\{\|\nabla\omega\|^2 + \varepsilon^2\,\|\omega\|_2^2\Big\} + c\,\varepsilon^{-2(4+n)/(4-n)}\,\|\omega\|^{2(8-n)/(4-n)}\ .
\end{aligned}
$$

Choosing $\delta_i = \min\{1/12, 1/24K\}$, $i = 1, ..., 4$, and using (58)–(61), we can transform (57) as follows

$$
\begin{aligned}
(62) \qquad U(t) + c\int_0^t \Big\{\|\nabla\omega\|^2 + \|\nabla\sigma\|^2 + \varepsilon^2\,\|\Delta\omega\|^2 + \|\omega\,\nabla\omega\|^2 + \frac{1}{\varepsilon^2}\,\Big\|\sqrt{\Psi_1}\,\omega\Big\|^2\Big\}\, dt' \le \\
\le c\,\varepsilon^{2M+1} + c\int_0^t \Big\{U(t') + \varepsilon^{-r}(U(t'))^{1+\lambda}\Big\}\, dt'\ .
\end{aligned}
$$

Here

$$
U(t) = \Big\{\|\omega\|^2 + \|\sigma\|^2\Big\}(t)\,, \qquad \lambda = 4/(4-n)\,, \qquad r = 2(4+n)/(4-n)\ .
$$

Let us fix a number $T_1 \in (0, T]$, $T < \infty$, and let $t \in [0, T_1]$. Then, according to the Gronwall lemma, (62) yields

$$
U(t) \le c\Big(\varepsilon^{2M+1} + \varepsilon^{-r}\int_0^{T_1} U^{1+\lambda}\, dt'\Big)\ .
$$

Let $z = \max\limits_{t\in[0,T_1]} U(t)$. Then

$$
z \le c\left(\varepsilon^{2M+1} + \varepsilon^{-r}\,T_1\,z^{1+\lambda}\right)\ . \tag{63}
$$

To analyze the last relation, we need the following lemma proved by V.P. Maslov and P.P. Mosolov.

Lemma 8. *Let positive numbers p, q, λ satisfy the estimate*

$$
q < \frac{\lambda}{1+\lambda}\Big(p(1+\lambda)\Big)^{-1/\lambda}\ . \tag{64}
$$

Then the solutions of the inequality $0 \le z \le q + p\,z^{1+\lambda}$ belong to the set $[0, Z_-] \cup [Z_+, \infty)$, where the numbers Z_+, Z_- are such that $0 \le Z_- < q(1+\lambda)/\lambda < Z_+$.

In our case $p = c\,T_1\,\varepsilon^{-r}$. Therefore, since ε is small enough, the inequality (64) holds for any $M \geq 2$. Since $z = z(T_1)$ continuously depends on T_1 and $z(0) \leq c\,\varepsilon^{2M+1} < c\,\varepsilon^{2M+1}(1+\lambda)/\lambda$, we obtain the estimate

$$\max_{t\in[0,T_1]} U(t) \leq c\,\varepsilon^{2M+1} \tag{65}$$

with a constant c independent of ε.

It is easy to see that (65) and (62) yield the estimates (47). This completes the proof of Theorem 4. ∎

Proof of Theorem 3: Let us choose the number $M' = M+1$, where $M \geq 2$. Then, by Theorem 4, we get

$$\theta = \theta^{\mathrm{as}}_{M'} + \varepsilon^{M+3/2}\,\sigma_{M'}\,, \qquad \varphi = \varphi^{\mathrm{as}}_{M'} + \varepsilon^{M+3/2}\,\omega_{M'}\ ,$$

where $\sigma_{M'}$, $\omega_{M'}$ are functions from $L^\infty(0,T;L^2(\Omega))$ uniformly bounded in ε. Nevertheless,

$$\begin{aligned}
&\theta^{\mathrm{as}}_{M'} = \theta^{\mathrm{as}}_{M} + \varepsilon^{M+1}\,\mathcal{Y}^{\theta}_{M+1}\,, \qquad \varphi^{\mathrm{as}}_{M'} = \varphi^{\mathrm{as}}_{M} + \varepsilon^{M+1}\,\mathcal{Y}^{\varphi}_{M+1}\ ,\\
&\mathcal{Y}^{\theta}_{M+1} = \theta_{M+1}(x,t) + U_{M+1}(\eta,x,t) + Y_{M+1}(\tau,x',t)\ ,\\
&\mathcal{Y}^{\varphi}_{M+1} = \varphi_{M+1}(x,t) + W_{M+1}(\eta,x,t) + Z_{M+1}(\tau,x',t)\ ,
\end{aligned}$$

where $\eta = (t-\psi(x))/\varepsilon$, $\tau = x_N/\varepsilon$. It is easy to calculate that

$$\Big\|\mathcal{Y}^{\theta}_{M+1};\, L^\infty(0,T;L^2(\Omega))\Big\| + \Big\|\mathcal{Y}^{\varphi}_{M+1};\, L^\infty(0,T;L^2(\Omega))\Big\| \leq c\ .$$

Therefore,

$$\theta = \theta^{\mathrm{as}}_{M} + \varepsilon^{M+1}\,\sigma^*_{M}\,, \qquad \varphi = \varphi^{\mathrm{as}}_{M} + \varepsilon^{M+1}\,\omega^*_{M}\ ,$$

where

$$\sigma^*_M = \mathcal{Y}^{\theta}_{M+1} + \sqrt{\varepsilon}\,\sigma_{M'}\ , \qquad \omega^*_M = \mathcal{Y}^{\varphi}_{M+1} + \sqrt{\varepsilon}\,\omega_{M'}$$

are functions such that the estimate

$$\begin{aligned}
&\Big\|\mathcal{Y}^{\theta}_{M+1};\, L^\infty(0,T;L^2(\Omega))\Big\| + \Big\|\mathcal{Y}^{\varphi}_{M+1};\, L^\infty(0,T;L^2(\Omega))\Big\| +\\
&\qquad + \Big\|\sigma_{M'};\, L^\infty(0,T;L^2(\Omega))\Big\| + \Big\|\omega_{M'};\, L^\infty(0,T;L^2(\Omega))\Big\| \leq \mathrm{const}
\end{aligned}$$

holds uniformly in ε. This estimate and Theorem 4 complete the proof of Theorem 3.

ACKNOWLEDGEMENT – One of the authors (G.O.) would like to thank IAEA and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where the most part of this work was done. He is cordially grateful to Professor M.S. Narasimhan for his invitation to ICTP. This work was partially supported by the Russian Foundation for Fundamental Research, grant N 96-01-01492.

REFERENCES

[1] Alt, H.W. and Pawlow, I. – *On the Entropy Principle of Phase Transition Models with a Conserved Order Parameter*, Rheinische Fridrich–Wilhelms–Universität, Bonn, Preprint N 365, 1994.

[2] Bates, P.W. and Fife, P.C. – Spectral comparison principles for the Cahn–Hilliard and phase–field equations, and time scale for coarsening, *Physica D,* 43 (1990), 335–348.

[3] Berger, M.S. and Fraenkel, L.E. – On the asymptotic solution of a nonlinear Dirichlet problem, *J. Math. Mech.,* 19:7 (1970), 553–585.

[4] Caginalp, G. – Stefan and Hele–Shaw type models as asymptotic limits of the phase-field equations, *Phys. Rew. A,* 39:11 (1989), 5887–5896.

[5] Caginalp, G. – A conserved phase system: implication for kinetic undercooling, *Phys. Rev. B,* 38 (1988), 789–891.

[6] Caginalp, G. – The dynamics of a conserved phase field system: Stefan like, Hele–Shaw, and Cahn–Hilliard models as asymptotic limits, *IMA J. Appl. Math.,* 44 (1990), 77–94.

[7] Danilov, V.G., Omel'yanov, G.A. and Radkevich, E.V. – Asymptotic solution of a phase field system and the modified Stefan problem, *Differential'nie Uravneniya,* 31:3 (1995), 483–491 (Russian); English transl. in *Differential equations*, 31:3 (1995).

[8] Danilov, V.G., Omel'yanov, G.A. and Radkevich, E.V. – Initial data's regularization of the modified Stefan problem, *Matem. Zametki,* 57:4 (1995) (Russian); English transl. in *Mathematical Notes*, 57:4 (1995).

[9] Elliott, C.M. and French, D.A. – Numerical studies of the Cahn–Hilliard equation for phase separation, *IMA J. Appl. Math.,* 38 (1987), 97–128.

[10] Ladyzhenskaya, O.A., Solonnikov, V.A. and Ural'tseva, N.N. – *Linear and Quasilinear Parabolic–type Equations*, Nauka, Moscow, 1967 (Russian).

[11] Maslov, V.P. and Tsupin, V.A. – *Shock wave propagation in isoentropic gas with small viscosity,* Itogi Nauki i Techniki: Sovremennie Problemy Mat., 8, VINITI, Moscow, 1976, 273–308 (Russian).

[12] Maslov, V.P. and Omel'yanov, G.A. – Asymptotic soliton-like solution of equations with small dispersion, *Uspekhi Mat. Nauk,* 36:3 (1981), 63–126 (Russian); English transl. in *Rus. Math. Surveys*, 36:3 (1981), 73–149.

[13] Maslov, V.P. and Omel'yanov, G.A. – Hugoniot-type conditions for infinitely narrow solutions of the equation for simple waves, *Sibirskii Mat. Zhurnal,* 24:5 (1983), 172–182 (Russian); English transl. in *Siberian Math. J.*, 24:5 (1983), 787–795.

[14] Maslov, V.P., Danilov, V.G. and Volosov, K.A. – *Mathematical Modeling of Heat and Mass Transfer Processes*, Kluwer, Dordrecht, 1995.

[15] Novick-Cohen, A. and Segel, L.A. – Nonlinear aspects of the Cahn–Hilliard equation, *Physica D,* 10 (1984), 277–298.

[16] Novick-Cohen, A. – The nonlinear Cahn–Hilliard equation: transition from spinodal decomposition to nucleation behavior, *J. Stat. Phys.,* 38:3/4 (1985), 707–722.

[17] Novick-Cohen, A. – *The Cahn–Hilliard Equation: Mathematical and Modelling Perspectives*, in "Surveys on Nonlinear Diffusion" (L.A. Peletier, Ed.), 1993.

[18] Omel'yanov, G.A., Radkevich, E.V. and Danilov, V.G. – Justification of the phase field system asymptotic solution and a modified Stefan problem, *Mat. Sb.,* 186:12 (1995), 62–80 (Russian).

[19] Omel'yanov, G.A., Radkevich, E.V. and Danilov, V.G. – Stability and "stabilization" of the solution of the phase field system, *Proceedings of Banach Center, Nonlinear Anal. and Appl.,* 6, (1995).

[20] Omel'yanov, G.A. – *Asymptotic solution of the non-isothermal Cahn–Hilliard system*, Int. Centre for Theor. Phys., Trieste, preprint IC/95/73, 1995.

[21] Omel'yanov, G.A. – *Asymptotic description of two metastable processes of solidification for the case of large relaxation time*, Int. Centre for Theor. Phys., Trieste, preprint IC/95/162, 1995.

[22] Pego, R.L. – Front migration in the non-linear Cahn–Hilliard equation, *Proc. R. Soc. Lond. A,* 422 (1989), 261–278.

[23] Radkevich, E.V. – Existence conditions for the classical solution of the modified Stefan problem (Gibbs–Thomson law), *Mat. Sb.,* 183:2 (1992), 77–101 (Russian).

[24] Radkevich, E.V. – About asymptotic solution of a phase field system, *Differential'nie Uravneniya,* 29:3 (1993), 487–500 (Russian); English transl. in Differential equations, 29:3 (1993).

[25] Stoth, B. – Convergence of the Cahn–Hilliard equation to the Mullins–Sekerka problem in spherical symmetry, *J. Diff. Eq.,* to appear.

G.A. Omel'yanov,
International Centre for Theoretical Physics,
I.C.T.P., PO BOX 586, 34100 Trieste – ITALY
Permanent address:
Moscow State Institute of Electronics and Mathematics,
B. Vuzovsky, 3/12, 109028 Moscow – RUSSIA

and

V.G. Danilov,
Moscow State Institute of Electronics and Mathematics,
B. Vuzovsky, 3/12, 109028 Moscow – RUSSIA

and

E.V. Radkevich,
Department of Mathematics and Mechanics, Moscow State University,
Vorobiovi Gori, 119899 Moscow – RUSSIA