PORTUGALIAE MATHEMATICA
Vol. 53 Fasc. 4 – 1996

# THE THEORY OF $\varepsilon$-GENERATORS AND SOME QUESTIONS IN ANALYSIS

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**Summary:** We introduce in this paper the fundamental ideas of the theory of $\varepsilon$-generators and illustrate their application to certain questions related to the classical Lusin's theorem, concerning the continuity of measurable functions when restricted to convenient subsets of their domains of definition. We also discuss some general results concerning sequence of functions and their $\varepsilon$-generators.

## I

**1.** In this paragraph we introduce some fundamental notions of non-deterministic analysis needed for the theory of $\varepsilon$-generators and their application. In particular we reproduce some results of A. Jansen [1] which are not of easy access in the literature. More informations about non-deterministic analysis can be seen at [2] and we advise the reader that in older papers what we call today $n$-functions, to be defined below, were called $g$-functions.

**2.** We call a *pair* $(X, V)$ a topological space $X$ together with a family $V$ of open coverings of $X$. When $X$ happens to be a subspace of another topological space $Y \supset X$ it is usually convenient to consider as a covering of $X$ the set of open sets in $Y$ whose intersection with $X$ is non-empty and not the set of open sets relative to $X$ whose union is $X$. All topological spaces involved will be at least $T_1$ although many results might be true for more general spaces.

**Definition I.** A non-deterministic function or a $n$-function

$$f\colon\ (X,V) \to (Y,V')$$

*Received*: April 9, 1993.

is:

**a**) a function

$$f_V\colon\ V \to V' \ ,$$

**b**) for each covering $\sigma \in V$, a function

$$f_\sigma\colon\ \sigma \to \sigma' = f_V(\sigma) \ .$$

**Definition II.** A $n$-function

$$f\colon\ (X,V) \to (Y,V')$$

is continuous if for each $\sigma, \tau \in V$, with $\tau > \sigma$, and $B \in \tau$, $A \in \sigma$ with $B \subset A$ we have:

**a**) $f_V(\tau) > f_V(\sigma)$,

**b**) $f_\tau(B) \subset f_\sigma(A)$,

where $\tau > \sigma$ means that $\tau$ refines $\sigma$, namely each $B \in \tau$ is contained in some $A \in \sigma$.

**Definition III.** A $n$-function

$$f\colon\ (X,V) \to (Y,V')$$

is point-wise cofinal if for any $x \in X$ and any open covering $\Gamma$ of $Y$ there is a $\sigma \in V$ and $A \in \sigma$ such that $x \in A$ and $f_\sigma(A)$ is contained in some open set of $\Gamma$.

**Definition IV.** A $n$-function

$$f\colon\ (X,V) \to (Y,V')$$

is fully-regular if for any $\sigma, \tau \in V$ and any $A \in \sigma$, $B \in \tau$, with $A \cap B \neq \emptyset$ we have

$$f_\sigma(A) \cap f_\tau(B) \neq \emptyset \ .$$

We also have the weaker concepts of cofinal and regular $n$-functions and similar notions introduced in [1] but we shall not need them in this paper. Strictly speaking the notion of continuous $n$-function will not be used here, but we include it above for completeness.

**Definition V.** A $n$-function

$$f\colon (X,V) \to (Y,V')$$

generates a continuous function

$$\varphi\colon X \to Y$$

if for any open set $W$ and any $x \in X$ with $\varphi(x) \in W$, there is a $\sigma \in V$ and $A \in \sigma$ with $x \in A$ and

**a**) $\varphi(x) \in \overline{f_\sigma(A)}$,

**b**) $f_\sigma(A) \subset W$.

The closure condition a) is due to technical reasons which we shall not discuss here.

Now we have the following result of A. Jansen which will be of fundamental importance in this paper:

**Theorem 1.** *Let the $n$-function*

$$f\colon (X,V) \to (Y,V')$$

*be fully regular and point-wise cofinal and $Y$ Hausdorff and regular. Then, there is a unique continuous function $\varphi\colon X \to Y$ generated by $f$.*

**Proof:**

Claim 1. For each $x_0 \in X$, there exists a point $y_0 \in Y$ such that each neighbourhood $N_{y_0}$ of $y_0$ intersects every set of form $f_\sigma(A)$, where $x_0 \in A \in \sigma \in V$.

Proof: If, for some $x_0 \in X$ there is no such point $y_0$, then

$$(1) \qquad \begin{cases} \text{For each } y \in Y\text{, there is a neighbourhood } N_y \text{ of } y \\ \text{and a cover } \sigma_y \in V \text{ such that for some } A_y \in \sigma_y, \\ \text{we have } x_0 \in A_y \text{ and } f_{\sigma_y}(A_y) \cap N_y = \emptyset. \end{cases}$$

The set of such $N_y$ form a cover $\alpha$ of $Y$. Take $\sigma \in V$ and $A \in \sigma$ such that $x_0 \in A$ and $f_\sigma(A) \subseteq N_{y'}$, for some $N_{y'} \in \alpha$ (which can be done by the point-wise cofinality of $f$). Now, if $x_0 \in B \in \tau \in V$ ($\tau$, $B$ arbitrary), then by full regularity

$$f_\tau(B) \cap f_\sigma(A) \neq \emptyset \implies f_\tau(B) \cap N_{y'} \neq \emptyset \ ,$$

i.e. any $f_\tau(B)$, with $x_\sigma \in B \in \tau$ intersects $N_{y'}$, which contradicts (1).

Claim 2. For each $x_0 \in X$, there is exactly one point $y_0$ satisfying Claim 1.

Proof: Suppose for $x_0 \in X$ there are two points $y_1$, $y_2$ satisfying Claim 1. Let $U_1$, $U_2$ be disjoint neighbourhoods of $y_1$, $y_2$ respectively, and let $V_1$, $V_2$ be neighbourhoods satisfying

$$y_1 \in V_1 \subseteq \overline{V}_1 \subseteq U_1\,, \qquad y_2 \in V_2 \subseteq \overline{V}_2 \subseteq U_2\ .$$

Let $\alpha$ be the cover $\{U_1, U_2, Y \backslash \overline{V_1 \cup V_2}\}$.

Now, by point-wise cofinality of $f$, we can find $\sigma_1 \in V$, $A_1 \in \sigma_1$ such that $x_0 \in A_1 \in \sigma_1$ and $f_{\sigma_1}(A_1) \subset C_1$ for some $C_1 \in \alpha$. Since $y_1$ satisfies Claim 1, $f_{\sigma_1}(A_1) \cap V_1 \neq \emptyset$, and so $f_{\sigma_1}(A_1) \subseteq U_1$ ($U_1$ being the only member of $\alpha$ which meets $V_1$). Similarly, we can find $A_2 \in \sigma_2 \in V$ with $x_0 \in A_2$, and $f_{\sigma_2}(A_2) \subseteq U_2$. Since $U_1 \cap U_2 = \emptyset$, we have $f_{\sigma_1}(A_1) \cap f_{\sigma_2}(A_2) = \emptyset$, which contradicts the full regularity of $f$. It now follows that there exists a function $\varphi\colon X \to Y$, where $\varphi(x) = y$ is the (unique) point having the property that every neighbourhood of $y$ intersects every set of form $f_\sigma(A)$ where $x \in A \in \sigma \in V$.

Claim 3. $\varphi$ is continuous.

Proof: Suppose $N_0$ is a neighbourhood of $y_0 = \varphi(x_0)$. We show $x_0$ has a neighbourhood $M$ such that $x \in M \Rightarrow \varphi(x) \in N_0$.

Take neighbourhoods $U_1$, $U_2$ of $y_0$ such that $y_0 \in U_1 \subseteq \overline{U}_1 \subseteq U_2 \subseteq \overline{U}_2 \subseteq N_0$. Let $\alpha$ be the cover $\{U_2, N_0 \backslash \overline{U}_1, Y \backslash \overline{U}_2\}$; by point-wise cofinality of $f$, there exists $\sigma_0 \in V$, $A_0 \in \sigma_0$ such that $x_0 \in A_0$ and $f_{\sigma_0}(A_0) \subseteq C$ for some $C \in \alpha$. Also, $x_0 \in A_0 \Rightarrow f_{\sigma_0}(A_0) \cap U_1 \neq \emptyset$. Hence $f_{\sigma_0}(A_0) \subseteq U_2$.

Now, if $x$ is any point in $A_0$, then by the above, every neighbourhood of $\varphi(x)$ intersects $f_{\sigma_0}(A_0)$, and therefore every neighbourhood of $\varphi(x)$ intersects $U_2$. Hence $\varphi(x) \in \overline{U}_2 \subseteq N_0$. Taking $A_0$ for $M$, we see that $\varphi$ is continuous.

Claim 4. $f$ generates $\varphi$.

We have from Claim 3, that if $N_0$ is any neighbourhood of $\varphi(x_0)$, then there exists $\sigma_0 \in V$, $A_0 \in \sigma_0$ such that $x_0 \in A_0$, $f_{\sigma_0}(A_0) \subseteq U_2 \subseteq \overline{U}_2 \subseteq N_0$. Thus $\overline{f_{\sigma_0}(A_0)} \subseteq N_0$. Also, $\varphi(x) \in \overline{f_{\sigma_0}(A_0)}$, since every neighbourhood of $\varphi(x)$ intersects $f_{\sigma_0}(A_0)$.

Claim 5.If $f$ generates $\psi\colon X \to Y$, then $\varphi = \psi$.

Proof: If $f$ generates $\psi$, then if $x \in X$, for every neighbourhood $W$ of $\psi(x)$ there exists $\sigma \in V$, $A \in \sigma$ such that $x \in A \in \sigma$, $\psi(x) \in \overline{f_\sigma(A)}$, $f_\sigma(A) \subseteq W$. Since $f$ is fully regular, if $x \in B \in \tau$ (arbitrary), then $f_0(A) \cap f_\tau(B) \neq \emptyset$. Hence $W \cap f_\tau(B) \neq \emptyset$, for any neighbourhood $W$ of $\psi(x)$, and any $B \in \tau$ for which $x \in B$. But by Claim 2, $\varphi(x)$ is the only point with this property, and so $\psi(x) = \varphi(x)$.

In [1], A. Jansen proves many other similar theorems which might as well produce results comparable to those obtained in the present work.

**3.** We shall need in this paper some results which are kind of "relative notions" of some definitions given above.

**Definition VI.** A $n$-function

$$f\colon (X,V) \to (Y,V')$$

**a**) is fully-regular relative to a pair $(K,L)$ where $K \subset X$ and $L \subset Y$ are subsets of $X$ and $Y$ respectively if for each $\sigma,\tau \in V$ and $A \in \sigma$, $B \in \tau$ with $A \cap B \cap K \neq \emptyset$ we have:

$$L \cap f_\sigma(A) \cap f_\tau(B) \neq \emptyset\ ;$$

**b**) is point-wise cofinal relative to a pair $(K,L)$ if for any covering $\Gamma$ of $L$ and any $x \in K$, there exists $\sigma \in V$ and $A \in \sigma$ with $x \in A$ and

$$\emptyset \neq L \cap f_\sigma(A) \subset M$$

for some open set $M \in \Gamma$;

**c**) generates a continuous function $\varphi\colon\ K \to L$ relative to the pair $(K,L)$ if for any $x \in K$ and any open set $W \subset Y$ with $\varphi(x) \in W \cap L$, there exists $\sigma \in V$ and $A \in \sigma$ such that $x \in A \cap K$ and

$$\varphi(x) \in \overline{f_\sigma(A)} \quad \text{and} \quad f_\sigma(A) \cap L \subset W \cap L\ .$$

We have the following theorem:

**Theorem 1′.** *If a $n$-function*

$$f\colon (X,V) \to (Y,V')$$

*is fully-regular and point-wise cofinal, relative to a pair $(K,L)$, $K \subset X$, $L \subset Y$ and $Y$ is Hausdorff regular there is a unique continuous function $\varphi :\ K \to L$ generated by $f$ relative to $(K,L)$.*

**Proof:** Analogous to the proof of Theorem 1, "mutatis mutandis". ∎

**4.** This relativization of the concepts like fully-regular, etc. can be extended to other concepts of similar nature but we do need them in this paper. We call attention to the fact that the relativization of such concepts is also useful to overcome the difficulties involved with the question of restriction of $n$-functions to sub-pairs. More precisely, if we have a $n$-function

$$f\colon (X,V) \to (Y,V')$$

and we consider to subsets $K \subset X$ and $L \subset Y$, in general it is not possible to define without ambiguity the restriction of $f$ to the pairs $(K,V_K)$, $(L,V'_L)$ for convenient defined families $V_K$ and $V'_L$. Let us illustrate this problem in a typical situation important for us in this paper. Suppose we give a continuous function

$$\varphi\colon X \to Y \ .$$

If $\varphi$ is surjective it is easy to define a $n$-function

$$f\colon (X,V) \to (Y,V')$$

generating $\varphi$ for a given family $V'$, cofinal in the set $\operatorname{Cov}(Y)$ of all open coverings of $Y$. Indeed, all we have to do is to consider for each $\sigma' \in V'$ and $A' \in \sigma'$ its inverse image $\varphi^{-1}(A')$ which is an open set in $X$ and then define

$$\begin{aligned}
&\sigma = \left\{\varphi^{-1}(A')\colon\ A' \in \sigma' \in V'\right\},\\
&V = \{\sigma\},\\
&f_V\colon V \to V'
\end{aligned}$$

by

$$f_V(\sigma) = \sigma' \qquad \forall\, \sigma \in V$$

and

$$\forall\, \sigma \in V\,, \qquad f_\sigma\colon \sigma \to \sigma'$$

by

$$f_\sigma(A) = A', \qquad \text{with} \ \ \varphi^{-1}(A') = A\ .$$

As easily seen such an $f$ generates $\varphi$.

What happens if $\varphi$ is not subjective? In this case in general we might have that two different open sets $A', B' \in \sigma'$ with the same intersection with $\varphi(X) \subset Y$ and then we have ambiguity in deciding what set to associate to $\varphi^{-1}(A') = \varphi^{-1}(B') = A$. Of course, we can select one of them arbitrarily and build $f$ in this way. But, in general $f$ does not generate $\varphi$ accordingly to Definition V. However we always have that $f$ generates $\varphi$ relative to the pair $(X, \varphi(X))$ accordingly to Definition VI c), no matter how we select arbitrarily $A'$ or $B'$, etc., corresponding to $A = \varphi^{-1}(A')$.

Let us summarize this discussion in the proposition:

**Proposition I:** Let $\varphi\colon X \to Y$ be a continuous function and let $V'$ be a family of open covering of $Y$ satisfying the following property: for each $y \in Y$ and an arbitrary open set $W$, $y \in W$, there is a $\sigma' \in V'$ and $A' \in \sigma'$ such that $y \in A' \subset W$. Then there exists a $n$-function

$$f\colon (X, Y) \to (Y, V')$$

generating $\varphi$ relative to the pair $(X, \varphi(X))$.

**Proof:** Take $\sigma' \in V'$ and consider all sets $A'$ in $\sigma'$ such that $A' \cap \varphi(X) \neq \emptyset$. This will define an open covering of $\varphi(X)$. If more than one $\sigma'$ define the same covering relative to $\varphi(X)$ select one of them arbitrarily and still call $V'$ the family of all such selected coverings. Now two different covering of $V'$ define different coverings relative to $\varphi(X)$. For each $\sigma' \in V'$ we obtain a covering $\sigma$ of $X$ given by all sets $\varphi^{-1}(A')$ for $A' \in \sigma'$ in $\sigma'$ and to each such $\sigma$ it is associated without ambiguity one covering $\sigma' \in V'$. Hence we have a function

$$f_V\colon V \to V' ,$$

where $V$ is the family of all $\sigma$ considered above when $\sigma'$ runs in $V'$, given by: $\forall\, \sigma \in V\colon f_V(\sigma) = \sigma'$.

Take any $\sigma' \in V$ and $A \in \sigma$. Then there is $A' \in \sigma'$ such that $A = \varphi^{-1}(A')$ and if different $A'$ have the same intersection with $\varphi(X)$ we take one of them arbitrarily and define

$$f_\sigma(A) = A'$$

what defines a function

$$f_\sigma\colon \sigma \to \sigma' = f_V(\sigma) .$$

Therefore we have a $n$-function

$$f\colon (X, V) \to (Y, V') ,$$

that is clearly fully-regular. Let us show that it is also point-wise cofinal relative to the pair $(X, \varphi(X))$. Indeed, take any open covering $\Gamma$ of $Y$ and any point $x \in X$. Look to $\varphi(x)$ and a set $M \in \Gamma$ with $\varphi(x) \in M$. By hypothesis there is $\sigma' \in V'$ and $A' \in \sigma'$ with $\varphi(x) \in A' \subset M$. If $\sigma'$ is not some of the selected coverings in the way indicated above let $\tau' \in V'$ be one of the selected coverings having a set $B'$ such that

$$A' \cap \varphi(X) = B' \cap \varphi(X) \ .$$

Let $\tau \in V$ be such that $f_V(\tau) = \tau'$ and look to $B = \varphi^{-1}(B') \in \tau$. Then $x \in B$ and

$$f_\tau(B) \cap \varphi(X) \subset M \cap \varphi(X) \subset M \in \Gamma$$

what shows that $f$ is point-wise cofinal relative to $(X, \varphi(X))$.

Let us show that $f$ generates $\varphi$ relative to $(X, \varphi(X))$. Indeed, let $x \in X$ and $W$ be open in $Y$ with $\varphi(x) \in W$. By the hypothesis of the theorem there is $\sigma' \in V'$ and $A' \in \sigma'$ with $\varphi(X) \in A' \subset W$. By the definition of $f$ there is a $\sigma \in V$ and $A \in \sigma$ with $x \in A$ such that

$$\varphi(X) \in f_\sigma(A) \cap \varphi(X) = A' \cap \varphi(X)$$

and this proves that $f$ generates $\varphi$, relative to $(X, \varphi(X))$. ∎

**Remark.** Clearly $\varphi$ might also be generated by other $n$-functions, but the one defined as indicated above will be called *canonical generator of* $\varphi$, relative to $(X, \varphi(X))$.

To finish this paragraph we introduce the notion of a function generated by a sequence of $n$-functions.

**Definition VII.** A sequence of $n$-functions $\{f^n\}_{n \geq 1}$

$$f^n \colon (X, V^n) \to (Y, V'^n) \ ,$$

generates a function $\varphi\colon X \to Y$ if there is a sequence $\{n_i\}_{i \geq 1}$ of positive integers with $\lim_{i\to\infty} n_i = +\infty$ such that for any $x \in X$ and for any open set $W \subset Y$ with $\varphi(x) \in W$, and for any integer $n_0 \geq 0$ there is $n_i > n_0$, $\sigma^{n_i} \in V^{n_i}$ and $A^{n_i} \in \sigma^{n_i}$ with $x \in A^{n_i}$ such that

$$f^{n_i}_{\sigma_{n_i}}(A^{n_i}) \subset W \ , \qquad \varphi(X) \in \overline{f^{n_i}_{\sigma_{n_i}}(A^{n_i})} \ .$$

We can also introduce, with obvious changes in the definition above, the notion of a function generated by a sequence of $n$-functions relative to a pair $(K, L)$, with $K \subset X$ and $L \subset Y$.

We call $\{n_i\}_{i \geq 1}$ of Definition VII, a *generating sequence for* $\varphi$.

Starting with Definition VII we could extend to sequence of $n$-functions the results established by A. Jansen in his Ph.D. thesis [1].

Another direction of research consists in generalizing Definition VII to $n$-distributions, in the sense defined in [2] and applications to several areas of topology.

An important problem is to establish sufficient conditions for a sequence of $n$-functions, or more generally, for a $n$-distribution to generate a function $\varphi$.

## II

**1.** We shall study now the basic facts of the theory of $\varepsilon$-generators to be used in this paper.

We denote by $(X, \mathcal{A}, m)$ a measure space where $X$ is a topological space, $\mathcal{A}$ a $\sigma$-algebra of subsets of $X$ and $m$ a measure in this $\sigma$-algebra. The outer measure generated by $m$ will be indicated by $m^*$ and as well-known it is given, for any $E \subset X$, by

$$m^*(E) = \inf_{\substack{G \in \mathcal{A} \\ G \supset E}} m(G) \ .$$

In this paper we shall always assume that $m$ is a *Borel outer regular measure*, namely, the family $B$ of all Borel sets of $X$ is contained in $\mathcal{A}$ and for any $E \in \mathcal{A}$ and $\varepsilon > 0$ there is an open set $A \supset E$ in $X$ such that

$$m(A - E) < \varepsilon \ .$$

We also say that $m$ *is inner regular* if for every $E \in \mathcal{A}$ and $\varepsilon > 0$ there is a closed set $F \subset E$ such that

$$m(E - F) < \varepsilon \ .$$

From now on we shall always assume that all topological spaces used in this paper are at least $T_1$.

**Definition I.** Let $\varphi: X \to Y$ be an arbitrary function and $(X, \mathcal{A}, m)$ a measure space. For a given $\varepsilon \geq 0$ we say that a $n$-function

$$f^\varepsilon \colon (X, V) \to (Y, V')$$

is an $\varepsilon$-generator of $\varphi$ if

$$m^*\Big( \bigcup_{\substack{\sigma \in V \\ A \in \sigma}} \Big\{ x \in A \in \sigma \colon \varphi(x) \notin f^\varepsilon_\sigma(A) \Big\} \Big) < \varepsilon \ .$$

If the set above, namely,

$$E^{\varepsilon}(X) = \bigcup_{\substack{\sigma \in V \\ A \in \sigma}} \Big\{ x \in A \in \sigma\colon\ \varphi(x) \notin f^{\varepsilon}_{\sigma}(A) \Big\}$$

is measurable, namely, $m^*[E(X)] = m[E(X)]$ we say that $f^{\varepsilon}$ is *measurable*.

For each $\sigma \in V$ and $A \in \sigma$ let us write

$$E^{\varepsilon}(A) = \Big\{ x \in A\colon\ \varphi(x) \notin f^{\varepsilon}_{\sigma}(A) \Big\}$$

an we call $E^{\varepsilon}(X)$ and $E^{\varepsilon}(A)$ associated sets of $f^{\varepsilon}$ respectively relative to $X$ and $A$.

Clearly,

$$E^{\varepsilon}(X) = \bigcup_{\substack{\sigma \in V \\ A \in \sigma}} E^{\varepsilon}(A)\ .$$

Usually we shall write only $E^{\varepsilon}$ instead of $E^{\varepsilon}(X)$ where no confusion is possible.

**Theorem 1.** *Let $\varphi\colon\ X \to Y$ be any function and for a given $\varepsilon > 0$ suppose that $\varphi$ has an $\varepsilon$-generator*

$$f^{\varepsilon}\colon\ (X, V) \to (Y, V')$$

*with $(X, \mathcal{A}, m)$ a measure space. If $f^{\varepsilon}$ generates a function $\psi\colon\ X \to Y$ then*

$$m^*\Big\{ x \in X\colon\ \varphi(x) \neq \psi(x) \Big\} < \varepsilon\ .$$

**Proof:** Let $E^{\varepsilon} = E^{\varepsilon}(X)$ be the associated set of $f^{\varepsilon}$ and let $F^{\varepsilon} = X - E^{\varepsilon}$. By definition $m^*(X - F^{\varepsilon}) < \varepsilon$ and if $x \in F^{\varepsilon}$ assume that $\varphi(x) \neq \psi(x)$. Take an open set $W$ in $Y$ containing $\psi(x)$ but not containing $\varphi(x)$. As $\psi$ is generated by $f^{\varepsilon}$ there is $\sigma \in V$ and $A \in \sigma$ such that

$$x \in A\,, \qquad f^{\varepsilon}_{\sigma}(A) \subset W\,, \qquad \psi(x) \in \overline{f^{\varepsilon}_{\sigma}(A)}\ .$$

But by definition of $F^{\varepsilon}$ we must have that

$$\varphi(x) \in f^{\varepsilon}_{\sigma}(A)\ ,$$

what is a contradiction and the theorem is proved. ∎

**2.** Now we discuss some questions connected with Lusin's theorem, to illustrate the use of $\varepsilon$-generators.

**Theorem 2.** *Let $(X, \mathcal{A}, m)$ and $(Y, B, \mu)$ be measure spaces with $m(X) < +\infty$ and $\varphi: X \to Y$ a $(\mathcal{M}, m)$-measurable function where $\mathcal{M} \subset B$ is the $\sigma$-algebra of Borel subsets of $Y$. Suppose that $Y$ is a regular Hausdorff space. Then for any given $\varepsilon > 0$ the two statements below are equivalent:*

**a**) *There is a set $F^\varepsilon \subset X$ such that $\varphi \,|\, F^\varepsilon$ is continuous and $m^*(X - F^\varepsilon) < \varepsilon$;*

**b**) *$\varphi$ has an $\varepsilon$-generator, which is point-wise cofinal relative to $(X, \varphi(X))$.*

**Proof: 1)** a)$\Rightarrow$b). Let $V'$ be an arbitrary family of open coverings of $Y$, cofinal in the set $\mathrm{Cov}(Y)$ of all open coverings of $Y$. As $\varphi \,|\, F^\varepsilon$ is continuous it has by Proposition I, §I, a canonical generator

$$f\colon\ (F^\varepsilon, V) \to (Y, V')$$

because $V'$ being cofinal in $\mathrm{Cov}(Y)$ satisfies the hypothesis of Proposition I. We can also assume that for each $\sigma \in V$ and $A \in \sigma$, the set $A$ is open in $X$ because to any open set $A_F$ in $F^\varepsilon$ it can be associated in many ways an open set in $X$ whose intersection with $F^\varepsilon$ is $A_F$.

If all coverings of $F^\varepsilon$, $\sigma \in V$, are also coverings of $X$ then as easily seen $f$ is also an $\varepsilon$-generator of $\varphi$, because the set $E^\varepsilon$ associated to $f = f^\varepsilon$ is contained in $X - F^\varepsilon$ and hence $m^*(E^\varepsilon) < \varepsilon$.

Otherwise there is, by the definition of outer measure of a regular measure, an open set $U$ containing $X - F^\varepsilon$ such that $m(U) < \varepsilon$. Now, from each $\sigma \in V$ pick up an arbitrary set $A \in \sigma$ and define $A^\varepsilon = A \cup U$ and $\sigma^\varepsilon$ as $\sigma$ plus the new set $A^\varepsilon$. Call $V^\varepsilon$ the family of all $\sigma^\varepsilon$ as $\sigma$ runs in $V$. Define

$$f^\varepsilon_{V^\varepsilon}\colon\ V^\varepsilon \to V'$$

by

$$\forall\, \sigma^\varepsilon \in V^\varepsilon\,, \qquad f^\varepsilon_{V^\varepsilon}(\sigma^\varepsilon) = f_V(\sigma)$$

and for each $\sigma^\varepsilon \in V^\varepsilon$ define

$$f^\varepsilon_{\sigma^\varepsilon}\colon\ \sigma^\varepsilon \to \sigma' = f^\varepsilon_{V^\varepsilon}(\sigma^\varepsilon) = f_V(\sigma)$$

by

$$\forall\, A^\varepsilon \in \sigma^\varepsilon\,, \qquad f^\varepsilon_{\sigma^\varepsilon}(A^\varepsilon) = f^\varepsilon_{\sigma^\varepsilon}(A \cap U) = f_\sigma(A)\ .$$

Again, as easily checked $f^\varepsilon$ is an $\varepsilon$-generator of $\varphi$, point-wise cofinal relative to $(X, \varphi(X))$ and this proves that a)$\Rightarrow$b).

**2**) b)$\Rightarrow$a). Let $f^\varepsilon$,

$$f^\varepsilon\colon\ (X, V) \to (Y, V')$$

be an $\varepsilon$-generator of $\varphi$, point-wise cofinal relative to $(X, \varphi(X))$. Then $m^*(E^\varepsilon) < \varepsilon$ and hence calling $F^\varepsilon = X - E^\varepsilon$ we have that $m^*(X - F^\varepsilon) < \varepsilon$.

Now let us show that $f^\varepsilon$ is fully-regular relative to $(F^\varepsilon, Y)$. Indeed, take any $A \in \sigma$ and $B \in \tau$ for arbitrary $\sigma, \tau \in V$ with $A \cap B \cap F^\varepsilon \neq \emptyset$. By the definition of $F^\varepsilon$ if $x \in A \cap B \cap F^\varepsilon$ we have

$$\varphi(x) \in f^\varepsilon_\sigma(A) \cap f^\varepsilon_\tau(B) \neq \emptyset$$

what proves that $f^\varepsilon$ is fully regular relative to $(F^\varepsilon, \varphi(X))$ and therefore by Jansen's Theorem 1 it generates a continuous function $\psi : F^\varepsilon \to Y$, relative to $(F^\varepsilon, \varphi(X))$.

Let us show that $\psi = \varphi \,|\, F^\varepsilon$. Suppose that for some $x \in F^\varepsilon$ we have $\psi(x) \neq \varphi(x) = (\varphi \,|\, F^\varepsilon)(x)$. Let $W$ be an open set in $Y$ containing $\psi(x)$ but not containing $\varphi(x)$. As $f^\varepsilon$ generates $\psi$ relative to $(F^\varepsilon, \varphi(X))$ there is $A \in \sigma \in V$, with $x \in A \cap F^\varepsilon$ such that

$$f^\varepsilon_\sigma(A) \cap \varphi(X) \subset W \ .$$

But by the definition of $F^\varepsilon$ we have that $\varphi(x) \in f^\varepsilon_\sigma(A)$ what is impossible. This completes the proof of the theorem. ∎

**Remarks.**

**1**) Classically, Lusin's theorem is stated with the assumption that the set $F^\varepsilon$ is a measurable set. In this case we have a similar theorem to Theorem 2 where a) and b) are changed to:

**a′**) There is a measurable set $F^\varepsilon \subset X$ such that $\varphi \,|\, F^\varepsilon$ is continuous and $m(X - F^\varepsilon) < \varepsilon$;

**b′**) $\varphi$ has a measurable $\varepsilon$-generator, which is point-wise cofinal relative to $(X, \varphi(X))$.

**2**) When $m$ is a Radon measure Theorem 2 can be improved by saying that

**a″**) For each compact set $K \subset X$ there is a compact set $K^\varepsilon \subset K$ such that $\varphi \,|\, K^\varepsilon$ is continuous and $m(K - K^\varepsilon) < \varepsilon$;

**b″**) $\varphi$ has a measurable $\varepsilon$-generator, point-wise cofinal relative to $(K, \varphi(K))$.

The crucial question at this point is the existence of point-wise cofinal $\varepsilon$-generator. We prove now one theorem of existence of $\varepsilon$-generators.

**Theorem 3.** *Let $(X, \mathcal{A}, m)$ and $(Y, B, \mu)$ be measure spaces where $B$ contains all Borel sets of $Y$ and $Y$ is a separable metric space. Let $\varphi\colon X \to Y$ be a $(\mathcal{M}, m)$-measurable function, where $\mathcal{M} \subset B$ is the set of all Borel sets of $Y$. Then for each*

$\varepsilon > 0$ $\varphi$ *has an $\varepsilon$-generator which is measurable and point-wise cofinal relative to* $(X, \varphi(X))$.

**Proof:** Let $V_Y$ be a countable family of open coverings of $Y$, where each $\sigma_Y^n \in V_Y$ is made up of countably many open balls of radius $\frac{1}{n}$, $n = 1, 2, \dots$. If several $\sigma_Y^n$, for different $n$, produce by intersection with $\varphi(X)$ the same covering of $\varphi(X)$ by open sets in $\varphi(X)$, we select one of them arbitrarily and indicate the family of all coverings of $Y$ selected in this way still by $V_Y$.

Take $\sigma_Y^n \in V_Y$ and $A_Y^n \in \sigma_Y^n$ and look to $\varphi^{-1}(A_Y^n)$ which is measurable in $(X, \mathcal{A}, m)$. As $m$ is outer regular there is an open set $A^n$ in $X$, called *associated to* $\varphi^{-1}(A_Y^n)$ such that

$$m(A^n - \varphi^{-1}(A_Y^n)) < \varepsilon(A_Y^n) \tag{1}$$

with $\varepsilon(A_Y^n)$ a positive number to be fixed later. As $A_Y^n$ runs in $\sigma_Y^n$ the set of all sets $A^n$ as above forms an open covering of $X$, denoted by $\sigma^n$, calling $V$ the family of all $\sigma^n$ as $\sigma_Y^n$ runs in $V_Y$.

In this way we have a function

$$f_V \colon V \to V'$$

given by

$$\forall\, \sigma^n \in V\,, \qquad f_V(\sigma^n) = \sigma_Y^n\ ,$$

which is well defined because no two coverings of $V_Y$ produce the same covering in $\varphi(X)$.

Take any $\sigma^n \in V$ and $A^n \in \sigma^n$. There is an open set $A_Y^n \in V_Y^n$ such that

$$m(A^n - \varphi^{-1}(A_Y^n)) < \varepsilon(A_Y^n)\ .$$

If several $A_Y^n$ produce the same $\varphi^{-1}(A_Y^n)$ select one of them arbitrarily. We still have the inconvenience that to different $\varphi^{-1}(A_Y^n)$ there corresponds the same associated $A^n$ and to avoid that we can use several ways. One of them consists in writing all $\varphi^{-1}(A_Y^n)$ inside the same $A^n$ in a row $\{\varphi^{-1}(A_Y^{n,i})\}_{i \geq 1}$ and for $i = 1$ select a point $x_1$ in

$$A^n - \varphi^{-1}(A_Y^{n,1})$$

if this set is not empty and $A_1^n = A^n - \{x_1\}$ is the set associated to $\varphi^{-1}(A_Y^{n,1})$. If

$$A^n - \varphi^{-1}(A_Y^{n,1}) = \emptyset$$

this means that $\varphi^{-1}(A_Y^{n,1})$ is open and no other set of the form $\varphi^{-1}(A_Y^n) \neq \varphi^{-1}(A_Y^{n,1})$ is contained in $A^n$.

Look to $\varphi^{-1}(A_Y^{n,2})$ and proceed as before by selecting

$$x_2 \in A^n - \varphi^{-1}(A_Y^{n,2})$$

with $x_1 \neq x_2$. Of course, we are disregarding the trivial case when either $X$ or $A^n$ are finite. Under this circumstance the sequence of points $\{x_i\}_{i \geq 1}$ is made up of distinct points, with

$$x_i \in A^n - \varphi^{-1}(A_Y^{n,i}) , \qquad A_i^n = A^n - \{x_i\} .$$

Now instead of $A^n$ we consider the sets $A_i^n$ whose union is $A^n$ as members of the covering $\sigma^n$ which is thus increased by at most countably many sets. Observe that

$$m(A_i^n - \varphi^{-1}(A_Y^{n,i})) < \varepsilon(A_Y^{n,i}) ,$$

where the $\varepsilon(A_Y^{n,i})$ had been originally fixed for $A^n$.

Now the function

$$f_{\sigma^n} \colon \sigma^n \to \sigma_Y^n ,$$

given by

$$\forall\, A^n \in \sigma^n , \qquad f_{\sigma^n}(A^n) = A_Y^n ,$$

where $A^n$ is the open set associated to $A_Y^n$ is well defined.

Therefore we have defined a $n$-function

$$f \colon (X, V) \to (Y, V_Y)$$

and selecting all $\varepsilon(A_Y^n)$ in such a way that

$$\sum_{\substack{A_Y^n \in \sigma_Y^n \\ \sigma_Y^n \in V_Y}} \varepsilon(A_Y^n) < \varepsilon$$

we conclude that $f$ is an $\varepsilon$-generator for $\varphi$, which we shall denote by $f^\varepsilon$. It is also point-wise cofinal relative to the pair $(X, \varphi(X))$. Indeed, let $\Gamma \in \operatorname{Cov}(Y)$ arbitrarily given and let $x \in X$ also arbitrary. Look to $\varphi(x)$ and $M \in \Gamma$ with $\varphi(x) \in M$. By the definition of $V_Y$ and because $Y$ is a metric space, for $n$ large enough there is a $\sigma_Y^n \in V_Y$ and $A_Y^n \in \sigma_Y^n$ with $\varphi(x) \in A_Y^n \subset M$ and hence

$$A_Y^n \cap \varphi(X) \subset M .$$

Now by the definition of $V$ there is a $\sigma^n \in V$ and $A^n \in \sigma^n$ such that $x \in A^n$ and

$$\varphi(X) \cap f_{\sigma^n}^\varepsilon(A^n) = A_Y^n \cap \varphi(X)$$

and therefore

$$f^{\varepsilon}_{\sigma^n}(A^n) \cap \varphi(X) \subset M \ ,$$

what proves that $f^{\varepsilon}$ is measurable and point-wise cofinal relative to $(X, \varphi(X))$. This completes the proof of the theorem. ∎

**Remarks.** **1)** We observe that Lusin's theorem where $Y$ is a separable metric space is already known in the literature, for instance [3], [4]. However, in all cases known to us there are always some restrictions put either in the measure, for instance Radom measure, or in the spaces used, for instance, local compactness. We are not aware of any proof as general as Theorem 2 above. As a matter of fact we do not believe that Lusin's theorem can be proved in more general classes then that of separable metric for the image space $Y$. Indeed, let us consider the examples below:

**Example 1:** Let $X = [0,1]$ and $Y$, the space of all real valued functions defined in $[0,1]$ with the topology of uniform convergence. As well known $Y$ is metric but not separable. Let $(X, \mathcal{A}, m)$, $(Y, B, \mu)$ be measure spaces where $\mathcal{A}$ is the $\sigma$-algebra of Lebesgue measurable sets in $[0,1]$, $m$, the Lebesgue measure, $B$ the $\sigma$-algebra of Borel sets in $Y$ and $\mu$ an arbitrary measure in $B$. Let $\varphi\colon\ X \to Y$ be defined as follows: for any $x_0 \in [0,1]$ define $\varphi(x_0)$ as the function of $x$:

$$f(x_0, x) \begin{cases} \sin \dfrac{1}{x - x_0} & x \neq x_0, \\ 0 & x = x_0\ . \end{cases}$$

This function $\varphi$ is clearly $(B - m)$-measurable but it is not continuous when restricted to any infinite compact set $K \in [0,1]$. Indeed, let $x_0$ be an accumulation point of $K$ and look to $(\varphi \,|\, K)(x_0) = \varphi(x_0)$.

Take in $Y$ a neighbourhood $W$ of $\varphi(x_0)$ given by

$$W = \Big\{ f \in Y\colon\ |\varphi(x_0) - f| < \alpha < 1 \Big\}$$

where $|a - b|$ denotes the distance of two elements $a, b \in Y$.

As easily seen, for any point $y_0 \in K$ sufficiently close to $x_0$ we always have $|\varphi(x_0) - \varphi(y_0)| > 1$ and hence $\varphi \,|\, K$ is not continuous. Therefore, Lusin's theorem is false for $\varphi$.

**Example 2:** This example will show that even strong compactness conditions on the spaces involved cannot guarantee the validity of Lusin's theorem outside the class of separable metric spaces. Let $X = [0,1]$ and $Y$ be the space of points in the square $[0,1] \times [0,1]$ with the topology of the lexicographic order.

More precisely, let us order $Y$ in the following way: $(x_1, y_1)$ will precede $(x_2, y_2)$, indicated by $(x_1, y_1) < (x_2, y_2)$ if either $x_1 < x_2$ or if $x_1 = x_2$, then $y_1 < y_2$. Introducing the topology in $Y$ induced by this order one can check that $Y$ is Hausdorff, regular, compact and satisfies the first axiom of countability. On top of that it is irreducible relative to connectivity between the points $(0,0)$ and $(1,1)$, being in this way a generalized are in the sense of Wilder (see [5]). Let $(X, \mathcal{A}, m)$ as before in Example 1 and $(Y, B, \mu)$ with $B$ the $\sigma$-algebra of Borel sets of $Y$ and $\mu$ an arbitrary measure. Define $\varphi\colon X \to Y$ by associating to any $x \in K$ the point $\varphi(X) = (x, \frac{1}{2}) \in Y$. Again, it is not hard to see that $\varphi\,|\,K$, where $K \subset X$ is any infinite compact set is not continuous and hence Lusin's theorem is false.

## III

**1.** To finish this paper we discuss a few facts concerning sequences of $\varepsilon$-generators. A more elaborated exposition of this question will be treated in forth coming papers.

**Theorem 4.** *Let $\varphi\colon X \to Y$ be an arbitrary function $m(X) < +\infty$, and $\{\varepsilon_n\}_{n\geq 1}$ a sequence of non-negative numbers such that*

$$\sum_{n\geq 1} \varepsilon_n < +\infty \ . \tag{1}$$

*Suppose that for each $\varepsilon_n$, $\varphi$ has an $\varepsilon_n$-generator $f^{\varepsilon_n}$ generating a function $\varphi_n : X \to Y$. Then there is a set $G \subset X$ with $m^*(G) = 0$ such that for each $x \in X - G$ there is an integer $n_0(x) > 0$ with the property that for all $n > n_0(x)$ we have*

$$\varphi_n(x) = \varphi(x) \ .$$

**Proof:** Call $G$ the set

$$G = \Big\{x \in X\colon \forall n_0\ \exists n > n_0 \ \text{ such that } \varphi_n(x) \neq \varphi(x)\Big\} \ .$$

Suppose that $m^*G = \delta > 0$. By Theorem 1, we have that for each $\varepsilon_n$,

$$m^*E^{\varepsilon_n} = m^*\Big\{x \in X\colon \varphi_n(x) \neq \varphi(x)\Big\} < \varepsilon_n \ .$$

If $n_0 \geq 1$ is large enough, by (1),

$$\sum_{n>n_0} \varepsilon_n < \delta \ ,$$

what implies that, with $\complement E$ denoting $X-E$, for any $E \subset X$,

$$m^*\Big(\bigcap_{n>n_0} \complement E^{\varepsilon_n}\Big) = m^*\Big(\complement \bigcup_{n>n_0} E^{\varepsilon_n}\Big) = m^*\Big(X - \bigcup_{n>n_0} E^{\varepsilon_n}\Big) \geq$$
$$\geq m^*X - m^*\Big(\bigcup_{n>n_0} E^{\varepsilon_n}\Big) > m^*X - \delta = m^*X - m^*G \ .$$

This implies that

$$G_0 = G \cap \Big(\bigcap_{n>n_0} \complement E^{\varepsilon_n}\Big) \neq \emptyset \ .$$

But if $x \in G_0$ this implies that $\varphi_n(x) = \varphi(x)$ for all $n > n_0$, as shown in the proof of Theorem 1, what contradicts the definition of $G$. Therefore $m^*G = 0$.

Now for each $x \in X-G$ we have $\varphi_n(x) = \varphi(x)$ and this proves the theorem. ∎

A general question in analysis is: given a sequence $\{\varphi_n\}_{n\geq 1}$ of functions $\varphi_n\colon X \to Y$ under what conditions this consequence converges, or has a convergent subsequence a.e., etc. An example of this situation will be given by the theorem which follows.

**Theorem 5.** *Let $\{\varphi_n\}_{n\geq 1}$ be a sequence of functions $\varphi_n : X \to Y$ and $\{\varepsilon_n\}_{n\geq 1}$ a sequence of positive numbers with*

$$\sum_{n\geq 1} \varepsilon_n < +\infty \ . \tag{1}$$

*Suppose that for each $\varepsilon_n$, $n \geq 1$, $\varphi_n$ has a $\varepsilon_n$-generator $f^{\varepsilon_n}$,*

$$f^{\varepsilon_n}\colon (X, V^n) \to (Y, V'^n)$$

*such that the sequence $\{f^{\varepsilon_n}\}_{n\geq 1}$ generates a function $\varphi\colon X \to Y$ with $\{n_i\}_{i\geq 1}$ for generating sequence in the sense of Definition VII. Then the sequence $\{\varphi_n\}_{n\geq 1}$ has a subsequence converging a.e. to $\varphi$.*

**Proof:** Let us indicate by $G$ the subset of $X$ given by

$$G = \Big\{x \in X\colon \exists\, W \text{ open in } Y \text{ and } \varphi(x) \in W$$
$$\text{such that } \forall\, n_0 \geq 0\ \exists\, n_i > n_0 \Rightarrow \varphi_{n_i}(x) \notin W\Big\} \ .$$

Suppose that $m^*G = \delta > 0$ and call as before for each $\varepsilon_n$, $n \geq 1$,

$$E^{\varepsilon_n} = \bigcup_{\substack{\sigma^n \in V^n \\ A^n \in \sigma^n}} \Big\{x \in A^n \in \sigma^n \in V^n\colon \varphi(x) \notin f^{\varepsilon_n}_{\sigma_n}(A^n)\Big\} \ ,$$

with $m^*(E^{\varepsilon_n}) < \varepsilon_n$, and let $F^{\varepsilon_n} = X - E^{\varepsilon_n}$.

For $n_0$ large enough we have, from (1),

$$\sum_{n>n_0} \varepsilon_n < \delta$$

what implies

$$m^*\Big(\bigcap_{n>n_0} F^{\varepsilon_n}\Big) = m^*\Big(\bigcap_{n>n_0} \complement E^{\varepsilon_n}\Big) = m^*\Big(X - \bigcup_{n>n_0} E^{\varepsilon_n}\Big) \geq$$
$$\geq m^*(X) - m^*\Big(\bigcup_{n>n_0} E^{\varepsilon_n}\Big) > m^*(X) - \delta = m^*(X) - m^*(G) \ .$$

Hence

$$G_0 = G \cap \bigcap_{n>n_0} F^{\varepsilon_0} \neq \emptyset$$

and therefore for each $x \in G_0$ we have by Definition VII and by definition of $F^{\varepsilon_n}$, $n \geq 1$, that for any $W$ open in $Y$ with $\varphi(x) \in W$ there is $n_i > n_0$, $\sigma^{n_i} \in V^{n_i}$, $A^{n_i} \in \sigma^{n_i}$ with $x \in A^{n_i}$ such that

$$f^{\varepsilon_{n_i}}_{\sigma^{n_i}}(A^{n_i}) \subset W \qquad \text{and} \qquad \varphi(x) \in \overline{f^{\varepsilon_{n_i}}_{\sigma_{n_i}}}(A^{n_i})$$

and also

$$\varphi_{n_i}(x) \in f^{\varepsilon_{n_i}}_{\sigma^{n_i}}(A^{n_i}) \ ,$$

what contradicts the definition of $G$ and hence $m^*(G) = 0$.

Take any point $x \in X - G$. Then for any $W$ open in $Y$ with $\varphi(x) \in W$, there is $n_0 \geq 0$ such that for any $n_i > n_0$ we have

$$\varphi_{n_i}(x) \in W \ .$$

Therefore the subsequence $\{\varphi_{n_i}\}_{i\geq 1}$ of the original sequence converges to $\varphi(x)$ in all points of $X - G$ and as $m^*(G) = 0$ this proves the theorem. ∎

**Theorem 6.** *Let $\{\varphi_n\}_{n\geq 1}$, $\varphi_n \colon X \to Y$ be a sequence, converging a.e. on $X$, with $m(X) < +\infty$. Suppose that $Y$ is a separable metric space.*

*Then for any $\varepsilon > 0$ there is an integer $n_0$ such that for all $n > n_0$, $\varphi_n$ and $\varphi$ have a common $\varepsilon$-generator.*

**Proof:** As $\Gamma$ is a separable metric space, $\varphi$ has an $\frac{\varepsilon}{2}$-generator $f^{\varepsilon/2}$,

$$f^{\frac{\varepsilon}{2}} \colon (X, V) \to (Y, V') \ .$$

Also under the hypothesis of the theorem we can use Egoroff's theorem and select a set $F \subset X$ such that $m(X - F) < \frac{\varepsilon}{2}$ and $\{\varphi_n\}_{n \geq 1}$ converges uniformly on $F$ to $\varphi \,|\, F$. Call $\psi_n = \varphi_n \,|\, F$ and $\psi = \varphi \,|\, F$. Then there is $n_0(\varepsilon)$ such that

$$(2) \qquad \forall\, x \in F\,, \qquad n > n_0(\varepsilon) \;\Rightarrow\; |\psi_n(x) - \psi(x)| = |\varphi_n(x) - \varphi(x)| < \frac{\varepsilon}{2}\,.$$

Recall that if $E^{\varepsilon/2}$ is the set associated to $f^{\varepsilon/2}$ we have $m^*(E^{\varepsilon/2}) < \frac{\varepsilon}{2}$.

Let us define a $n$-function

$$g^\varepsilon \colon (X, V) \to (Y, V'')$$

as follows: to each $\sigma \in V$ and $A \in \sigma$ look to $A' = f_\sigma(A) \in \sigma' \in V'$ and cover this set with open balls of diameter equal to $\frac{\varepsilon}{2}$ and call $A''$ this new set. Call $\sigma''$ the open covering of $Y$ made up of all $A''$ as $A$ runs in $\sigma$ and call $V''$ the family of all $\sigma''$.

Put

$$g^\varepsilon_V \colon V \to V'$$

by associating to each $\sigma \in V$ the covering $\sigma'' \in V''$ and to each $\sigma \in V$ define

$$g_\sigma \colon \sigma \to \sigma''$$

by associating to each $A \in \sigma$ the set $A'' \in \sigma''$. All this defines the $n$-function $g^\varepsilon$. If $E^\varepsilon$ is the set associated to $g^\varepsilon$ we have $E^\varepsilon \cap F \subset E^{\varepsilon/2} \cap F$ what gives

$$m^*(E^\varepsilon \cap F) \leq m^*(E^{\varepsilon/2} \cap F) < \frac{\varepsilon}{2}\,,$$

$$m^*(E^\varepsilon \cap \complement F) \leq m^*(\complement F) < \frac{\varepsilon}{2}\,,$$

hence

$$m^*(E^\varepsilon) < \varepsilon$$

and $g^\varepsilon$ is an $\varepsilon$-generator for $\varphi$.

Now if $E^\varepsilon_n$ is the set

$$E^\varepsilon_n = \bigcup_{\substack{A \in \sigma \\ \sigma \in V}} \Big\{ x \in A \colon \varphi_n(x) \notin g^\varepsilon(A) \Big\}$$

we have due to (1) that $E^\varepsilon_n \cap F \subset E^{\varepsilon/2} \cap F$, what gives

$$m^*(E^\varepsilon_n \cap F) < \frac{\varepsilon}{2}\,,$$

$$m^*(E^\varepsilon_n \cap \complement F) < \frac{\varepsilon}{2}$$

and hence

$$m^*(E_n^\varepsilon) < \varepsilon$$

what proves that $g^\varepsilon$ is also an $\varepsilon$-generator for all $\varphi_n$ with $n > n_0$. This completes the proof. ∎

**2.** We discuss now a characterization of quasi-continuous functions in terms of $\varepsilon$-generators. As well known a function $\varphi : X \to Y$ is quasi-continuous, abbreviated $q$-continuous, if there is a subset $Q \subset X$ such that $m(X - Q) = 0$ and $\varphi \,|\, Q$ is continuous. For instance, Dirichlet's function equal zero for rationals and one for irrationals on the real line is $q$-continuous. Even more, its restriction to $Q = \{\text{set of irrationals on the real line}\}$ can be extended to a continuous function to all reals. That is not always the case as shown by the function $y = \sin \frac{1}{x}$ for $x \neq 0$ and $y = 0$ for $x = 0$. Finally, a function $\varphi$ can be measurable and not $q$-continuous on the real line.

**Theorem 7.** *A function $\varphi$: $X \to Y$ is $q$-continuous if and only if there is a set $Q \subset X$, with $m^*(X - Q) = 0$, such that $\varphi \,|\, Q$ has a sequence of $\varepsilon_n$-generators $f^{\varepsilon_n}$, $n \geq 1$,*

$$f^{\varepsilon_n} \colon (Q, V) \to (Y, V')$$

*with*

$$\sum_{n \geq 1} \varepsilon_n < +\infty \ ,$$

*where each $f^{\varepsilon_n}$ generates a function $\psi_n : Q \to Y$ whose sequence has a limit $\psi$: $Q \to Y$ continuous.*

**Proof:** If $\varphi$: $X \to Y$ is $q$-continuous, then there is $Q \in X$ with $m^*(X - Q) = 0$ and $\varphi \,|\, Q$ is continuous. Then taking for $\varphi \,|\, Q$ for each $n$ as $\varepsilon_n$-generator $f^{\varepsilon^n}$, with $\varphi_n = 0$ the canonical generator of $\varphi$, according to the remark at the end of Proposition I, §I, we have that each $f^{\varepsilon_n}$ generates $\varphi_n = \varphi \,|\, Q$ and then the result follows.

Conversely if each $f^{\varepsilon_n}$ as in the hypothesis of the theorem generates $\psi_n$: $Q \to Y$ then by Theorem 4 $\lim \psi_n$ exists and it is equal to $\varphi \,|\, Q$, a.e. and as, by hypothesis, this limit is continuous, we conclude that $\varphi \,|\, Q$ is continuous except in a set of measure zero and consequently, as $m(X - Q) = 0$, it is $q$-continuous. This completes the proof. ∎

Finally, let us investigate the question: when a sequence of $q$-continuous functions has a limit which is also $q$-continuous?

Initially let us observe that a sequence of $q$-continuous functions might have a limit which is not $q$-continuous. Indeed, let $X = [0,1]$, $Y =$ reals and $F \subset X$ such that $mF = \frac{1}{2}$, and $F$ is closed, with empty interior in $X$. The existence of such a set can be seen, for instance, in [6], p. 291. Now let $\varphi\colon X \to Y$ be given by

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F\ . \end{cases}$$

As easily seen such a function $\varphi$ is not $q$-continuous but it is the limit of a sequence of continuous functions $\varphi_n\colon X \to Y$. Indeed as $X - F$ is open we can find for each $n$ a closed set $F_n \subset X - F$ whose distance from $F$ is less than $\frac{1}{n}$. Let by Urysohn's lemma $\varphi_n$ be a continuous function in $X$ with real values such that $\varphi_n \,|\, F_n = 0$ and $\varphi_n \,|\, F = 1$. As easily seen $\varphi_n(x)$ converges to $\varphi(x)$ at each $x \in X$.

**Theorem 8.** *Let $\{\varphi_i\}_{i\geq 1}$ be a sequence of $q$-continuous functions $\varphi_i\colon X \to Y$, $Y$ separable metric space, converging to a function $\varphi\colon X \to Y$. Let for each $n \geq 1$ $f^{\varepsilon_n}$ be a common $\varepsilon_n$-generator for $\varphi_{i_n}$ and $\varphi$, which exists by Theorem 6, with $\lim_{n\to\infty} i_n = +\infty$. Then if the sequence $\{f^{\varepsilon_n}\}_{n\geq 1}$ generates a continuous function $\psi\colon X \to Y$, the function $\varphi$ is $q$-continuous, assuming that*

$$\sum_{n\geq 1} \varepsilon_n < +\infty\ .$$

**Proof:** By Theorem 5, the sequence $\{\varphi_{i_n}\}_{n\geq 1}$, whose limit is also $\varphi$ has a subsequence converging a.e. to $\psi$, which is then also limit a.e. of the original sequence $\{\varphi_i\}_{i\geq 1}$. As $\psi$ is continuous by hypothesis we conclude that $\varphi = \psi$ a.e. namely $\varphi$ is $q$-continuous and this completes the proof. ∎

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