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JACOBI ACTIONS OF $\mathrm{SO}(2) \times \mathbb{R}^2$ AND $\mathrm{SU}(2, \mathbb{C})$ ON TWO JACOBI MANIFOLDS

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Abstract: We take a sphere S of the dual space $\mathcal{G}^*$ of $\mathcal{G} = \mathrm{so}(2) \times \mathbb{R}^2$ with the Jacobi manifold structure obtained by quotient by the homothety group of the Lie–Poisson structure in $\mathcal{G}^* \backslash \{0\}$ and we study the actions of two subgroups of $\mathrm{SO}(2) \times \mathbb{R}^2$ on S.

We show that the natural action of $\mathrm{SU}(2, \mathbb{C})$ on the unitary 3-sphere of $\mathbb{C}^2$ with the Jacobi structure determined by its canonical contact structure is a Jacobi action that admits an unique Ad^*-equivariant momentum mapping.

1 – Introduction

The notions of *Jacobi manifold* and *Jacobi conformal manifold* were introduced by A. Lichnerowicz ([5]) in 1978. A. Kirillov ([3]) also studied these structures under the name of *local Lie algebras*, when defined on the space of the differentiable sections of a vector bundle with 1-dimensional fibres.

Let $\mathcal{G}^*$ be the dual of the Lie algebra of a finite dimensional Lie group, with its Lie–Poisson structure ([6]), and take the quotient of $\mathcal{G}^* \backslash \{0\}$ by the homothety group. A. Lichnerowicz ([6]) showed that the Lie–Poisson structure defines on the quotient space (which can be identified with an unitary sphere of $\mathcal{G}^*$) a Jacobi structure.

Finally, let us recall that the notion of *momentum mapping*, introduced by J.-M. Souriau ([11]) and B. Kostant ([4]) in the symplectic manifold context, can be extended to the Jacobi manifolds (cf. [8]), when a *Jacobi action* or a *conformal Jacobi action* ([9]) of a Lie group on a Jacobi manifold takes place.

In Appendix we summarize some of the basic concepts useful for a better understanding of the paper.

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2 – A Jacobi action of the Lie group $\mathrm{SO}(2) \times \mathbb{R}^2$ on the unitary sphere of the dual of its Lie algebra

Let G be the Lie group of the euclidean displacements, that is, the semidirect product of $\mathrm{SO}(2)$ with $\mathbb{R}^2$. The product of two elements (g,x) and (h,y) in $G = \mathrm{SO}(2) \times \mathbb{R}^2$ is given by

$$(g,x)\cdot(h,y) = (gh, gy+x) \ . \tag{1}$$

We can write the elements (g,x) of G as 3×3 matrices of the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & x_1 \\ \sin\alpha & \cos\alpha & x_2 \\ 0 & 0 & 1 \end{pmatrix} \equiv (g_\alpha, x) \ ,$$

where $\alpha \in \mathbb{R}$, $(x_1,x_2) \in \mathbb{R}^2$, the composition law (1) in G corresponding to the product of the two respective matrices.

The Lie group G acts on the plane $\mathbb{R}^2$ by an action ϕ given by

$$\phi\colon\ ((g_\alpha,x),y) \in G\times\mathbb{R}^2 \to (g_\alpha y + x) \in \mathbb{R}^2$$

which can be expressed in matricial form by the following product of matrices:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & x_1 \\ \sin\alpha & \cos\alpha & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1\cos\alpha - y_2\sin\alpha + x_1 \\ y_1\sin\alpha + y_2\cos\alpha + x_2 \\ 1 \end{pmatrix} \ .$$

This action corresponds to an α-rotation of the point (y_1,y_2) about the origin followed by a translation by the vector of components (x_1,x_2).

Let $\mathcal{G} \equiv \mathrm{so}(2)\times\mathbb{R}^2$ be the Lie algebra of G. An element (a,v) of $\mathcal{G}$ can be written as

$$\begin{pmatrix} 0 & a & v_1 \\ -a & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \equiv (a,v) \ ,$$

where $a\in\mathbb{R}$ and $(v_1,v_2)\in\mathbb{R}^2$.

The set $\mathcal{B}$ of elements

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is a basis of $\mathcal{G} = \mathrm{so}(2)\times\mathbb{R}^2$. Let

$$\left\{ \frac{\partial}{\partial B_1}, \frac{\partial}{\partial B_2}, \frac{\partial}{\partial B_3} \right\}$$

be the basis of $\mathcal{G}^*$, dual of $\mathcal{B}$. Once we have

$$[B_1,B_2]=-B_3\,,\qquad [B_1,B_3]=B_2\qquad\text{and}\qquad [B_2,B_3]=0\ ,$$

if we put

$$\Lambda=-B_3\,\frac{\partial}{\partial B_1}\wedge\frac{\partial}{\partial B_2}+B_2\,\frac{\partial}{\partial B_1}\wedge\frac{\partial}{\partial B_3}$$

and

$$Z=\sum_{i=1}^{3}B_i\,\frac{\partial}{\partial B_i}\ ,$$

the couple (Λ,Z) defines an *homogeneous* Lie–Poisson structure on $\mathcal{G}^*$. (Homogeneous means that $[\Lambda,Z]=-\Lambda$, $[\ ,\ ]$ being the Schouten bracket ([10]); Z is called the *Liouville* vector field.)

From now on, we will identify $\mathcal{G}^*=(\mathrm{so}(2)\times\mathbb{R}^2)^*$ with the product $(\mathrm{so}(2))^*\times(\mathbb{R}^2)^*$. Thus, an arbitrary element of $\mathcal{G}^*$ will be expressed by a couple (ξ,p) with $\xi\in(\mathrm{so}(2))^*$ and $p\in(\mathbb{R}^2)^*$.

Let us suppose that $\mathcal{G}^*$ is endowed with the usual Euclidean norm. If $\eta=(\xi,p)$ is an element of $\mathcal{G}^*$ with coordinates (η_1,η_2,η_3) in the basis $\{\frac{\partial}{\partial B_i}\}$, we define the *norm* of η, by putting

$$\|\eta\|^2=\sum_{i=1}^{3}(\eta_i)^2\ .$$

Let S be the unitary sphere of $\mathcal{G}^*$,

$$S=\Big\{\eta\in\mathcal{G}^*:\ \|\eta\|^2=1\Big\}\ ,$$

and suppose that S is supplied with the Jacobi structure obtained by quotient of the Lie–Poisson structure of $\mathcal{G}^*_0=\mathcal{G}^*\backslash\{0\}$ by the homothety group. On the open subsets

$$U_i^+=\Big\{(B_1,B_2,B_3)\in S:\ B_i>0\Big\}$$

and

$$U_i^-=\Big\{(B_1,B_2,B_3)\in S:\ B_i<0\Big\}\,,\qquad i=1,2,3$$

of S, we take the coordinate functions

$$\Big(x_1=B_1,\ \ \widehat{x}_i=\widehat{B}_i,\ \ x_3=B_3\Big)\,,\qquad i=1,2,3\ ,$$

where "$\widehat{\ }$" means absence.

The Jacobi structure (C,E) of S is given, in the local charts taken above, in the following Table, where

$$\varepsilon=\begin{cases}+1, & \text{on } U_i^+,\\ -1, & \text{on } U_i^-\ .\end{cases}$$

$\left(U_1^{\pm},(x_2,x_3)\right)$	$E=-\varepsilon\, x_3\,\sqrt{1-(x_2)^2-(x_3)^2}\,\frac{\partial}{\partial x_2}+\varepsilon\, x_2\,\sqrt{1-(x_2)^2-(x_3)^2}\,\frac{\partial}{\partial x_3}$ $C=-\varepsilon\,\sqrt{1-(x_1)^2-(x_3)^2}\,\left((x_2)^2+(x_3)^2\right)\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}$
$\left(U_2^{\pm},(x_1,x_3)\right)$	$E=\varepsilon\, x_1\,\sqrt{1-(x_1)^2-(x_3)^2}\,\frac{\partial}{\partial x_3}$ $C=\varepsilon\,\sqrt{1-(x_1)^2-(x_2)^2}\,\left(1-(x_1)^2\right)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}$
$\left(U_3^{\pm},(x_1,x_2)\right)$	$E=-\varepsilon\, x_1\,\sqrt{1-(x_1)^2-(x_2)^2}\,\frac{\partial}{\partial x_2}$ $C=\varepsilon\,\sqrt{1-(x_1)^2-(x_2)^2}\,\left((x_1)^2-1\right)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}$

V. Guillemin and S. Sternberg ([2]) showed that the coadjoint action Ad^* of G on the dual $\mathcal{G}^*$ of its Lie algebra is given by

$$\mathrm{Ad}^*_{(g_\alpha,x)}(\xi,p)=\Big(\xi+(g_\alpha p)\otimes x,\, g_\alpha p\Big)\ , \tag{2}$$

for every $(g_\alpha,x)\in G$ and $(\xi,p)\in\mathcal{G}^*$, where $\otimes$ is a mapping from $(\mathbb{R}^2)^*\times\mathbb{R}^2$ to $(\mathrm{so}(2))^*$,

$$(p,x)\in(\mathbb{R}^2)^*\times\mathbb{R}^2\ \to\ p\otimes x\in(\mathrm{so}(2))^*\ ,$$

such that

$$\langle p\otimes x,a\rangle=\langle p,ax\rangle\ ,$$

for all $a\in\mathrm{so}(2)$.

The restriction to S of the coadjoint action of G on $\mathcal{G}^*$ doesn't preserve the sphere S. However, we can take the *quotient coadjoint action* ([6]) $\overline{\mathrm{Ad}}$ of G on S which is given, for every $(g_\alpha,x)\in G$, by

$$\pi\circ\mathrm{Ad}^*_{(g_\alpha,x)}=\overline{\mathrm{Ad}}_{(g_\alpha,x)}\circ\pi\ ,$$

where $\pi:\ \mathcal{G}_0^*\to S$ is the canonical projection of $\mathcal{G}_0^*$ on the sphere S, this one being identified with the quotient of $\mathcal{G}_0^*$ by the homothety group.

Let

$$H=\Big\{(g_\alpha,0),\ g_\alpha\in\mathrm{SO}(2)\Big\}$$

be the 1-dimensional Lie subgroup of G corresponding to the plane rotations about the origin and whose elements can be written on the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv (g_\alpha, 0) , \qquad \alpha \in \mathbb{R} .$$

From (2), we may conclude that the restriction Ad^{*H} to the Lie subgroup H, of the coadjoint action of G on $\mathcal{G}^*$ is given by

$$\mathrm{Ad}^{*H}_{(g_\alpha,0)}(\xi, p) = (\xi, g_\alpha p) ,$$

with $(\xi, p) \in (\mathrm{so}(2))^* \times (\mathbb{R}^2)^*$ and $(g_\alpha, 0) \in H$.

As the Ad^{*H} action preserves the sphere S (in fact if $(\xi, p) \in S$ then $(\xi, g_\alpha p) \in S$, since $\|(\xi, p)\| = \|(\xi, g_\alpha p)\|$), the restriction to the subgroup H of the quotient action $\overline{\mathrm{Ad}}$ of G on S, coincides with the restriction to S of the Ad^{*H} action,

$$\mathrm{Ad}^{*H} = \overline{\mathrm{Ad}}_{|H} \colon\ H \times S \to S .$$

Proposition. *The restriction to the subgroup H of the quotient coadjoint action of G on S is a Jacobi action.*

Proof: The Lie algebra of H being generated by the element

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the basis $\mathcal{B}$ of $\mathcal{G}$, the Ad^{*H} action is a Jacobi action of H on S if

$$\Big[(B_1)_S, E\Big] = 0 \quad \text{and} \quad \Big[(B_1)_S, C\Big] = 0 ,$$

where $(B_1)_S$ is the fundamental vector field associated with B_1 ([9]) and in the last equality [,] is the Schouten bracket ([10]). But, if X_{x_1} is the hamiltonian vector field ([7]) associated with $x_1 \in C^\infty(S, \mathbb{R})$, we have

$$(B_1)_S = X_{x_1} ,$$

because B_1, as a function from $\mathcal{G}^*$ to $\mathbb{R}$, is homogeneous with respect to the Liouville vector field and projects into S, its projection being the function x_1. We have then

$$\Big[(B_1)_S, E\Big] = \Big[X_{x_1}, E\Big] = X_{-(E.x_1)}$$

and

$$\left[(B_1)_S, C\right] = \left[X_{x_1}, C\right] = -(E.x_1)\, C \ .$$

If we look at the expression of the vector field E in the local charts of S on the preceding Table, we can see that

$$E.x_1 = 0 \ ,$$

in all cases. Thus, we have

$$\left[(B_1)_S, E\right] = \left[(B_1)_S, C\right] = 0$$

and the $\mathrm{Ad}^{*H} \equiv \overline{\mathrm{Ad}}_{|H}$ action is a Jacobi action of H on S. ∎

If instead of H we take the 2-dimensional subgroup H^1 of G that corresponds to the plane translations and whose elements are of the form

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

where $(x_1, x_2) \in \mathbb{R}^2$, the restriction to H^1 of the quotient coadjoint action of G on the sphere S is a conformal Jacobi action. In fact, the Lie algebra of H^1 being generated by the elements

$$B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

of the basis $\mathcal{B}$ of $\mathcal{G}$, we have

$$\begin{cases} \left[(B_2)_S, E\right] = \left[X_{x_2}, E\right] = X_{-(E.x_2)} \\ \left[(B_2)_S, C\right] = \left[X_{x_2}, C\right] = -(E.x_2) \end{cases}$$

and also

$$\begin{cases} \left[(B_3)_S, E\right] = \left[X_{x_3}, E\right] = X_{-(E.x_3)} \\ \left[(B_3)_S, C\right] = \left[X_{x_3}, C\right] = -(E.x_3) \ . \end{cases}$$

Thus, the action $\overline{\mathrm{Ad}}_{|H^1}$ is a conformal Jacobi action of H^1 on the Jacobi manifold S.

3 – A Jacobi action of SU(2, $\mathbb{C}$) on the unitary 3-sphere of $\mathbb{C}^2$

Let (z_1, z_2) be the canonical coordinates on $\mathbb{C}^2$. We take $\mathbb{C}^2$ with the following hermitian product

$$\Big((z_1, z_2) \mid (z_1', z_2')\Big) = z_1\,\overline{z}_1' + z_2\,\overline{z}_2' \ .$$

By means of this hermitian product, we can define a norm in $\mathbb{C}^2$ by putting

$$\|(z_1, z_2)\|^2 = \Big((z_1, z_2) \mid (z_1, z_2)\Big) = z_1\,\overline{z}_1 + z_2\,\overline{z}_2 \ .$$

Let

$$S^3 = \Big\{(z_1, z_2) \in \mathbb{C}^2 \colon\ z_1\,\overline{z}_1 + z_2\,\overline{z}_2 = 1\Big\}$$

be the unitary sphere of $\mathbb{C}^2$ and let α be the 1-form in $\mathbb{C}^2$ given by

$$\alpha = \mathrm{Re}\left[\frac{1}{i}\Big(z_1\,d\overline{z}_1 + z_2\,d\overline{z}_2\Big)\right] \ .$$

The restriction of α to S^3 defines a contact structure on the sphere ([11]).

If we identify the space $\mathbb{C}^2$ with $\mathbb{R}^4$, making the correspondence between the couple of complexes $(z_1 = x_1 + ix_3,\ z_2 = x_2 + ix_4)$ and the real quadruple (x_1, x_2, x_3, x_4), the 1-form α express as

$$\alpha = -x_3\,dx_1 - x_4\,dx_2 + x_1\,dx_3 + x_2\,dx_4 \ .$$

Since every contact manifold is a Jacobi manifold ([5]), we can take the sphere S^3 as a Jacobi manifold whose structure is given by

$$\begin{aligned}
E &= -x_3\,\frac{\partial}{\partial x_1} - x_4\,\frac{\partial}{\partial x_2} + x_1\,\frac{\partial}{\partial x_3} + x_2\,\frac{\partial}{\partial x_4} \ ,\\
C &= \frac{1}{2}(x_1x_4 - x_2x_3)\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_4}\right)\\
&\quad - \frac{1}{2}(x_1x_2 + x_3x_4)\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right)\\
&\quad - \frac{1}{2}\Big((x_1)^2 + (x_3)^2 - 1\Big)\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}\right)\\
&\quad - \frac{1}{2}\Big((x_2)^2 + (x_4)^2 - 1\Big)\left(\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_4}\right) \ .
\end{aligned} \tag{3}$$

Let's take the Lie group SU(2, $\mathbb{C}$) — which is a Lie subgroup of GL(2, $\mathbb{C}$) of dimension (real) 3 — and its Lie algebra su(2, $\mathbb{C}$). According to its definition, SU(2, $\mathbb{C}$) preserves the norm in $\mathbb{C}^2$ and acts on S^3 by the natural action

$$\Big(A, (z_1, z_2)\Big) \in \mathrm{SU}(2, \mathbb{C}) \times S^3 \to A.\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in S^3 \ .$$

The elements

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

that verify

$$[X_1, X_2] = -2X_3, \quad [X_1, X_3] = -2X_2 \quad \text{and} \quad [X_2, X_3] = -2X_1,$$

set up a basis of $\mathrm{su}(2, \mathbb{C})$. Taking in account the preceding identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, we can write these elements on the following form:

$$\text{(4)} \qquad \begin{cases} X_1 = -x_4 \dfrac{\partial}{\partial x_1} - x_3 \dfrac{\partial}{\partial x_2} + x_2 \dfrac{\partial}{\partial x_3} + x_1 \dfrac{\partial}{\partial x_4}, \\ X_2 = x_2 \dfrac{\partial}{\partial x_1} - x_1 \dfrac{\partial}{\partial x_2} + x_4 \dfrac{\partial}{\partial x_3} - x_3 \dfrac{\partial}{\partial x_4}, \\ X_3 = x_3 \dfrac{\partial}{\partial x_1} - x_4 \dfrac{\partial}{\partial x_2} - x_1 \dfrac{\partial}{\partial x_3} + x_2 \dfrac{\partial}{\partial x_4}. \end{cases}$$

Proposition. *The natural action of* $\mathrm{SU}(2, \mathbb{C})$ *on the sphere* (S^3, C, E) *is a Jacobi action.*

Proof: The set $\{X_1, X_2, X_3\}$ being a basis of $\mathrm{su}(2, \mathbb{C})$, we only must show that

$$\Big[(X_i)_{S^3}, E\Big] = \Big[(X_i)_{S^3}, C\Big] = 0, \qquad \text{for } i = 1, 2, 3,$$

where $(X_i)_{S^3}$ is the fundamental vector field associated with X_i, with respect to the action of $\mathrm{SU}(2, \mathbb{C})$ on S^3. But, this action being the natural action, we have, for $i = 1, 2, 3$,

$$(X_i)_{S^3} = -X_i.$$

From (3) and (4), we can easily prove that

$$[X_i, E] = [X_i, C] = 0, \qquad i = 1, 2, 3. \ \blacksquare$$

The action of $\mathrm{SU}(2, \mathbb{C})$ on S^3 admits a momentum mapping that we're going to evaluate. Let A be an arbitrary element of $\mathrm{SU}(2, \mathbb{C})$. Then A is a matrix of the form

$$A = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix},$$

where $(a, b, c, d) \in \mathbb{R}^4$ and $a^2 + b^2 + c^2 + d^2 = 1$.

Let ξ be an element of su$^*(2,\mathbb{C})$ of coordinates (ξ_1,ξ_2,ξ_3) on the dual basis of $\{X_1,X_2,X_3\}$. Then, for every $X\in$ su$(2,\mathbb{C})$, we have

$$\langle \mathrm{Ad}^*_A\,\xi, X\rangle = \langle \xi, \mathrm{Ad}_{A^{-1}}(X)\rangle = \langle \xi, A^{-1}XA\rangle = \langle \xi, (\overline{A})^{\mathrm{T}}XA\rangle \ .$$

We also have, for the elements X_1, X_2 and X_3 of the su$(2,\mathbb{C})$ basis,

$$\begin{cases} \langle \mathrm{Ad}^*_A\,\xi, X_1\rangle = \Big\langle \xi, (a^2-b^2-c^2+d^2)\,X_1 + 2(ab+cd)\,X_2 + 2(ac-bd)\,X_3\Big\rangle, \\ \langle \mathrm{Ad}^*_A\,\xi, X_2\rangle = \Big\langle \xi, 2(cd-ab)\,X_1 + (a^2-b^2+c^2-d^2)\,X_2 + 2(-ad-bc)\,X_3\Big\rangle, \\ \langle \mathrm{Ad}^*_A\,\xi, X_3\rangle = \Big\langle \xi, 2(-ac-bd)\,X_1 + 2(ad-bc)\,X_2 + (a^2+b^2-c^2-d^2)\,X_3\Big\rangle \ . \end{cases}$$

So,

$$\mathrm{Ad}^*_A\,\xi = \begin{pmatrix} \xi_1(a^2-b^2-c^2+d^2) + 2\xi_2(ab+cd) + 2\xi_3(ac-bd) \\ 2\xi_1(cd-ab) + \xi_2(a^2-b^2+c^2-d^2) + 2\xi_3(-ad-bc) \\ 2\xi_1(-ac-bd) + 2\xi_2(ad-bc) + \xi_3(a^2+b^2-c^2-d^2) \end{pmatrix} \ .$$

Proposition. *Let $J\colon S^3\to$ su$^*(2,\mathbb{C})$ be the mapping given by*

$$\begin{cases} \langle J, X_1\rangle\,(x_1+ix_3, x_2+ix_4) = 2(-x_1x_2 - x_3x_4), \\ \langle J, X_2\rangle\,(x_1+ix_3, x_2+ix_4) = 2(-x_1x_4 + x_2x_3), \\ \langle J, X_3\rangle\,(x_1+ix_3, x_2+ix_4) = (x_1)^2 - (x_2)^2 + (x_3)^2 - (x_4)^2 \ , \end{cases}$$

where X_1, X_2 and X_3 are the elements of the su$(2,\mathbb{C})$ *basis defined above. Then J is the unique* Ad**-equivariant momentum mapping of the natural Jacobi action of* SU$(2,\mathbb{C})$ *on S^3.*

Proof: If we calculate the hamiltonian vector fields $X_{\langle J,X_i\rangle}$ $(i=1,2,3)$ corresponding to the functions $\langle J,X_i\rangle$, we obtain

$$X_{\langle J,X_i\rangle} = -X_i \ .$$

But, as we have already remarked, $(X_i)_{S^3} = -X_i$. The mapping J is then a momentum mapping of the action of SU$(2,\mathbb{C})$ on S^3.

Let $A = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \in$ SU$(2,\mathbb{C})$ and $z_1 = x_1+ix_3$, $z_2 = x_2+ix_4 \in S^3$, be arbitrary elements. Then, we have

$$
\begin{aligned}
J\left(A.\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) &= J\begin{pmatrix} (ax_1 - bx_3 + cx_2 - dx_4) + i(ax_3 + bx_1 + cx_4 + dx_2) \\ (-cx_1 - dx_3 + ax_2 + bx_4) + i(-cx_3 + dx_1 - bx_2 + ax_4) \end{pmatrix} \\
&= \begin{pmatrix} -2(ax_1 - bx_3 + cx_2 - dx_4)\,(-cx_1 - dx_3 + ax_2 + bx_4) - \\ \quad - 2(ax_3 + bx_1 + cx_4 + dx_2)\,(-cx_3 + dx_1 - bx_2 + ax_4) \\ -2(ax_1 - bx_3 + cx_2 - dx_4)\,(-cx_3 + dx_1 - bx_2 + ax_4) + \\ \quad + 2(-cx_1 - dx_3 + ax_2 + bx_4)\,(ax_3 + bx_1 + cx_4 + dx_2) \\ (ax_1 - bx_3 + cx_2 - dx_4)^2 - (-cx_1 - dx_3 + ax_2 + bx_4)^2 + \\ \quad + (ax_3 + bx_1 + cx_4 + dx_2)^2 - (-cx_3 + dx_1 - bx_2 + ax_4)^2 \end{pmatrix} \\
&= \mathrm{Ad}^*_A \begin{pmatrix} -2(x_1x_2 + x_3x_4) \\ -2(x_1x_4 - x_2x_3) \\ (x_1)^2 - (x_2)^2 + (x_3)^2 - (x_4)^2 \end{pmatrix} \\
&= \mathrm{Ad}^*_A\Big(J(x_1 + ix_3,\, x_2 + ix_4)\Big) \ .
\end{aligned}
$$

So, J is an Ad^*-equivariant momentum mapping.

Finally remark that, as $\mathrm{su}(2,\mathbb{C})$ equals its derived algebra, if an Ad^*-equivariant momentum mapping exists, it is unique. ∎

APPENDIX

In what follows, M is a differentiable connected finite dimensional manifold.

I) Let A (resp. B) be a p-times (resp. q-times) contravariant skew-symmetric tensor field on M. The *Schouten bracket* ([10]) of A and B is a $(p+q-1)$-times contravariant skew-symmetric tensor field on M, denoted by $[A,B]$, such that for any closed $(p+q-1)$-form β,

$$i([A,B])\,\beta = (-1)^{(p+1)q}\, i(A)\, di(B)\,\beta + (-1)^p\, i(B)\, di(A)\,\beta \ ,$$

where i is the interior product.

Some of the properties of the Schouten bracket are:

i) If $p=1$, $[A,B] = \mathcal{L}(A)B$ is the Lie derivative of B with respect to A;

ii) $[A,B] = (-1)^{pq}[B,A]$;

iii) If C is an r-contravariant skew-symmetric tensor field,

$$S(-1)^{pq}\Big[[B,C],A\Big] = 0 \ ,$$

where S means sum after circular permutation;

iv) $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(p+1)q} B \wedge [A, C]$.

II) Let C be a two times contravariant skew-symmetric tensor field on M and E a vector field on M. For any couple (f, h) of functions on M, we set

$$\{f, h\} = C(df, dh) + f(E.h) - h(E.f)$$

and define a bilinear and skew-symmetric internal law on $C^\infty(M, \mathbb{R})$. This law satisfies the *Jacobi identity* (i.e., $S\{\{f, h\}, g\} = 0$) if and only if

$$[C, C] = 2E \wedge C \quad \text{and} \quad [E, C] = 0 \quad ([5]) \ ,$$

the bracket $[\ ,\ ]$ being the Schouten bracket. In this case, we say that $\{\ ,\ \}$ is a *Jacobi bracket* and (M, C, E) is a *Jacobi manifold*. The space $C^\infty(M, \mathbb{R})$ with a Jacobi bracket is a *local Lie algebra*. If $E = 0$, the Jacobi manifold is a *Poisson manifold*.

If (M, C, E) is a Jacobi manifold, there exists a vector bundle morphism

$$^\#(\ )\colon\ (TM)^* \to TM$$

that is given, for all α and β in the same fiber of $(TM)^*$, by

$$\langle \beta, {}^\#\alpha \rangle = C(\alpha, \beta) \ .$$

If $f \in C^\infty(M, \mathbb{R})$, we call $X_f = {}^\#df + fE$ the *hamiltonian vector field* associated with f ([7]).

Let (M, C, E) be a Jacobi manifold and $a \in C^\infty(M, \mathbb{R})$ a differentiable function that never vanishes. For all f and h elements of $C^\infty(M, \mathbb{R})$, we set

$$\{f, h\}^a = \frac{1}{a} \{af, ah\} \ .$$

The bracket $\{\ ,\ \}^a$ is a Jacobi bracket and defines on M a new Jacobi structure (C^a, E^a), with

$$C^a = a\, C \quad \text{and} \quad E^a = {}^\#da + aE \ .$$

We say that the structure (C^a, E^a) is a-conformal to (C, E). The equivalence class of all Jacobi structures on M, conformal to a given structure is called a *conformal Jacobi structure* on M.

Let (M_1, C_1, E_1) and (M_2, C_2, E_2) be two Jacobi manifolds. A differentiable mapping $\phi\colon\ M_1 \to M_2$ is called a *Jacobi morphism* if

$$\{f, h\}_{M_2} \circ \phi = \{f \circ \phi, h \circ \phi\}_{M_1} \ ,$$

for all $f, h \in C^\infty(M_2, \mathbb{R})$. We call ϕ an *a-conformal Jacobi morphism* if there exists a function $a \in C^\infty(M_1, \mathbb{R})$ that never vanishes, such that ϕ is a Jacobi morphism of (M_1, C_1^a, E_1^a) into (M_2, C_2, E_2).

A vector field X on a Jacobi manifold (M, C, E) is an *infinitesimal Jacobi automorphism* (resp. *infinitesimal conformal Jacobi automorphism*) if and only if $[X, C] = 0$ and $[X, E] = 0$ (resp. if and only if there exists a function $a \in C^\infty(M, \mathbb{R})$ such that $[X, C] = aC$ and $[X, E] = {}^\# da + aE$).

III) Let (M, C, E) be a Jacobi manifold and G a Lie group acting on the left on M, by an action ϕ. Suppose that for each $g \in G$ there exists a function $a_g \in C^\infty(M, \mathbb{R})$ that never vanishes and such that the mapping

$$\phi_g \colon x \in M \to \phi(g, x) \in M$$

is an a_g-conformal Jacobi morphism. Then the action ϕ is called a *conformal Jacobi action*. When, for all $g \in G$, the function $a_g \in C^\infty(M, \mathbb{R})$ is constant and equals 1, the action ϕ is called a *Jacobi action*. In this case, for any $g \in G$, the mapping ϕ_g is a Jacobi morphism.

Given an element X of the Lie algebra $\mathcal{G}$ of G, the *fundamental vector field* associated with X for the action ϕ ([9]), is the vector field X_M on M, such that, for all $x \in M$,

$$X_M(x) = \frac{d}{dt}\Big(\phi(\exp(-tX), x)\Big)_{|t=0} .$$

If G is a connected Lie group, the action ϕ of G on M is a *Jacobi action* (resp. *conformal Jacobi action*) if and only if for all $X \in \mathcal{G}$, the fundamental vector field X_M associated with X is an infinitesimal Jacobi automorphism (resp. infinitesimal Jacobi conformal automorphism).

IV) Let G be a finite dimensional Lie group and $\mathcal{G}$ its Lie algebra. On the dual $\mathcal{G}^*$ of $\mathcal{G}$ we can define a Poisson structure, called the *Lie–Poisson structure* ([6]), by setting for all $f, h \in C^\infty(\mathcal{G}^*, \mathbb{R})$ and $\xi \in \mathcal{G}^*$,

$$\{f, h\}(\xi) = \Big\langle \xi, \Big[df(\xi), dh(\xi)\Big]\Big\rangle ,$$

with $[\ ,\ ]$ the Lie bracket on $\mathcal{G}$, $\langle\ ,\ \rangle$ the duality product of $\mathcal{G}$ and $\mathcal{G}^*$ and where we identify the elements of $\mathcal{G}$ with linear mappings of $\mathcal{G}^*$ into $\mathbb{R}$.

If Z is the *Liouville vector field* on $\mathcal{G}^*$ and Λ is the Lie–Poisson tensor field on $\mathcal{G}^*$, one can show ([6]) that

$$[\Lambda, Z] = -\Lambda ,$$

i.e., $(\mathcal{G}^*, \Lambda, Z)$ is an *homogeneous* Lie–Poisson structure.

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