PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 103(117) (2018), 223–236 DOI: https://doi.org/10.2298/PIM1817223T

ON BICONSERVATIVE HYPERSURFACES IN PSEUDO-RIEMANNIAN SPACE FORMS AND THEIR GAUSS MAP

Nurettin Cenk Turgay

Abstract. We first present a survey about recent results on biconservative hypersurfaces in the Minkowski space $\mathbb{E}^4_1$, pseudo-Euclidean space $\mathbb{E}^5_2$ and Rieamnnian space-form $\mathbb{H}^4$. Then we obtain some geometrical properties of these hypersurface families concerning their mean curvature and Gauss map.

1. Introduction

Let $\phi : M \to N$ be a smooth mapping between two Riemannian manifolds (M^n, g), $(N^m, \tilde{g})$ and suppose that its tension field is denoted by $\tau(\phi)$. Then, ϕ is said to be biharmonic if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g.$$

In [**9**], Jiang proved that ϕ is biharmonic if and only if it satisfies the system of fourth order elliptic partial differential equations given by $\tau_2(\phi) = 0$, where $\tau_2(\phi) = \Delta\tau(\phi) - tr\tilde{R}(d\phi, \tau(\phi))d\phi$ is the bitension field. If $\phi : M \to N$ is a biharmonic isometric immersion, then M is said to be a biharmonic submanifold of N.

As a generalization of biharmonic submanifolds, the following definition was given.

Definition 1.1. A submanifold M of N is said to be biconservative if the isometric immersion $\phi : M \to N$ satisfies $\tau_2(\phi)^\top = 0$, where $\tau_2(\phi)^\top$ denotes the tangential part of $\tau_2(\phi)$.

Before we proceed, we state the following proposition (see [**2, 8, 9**]).

Proposition 1.1. *Let $\phi : M \to N$ be an isometric immersion between Riemannian manifolds. Then the following statements are equivalent.*

(1) *ϕ (or M) is biconservative;*
(2) *The stress-energy tensor S_2 of ϕ satisfies* $\operatorname{div} S_2 = 0$*;*

2010 *Mathematics Subject Classification:* Primary 53C40; Secondary 53C42, 53C50.
Key words and phrases: biharmonic submanifolds, Lorentzian hypersurfaces, biconservative hypersurfaces, finite type submanifolds.

(3) *The equation*

$$m\nabla\|H\|^2 + 4\,\mathrm{trace}\,A_{\nabla^\perp_\cdot H}(\cdot) + 4\,\mathrm{trace}\left(\tilde{R}(\cdot, H)\,\cdot\,\right)^T = 0$$

is satisfied, where H, $\nabla^\perp$ and A are the mean curvature vector, normal connection and the shape operator of M, respectively.

In this paper, we give a short survey of biconservative surfaces on pseudo-Riemannian space-forms $\mathbb{E}^4_1$, $\mathbb{E}^5_2$ and $\mathbb{H}^4$. Then, we study some of geometrical properties of these hypersurface families considering their Gauss map and mean curvature. The organization of this paper is as following. In Section 2, we first describe some of the basic facts. In Section 3, we present explicit parametrization of biconservative hypersurfaces obtained in [**7, 13, 14**]. In Section 4, we obtain some new results of biconservative hypersurfaces in Minkowski spaces. In Section 5, we get some classification results considering Gauss map of hypersurface families presented in Section 3.

2. Preliminaries

Let $\mathbb{E}^m_t$ denote the semi-Euclidean m-space with the canonical semi-Euclidean metric tensor of index t given by

$$\tilde{g} = \langle , \rangle = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2.$$

Pseudo-Riemannian space-forms are defined by

$$\mathbb{S}^n_t(r^2) = \{x \in \mathbb{E}^{n+1}_t : \langle x, x\rangle = r^{-2}\},$$
$$\mathbb{H}^n_{t-1}(-r^2) = \{x \in \mathbb{E}^{n+1}_t : \langle x, x\rangle = -r^{-2}\}.$$

For the particular case $t = 1$ and $r = 1$, we put $\mathbb{H}^n_0(-1) = \mathbb{H}^n$ which is called the anti-de Sitter space-time when $n = 4$.

Consider an n-dimensional semi-Riemannian submanifold M of the pseudo-Euclidean space $\mathbb{E}^m_t$. We denote the Levi-Civita connections of $\mathbb{E}^m_t$ and M by $\widetilde{\nabla}$ and ∇, respectively. Note that the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\widetilde{\nabla}_X \zeta = -A_\zeta(X) + \nabla^\perp_X \zeta,$$

respectively, for all tangent vectors fields X, Y and normal vector fields ζ, where h, $\nabla^\perp$ and A denote the second fundamental form, the normal connection and the shape operator of M, respectively. The Gauss and Codazzi equations are given, respectively, by

$$R(X, Y)Z = A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{2.1}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.2}$$

where R is the curvature tensor associated with connection ∇ and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Gauss map of hypersurfaces in semi-Euclidean spaces. Consider an oriented hypersurface M of a (semi-)Euclidean space and let N be its Gauss map. By definition, M is said to have pointwise 1-type Gauss map if the Laplacian of its Gauss map takes the form

$$\Delta N = \psi(N + C) \tag{2.3}$$

for a smooth function ψ and constant vector C. More precisely, a pointwise 1-type Gauss map is called *of the first kind* if (2.3) is satisfied for $C = 0$, and *of the second kind* if $C \neq 0$. Moreover, if (2.3) is satisfied for a nonconstant function ψ, then M is said to have *proper* pointwise 1-type Gauss map. Otherwise, G is said to be (global) 1-type, [**3, 4, 6**].

3. Recent classifications of biconservative hypersurfaces

In this section, we would like to present some recent results on biconservative hypersurfaces.

3.1. Biconservative hypersurfaces in the Minkowski space $\mathbb{E}^4_1$. In [**7**], the author and Yu Fu considered biconservative hypersurfaces in the Minkowski 4-space with diagonalizable shape operator. They obtained the following results.

Proposition 3.1. [**7**] *Let M be a hypersurface in $\mathbb{E}^4_1$ given by*

$$\begin{aligned} x(s,t,u) = \bigg(&\frac{1}{2}s(t^2+u^2) + au^2 + s + \phi(s), st, (s+2a)u, \\ &\frac{1}{2}s(t^2+u^2) + au^2 + \phi(s)\bigg), \quad a \neq 0. \end{aligned} \tag{3.1}$$

Then, M is biconservative if and only if either M is Riemannian and

$$\phi(s) = c_1\left(\ln(s+2a) - \ln s - \frac{a}{s} - \frac{a}{s+2a}\right) - \frac{s}{2}$$

or it is Lorentzian and

$$\phi(s) = c_1 \int_{s_0}^{s} \left(\xi(\xi+2a)\right)^{2/3} d\xi - \frac{s}{2},$$

where $c_1 \neq 0$ and s_0 are some constants.

Theorem 3.1. [**7**] *Let M be a hypersurface in $\mathbb{E}^4_1$ with diagonalizable shape operator and three distinct principal curvatures. Then M is biconservative if and only if it is congruent to one of hypersurfaces*

(1) *A generalized cylinder $M^2_0 \times \mathbb{E}^1_1$ where M is a biconservative surface in $\mathbb{E}^3$;*
(2) *A generalized cylinder $M^2_0 \times \mathbb{E}^1$ where M is a biconservative Riemannian surface in $\mathbb{E}^3_1$;*
(3) *A generalized cylinder $M^2_1 \times \mathbb{E}^1$, where M is a biconservative Lorentzian surface in $\mathbb{E}^3_1$;*

(4) *A Rimennian surface given by*

$$x(s,t,u) = (s\ \cosh t, s\ \sinh t, f_1(s)\cos u, f_1(s)\sin u) \tag{3.2}$$

for a function f_1 *satisfying*

$$\frac{f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$

(5) *A Lorentzian surface with the parametrization given in* (3.2) *for a function* f_1 *satisfying*

$$\frac{-3 f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$

(6) *A Rimennian surface given by*

$$x(s,t,u) = (s\ \sinh t, s\ \cosh t, f_2(s)\cos u, f_2(s)\sin u) \tag{3.3}$$

for a function f_2 *satisfying*

$$\frac{f_2''}{f_2'^2 + 1} = \frac{f_2 f_2' + s}{s f_2};$$

(7) *A surface given in Proposition 3.1.*

Recently, Kumari studied biconservative hypersurfaces with nondiagonalizable shape operator and obtain the following result, [**10**].

Theorem 3.2. [**10**] *Let* M_1^n *in* $\mathbb{E}_1^{n+1}$ *be a biconservative Lorentz hypersurface having nondiagonalizable shape operator with complex eigenvalues and with at most five distinct principal curvatures. Then* M_1^n *has constant mean curvature.*

3.2. Biconservative hypersurfaces in the pseudo-Euclidean space $\mathbb{E}_2^5$. In [**14**], the author and Upadhyay studied biconservative hypersurfaces with index 2 in the pseudo-Euclidean space $\mathbb{E}_2^5$. They obtained the following result.

Theorem 3.3. [**14**] *Let* M *be an oriented hypersurface of index* 2 *in the pseudo-Euclidean space* $\mathbb{E}_2^5$*. Assume that its shape operator has the form*

$$S = \operatorname{diag}(k_1, k_2, k_2, k_4), \quad k_4 \neq k_2$$

for some nonvanishing smooth functions k_1, k_2, k_4*. Then, it is congruent to one of the following eight type of hypersurfaces for some smooth functions* $\phi_1 = \phi_1(s)$ *and* $\phi_2 = \phi_2(s)$*.*

(1) $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \cosh t, \phi_1 \sinh t \cos u, \phi_1 \sinh t \sin u, \phi_2 \cosh v)$, $\phi_1'^2 - \phi_2'^2 = 1$;
(2) $x(s,t,u,v) = (\phi_2 \cos v, \phi_2 \sin v, \phi_1 \cos t, \phi_1 \sin t \cos u, \phi_1 \sin t \sin u)$, $\phi_1'^2 - \phi_2'^2 = -1$;
(3) $x(s,t,u,v) = (\phi_1 \cosh t \sin u, \phi_1 \cosh t \cos u, \phi_1 \sinh t, \phi_2 \cos v, \phi_2 \sin v)$, $\phi_1'^2 - \phi_2'^2 = 1$;
(4) $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \cosh v)$, $\phi_1'^2 + \phi_2'^2 = 1$;
(5) $x(s,t,u,v) = (\phi_2 \cosh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \sinh v)$, $\phi_1'^2 - \phi_2'^2 = -1$;

(6) $x(s,t,u,v) = (\phi_1 \sinh t cosu, \phi_1 \sinh t \sin u, \phi_1 \cosh u, \phi_2 \cos v, \phi_2 \sin v)$, $\phi_1'^2 + \phi_2'^2 = 1$;

(7) *A hypersurface given by*

$$x(s,t,u,v) = \Big(\frac{s}{2}(t^2+u^2-v^2) - av^2 + \psi, v(2a+s), st, su,$$
$$\frac{s}{2}(t^2+u^2-v^2) - av^2 + \psi - s\Big)$$

for a nonzero constants a and a smooth function $\psi = \psi(s)$ such that $1 - 2\psi' < 0$*;*

(8) *A hypersurface given by*

$$x(s,t,u,v) = \Big(\frac{s(t^2-u^2-v^2)}{2} + av^2 + \psi, st, su, v(s-2a),$$
$$\frac{s(t^2-u^2-v^2)}{2} + av^2 + \psi + s\Big)$$

for a nonzero constants a and a smooth function $\psi = \psi(\tilde{s})$ such that $1 + 2\psi' < 0$.

3.3. Biconservative hypersurfaces in the Riemannian space-form $\mathbb{H}^4$. In this subsection, we want to announce the biconservative hypersurfaces in $\mathbb{H}^4$ that was recently obtained in a joint work with Upadhyay in [**13**].

If M is a hypersurface in a 4-dimensional Riemannian space-form, then it is biconservative if and only if the equation

$$S(\nabla H) = -2H\nabla H \tag{3.4}$$

is satisfied, where S is the shape operator of M.

Example 3.1. Consider the hypersurface in $\mathbb{H}^4$ given by

$$x(s,t,u) = \Big(\frac{aA(s)^2 + a}{s} + asu^2 + \frac{s}{4a}, su, A(s)\cos t, A(s)\sin t, \tag{3.5}$$
$$\frac{aA(s)^2 + a}{s} + asu^2 - \frac{s}{4a}\Big)$$

for a smooth nonvanishing function A. We would like to note that if A is chosen properly then the hypersurface given by (3.5) satisfies (3.4).

Example 3.2. Consider the hypersurface in $\mathbb{H}^4$ given by

$$x(s,t,u) = \Big(\frac{aA(s)^2}{s} + as(t^2+u^2) + \frac{s}{4a} + \frac{a}{s}, st, su, A(s), \tag{3.6}$$
$$\frac{aA(s)^2}{s} + as(t^2+u^2) - \frac{s}{4a} + \frac{a}{s}\Big)$$

for a smooth nonvanishing function A. A direct computation yields that M has two distinct principle curvatures. Furthermore, if A is chosen properly then the hypersurface given by (3.6) satisfies (3.4).

4. Biconservative hypersurfaces in Minkowski spaces

In this section, we get new classifications of biconservative hypersurfaces in Minkowski spaces. Note that a hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ is biconservative if and only if the differential equation

$$S(\nabla H) = \frac{-\varepsilon n H}{2} \nabla H \tag{4.1}$$

is satisfied, where $\varepsilon = \langle N, N \rangle$.

4.1. A classification of biconservative hypersurfaces with more than 3 distinct principle curvatures. In this subsection, we construct an example of biconservative hypersurfaces with arbitrary number of distinct principle curvatures.

We put by $\vec{a_1} \cdot \vec{a_2} = \sum_i a_{1i} a_{2i}$, where $a_j = (a_{j1}, a_{j2}, \ldots, a_{j(n-1)})$. Consider a hypersurface M in $\mathbb{E}^{n+1}_1$ given by

$$\begin{aligned} x(s, \vec{t}) = \Big(& \frac{1}{2} s \vec{t} \cdot \vec{t} + \vec{a} \cdot \vec{t} + s + \phi(s), s t_1, (s + 2a_2) t_2, \ldots, \\ & \frac{1}{2} s \vec{t} \cdot \vec{t} + \vec{a} \cdot \vec{t} + \phi(s) \Big) \end{aligned} \tag{4.2}$$

for a smooth function ϕ, where $\vec{t} = (t_1, t_2, \ldots, t_{n-1})$, $\vec{a} = (a_1, a_2, a_3, \ldots, a_{n-1})$ for some constants $a_1 = 0, a_2, \ldots, a_n$. Note that

$$e_1 = \frac{1}{\sqrt{-\epsilon(2\phi'(s) + 1)}} \partial_s, \quad e_l = \frac{1}{s + a_{l-1}} \partial_{t_{l-1}}, \quad l = 2, 3, \ldots, n$$

form an orthonormal frame field and the normal vector field of M is

$$N = \frac{1}{\sqrt{-\epsilon(2\phi'(s) + 1)}} \left(\vec{t} \cdot \vec{t} - 2\phi'(s), s t_1, (s + 2a_2) t_2, \ldots, \vec{t} \cdot \vec{t} - 2\phi'(s) - 2 \right),$$

where we put $\varepsilon = \langle e_1, e_1 \rangle$.

A direct computation yields that the matrix representation of the shape operator of M is

$$S = \begin{pmatrix} k_1 & 0 & \ldots & 0 \\ 0 & k_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & k_n \end{pmatrix},$$

where

$$\begin{aligned} k_1 &= -\frac{\epsilon \phi''(s)}{(-\epsilon(2\phi'(s) + 1))^{3/2}}, \\ k_l &= -\frac{1}{(2a_{l-1} + s)\sqrt{-\epsilon(2\phi'(s) + 1)}}, \quad l = 2, 3, \ldots, n. \end{aligned} \tag{4.3}$$

Note that the tangent vector fields ∂_s is proportional to ∇H and it is a principle direction of M. Therefore, M is biconservative if and only if $-2\varepsilon k_1 = nH = k_1 + k_2 + \cdots + k_n$ because of (4.1). Hence, M is biconservative if and only if either $\epsilon = 1$ and $k_1 = k_2 + k_3 + \cdots + k_n$ or $\epsilon = -1$ and $-3k_1 = k_2 + k_3 + \cdots + k_n$. By considering (4.3), we obtain the following result.

THEOREM 4.1. *Let M be a hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ given by* (4.2) *for a smooth function ϕ. Then, M is biconservative if and only if either M is Riemannian and ϕ is the function given by*

$$\phi(s) = \int_{s_0}^{s} \frac{c_1}{\xi^2(2a_2+\xi)^2\dots(2a_{n-1}+\xi)^2} d\xi - \frac{s}{2}$$

for some constants $c_1 \neq 0, s_0$ or M is Lorentzian and ϕ is the function given by

$$\phi(s) = \int_{s_0}^{s} c_2\xi^{2/3}(2a_2+\xi)^{2/3}\dots(2a_{n-1}+\xi)^{2/3} d\xi - \frac{s}{2}$$

for a constant $c_2 \neq 0, s_0$.

4.2. A characterization of biconservative hypersurfaces with nondiagonalizable shape operator. In this subsection, we give a characterization of biconservative hypersurfaces in $\mathbb{E}^{n+1}$ with nondiagonalizable shape operator S. We focus on the case of having minimal polynomial of $P(\lambda) = (\lambda - k_1)(\lambda - k_2)^2$ of S for some smooth function k_1, k_2.

We would like to note that in this case M is Lorentzian. Thus, (4.1) becomes

$$S(\nabla H) = \frac{-3H}{2}\nabla H. \tag{4.4}$$

THEOREM 4.2. *Let M be a hypersurface in the Minkowski space $\mathbb{E}^4_1$ and S its shape operator with nonconstant mean curvature. Assume that the minimal polynomial of S is $P(\lambda) = (\lambda - k_1)(\lambda - k_2)^2$ for some smooth function k_1, k_2. Then, M is biconservative if and only if there exists a frame field $\{e_1, e_2, e_3\}$ on M with $\langle e_1, e_1\rangle = 1$, $\langle e_a, e_b\rangle = \delta_{ab} - 1$, $\langle e_1, e_a\rangle = 0$, $a, b = 2, 3$ satisfying the following conditions.*

(1) *The Levi-Civita connection of M is*

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_1}e_2 = \frac{3(25AH^2 + 4e_1(H))}{40H}e_2,\\
&\nabla_{e_2}e_1 = -\frac{3e_1(H)}{5H}e_2, && \nabla_{e_2}e_2 = 0,\\
&\nabla_{e_3}e_1 = -Ae_2 - \frac{3e_1(H)}{5H}e_3, && \nabla_{e_3}e_2 = -Be_2 - \frac{3e_1(H)}{5H}e_1,\\
&\nabla_{e_1}e_3 = -\frac{3(25AH^2 + 4e_1(H))}{40H}e_3, && \nabla_{e_2}e_3 = -\frac{3e_1(H)}{5H}e_1,\\
&\nabla_{e_3}e_3 = -Ae_1 + Be_3,
\end{aligned} \tag{4.5}$$

for some functions A, B and H satisfying

$$\begin{aligned}
e_2(H) = e_3(H) = e_2(A) = 0,&\\
40He_1e_1(H) - 64e_1(H)^2 + 225H^4 = 0,&\\
\frac{9e_1(H)^2}{25H^2} + \frac{81H^2}{16} = e_2(B),&\\
\frac{3e_1(H)}{5H}A - \frac{3}{4}H(5A^2+2) = e_1(A),&
\end{aligned} \tag{4.6}$$

$$-75H^2AB + 12e_1(H)B = 40He_1(B) + 75H^2e_3(A).$$

(2) *The shape operator of M has the matrix representation*

$$S = \begin{pmatrix} -\frac{3H}{2} & 0 & 0 \\ 0 & \frac{9H}{4} & 1 \\ 0 & 0 & \frac{9H}{4} \end{pmatrix} \tag{4.7}$$

Proof. In order to prove the neessary condition, we assume that M is a biconservative hypersurface in $\mathbb{E}^{n+1}_1$ with the minimal polynomial $P(\lambda) = (\lambda - k_1)(\lambda - k_2)^2$ for some smooth function k_1, k_2. Furthermore, the results obtained in [**12**] yields that ∇H can not be light-like. Therefore, (4.1) implies that $e_1 = \frac{\nabla H}{\langle \nabla H, \nabla H\rangle^{1/2}}$ is an eigenvector of S with corresponding eigenvalue $k_1 = -3H/2$, where H is the mean curvature of M and we have $X(H) = 0$ whenever $\langle X, e_1\rangle = 0$. Moreover, because of (4.4), if a frame field $\{e_1, e_2, e_3\}$ with $\langle e_1, e_1\rangle = 1$, $\langle e_a, e_b\rangle = \delta_{ab} - 1$, $\langle e_1, e_a\rangle = 0$, $a, b = 2, 3$ is chosen properly, then the matrix representation of S becomes as given in (4.7) (See [**11**]), where H is the mean curvature of M. Note that we have $e_2(H) = e_3(H) = 0$.

The Levi-Civita connection ∇ of M satisfies

$$\begin{aligned} \nabla_{e_i} e_1 &= -\omega_{13}(e_i)e_2 - \omega_{12}(e_i)e_3, \\ \nabla_{e_i} e_2 &= -\omega_{12}(e_i)e_1 - \omega_{23}(e_i)e_2, \\ \nabla_{e_i} e_3 &= -\omega_{13}(e_i)e_1 + \omega_{23}(e_i)e_3, \end{aligned} \tag{4.8}$$

where we put $\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle$.

We apply the Codazzi equation (2.2) for $X = e_i$, $Y = e_j$, $Z = e_k$ for $X = e_i$, $Y = e_j$, $Z = e_k$ for different choices of (i, j, k) and combine equations obtained with (4.8) to get

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = 0, \quad \omega_{23}(e_2) = \omega_{12}(e_2) = 0, \\ &\omega_{12}(e_3) = \frac{3e_1(H)}{5H}, \qquad \omega_{23}(e_1) = -\frac{3\left(25H^2\omega_{13}(e_3) + 4e_1(H)\right)}{40H} \end{aligned} \tag{4.9}$$

By combining (4.9) with (4.8), we obtain (4.5) for $A = \omega_{13}(e_3)$ and $B = \omega_{23}(e_3)$.

Next, by taking into account (4.5), we use Gauss equation (2.1) for $X = e_i$, $Y = e_j$, $Z = e_k$ for different triplets of (i, j, k) to get (4.6). □

Remark 4.1. We would like to note that obtaining hypersurfaces given in Theorem 4.2 is still an open problem. However, it was proved in [**1**] that there is no biharmonic hypersurface in the Minkowski space-time $\mathbb{E}^4_1$ with nondiagonalizable shape operator.

5. Gauss map of hypersurfaces in $\mathbb{E}^4_1$ and $\mathbb{H}^4$

In this section, we consider some of hypersurface families mentioned in Section 3. We get some classification results considering their Gauss map and obtain hypersurfaces whose Gauss map N satisfies (2.3).

5.1. Gauss map of hypersurfaces in the Minkowski space $\mathbb{E}^4_1$. In [**5**], Dursun studied rotational hypersurfaces with pointwise 1-type Gauss map in a Minkowski space with arbitrary dimension. In this subsection, we firstly obtain the following theorem by considering Gauss map of hypersurfaces in $\mathbb{E}^4_1$ given by (3.2).

THEOREM 5.1. *Let M be a hypersurface in $\mathbb{E}^4_1$ given by* (3.2) *for a smooth function f_1. Then, M has pointwise 1-type Gauss map if and only if it has constant mean curvature.*

PROOF. Let M be a hypersurface in $\mathbb{E}^4_1$ given by (3.2). We put $f = f_1$. Note that tangent vector fields given by

$$e_1 = \frac{1}{\sqrt{\varepsilon\left(f'(s)^2 - 1\right)}}\partial_s, \quad e_2 = \frac{1}{s}\partial_t, \quad e_3 = \frac{1}{f(s)}\partial_u$$

form an orthonormal frame field for the tangent space of M and the unit normal vector field of M is

$$N = \frac{1}{\sqrt{\varepsilon(f'(s)^2 - 1)}}\left(\cosh t f'(s), \sinh t f'(s), \cos u, \sin u\right), \tag{5.1}$$

where we put $\varepsilon = \langle e_1, e_1\rangle$. By a direct computation, we obtain

$$\begin{aligned}
&\nabla_{e_1} e_i = 0, \quad i = 1, 2, 3,\\
&\nabla_{e_2} e_1 = Ae_2, \qquad \nabla_{e_2} e_2 = -\varepsilon A e_1, \quad \nabla_{e_2} e_3 = 0,\\
&\nabla_{e_3} e_1 = Be_3, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = -Be_1
\end{aligned} \tag{5.2}$$

for smooth functions $A = \dfrac{1}{s\sqrt{\varepsilon\left(f'(s)^2 - 1\right)}}$ and $B = \dfrac{f'(s)}{f(s)\sqrt{\varepsilon\left(f'(s)^2 - 1\right)}}$. Furthermore, the shape operator of M becomes

$$S = \begin{pmatrix} \frac{\varepsilon f''(s)}{(\varepsilon(f'(s)^2-1))^{3/2}} & 0 & 0\\ 0 & -\frac{f'(s)}{s\sqrt{\varepsilon(f'(s)^2-1)}} & 0\\ 0 & 0 & -\frac{1}{f(s)\sqrt{\varepsilon(f'(s)^2-1)}} \end{pmatrix} \tag{5.3}$$

By considering (5.1), (5.2) and (5.3), we obtain

$$\Delta N = e_1(k_1 + k_2 + k_3)e_1 + (k_1^2 + k_2^2 + k_3^2)N, \tag{5.4}$$

where k_1, k_2, k_3 are principle curvatures of M given in (5.3). Note that the mean curvature of M is $H = \frac{1}{3}(k_1 + k_2 + k_3)$.

Now, in order to prove the necessary condition, we assume that (2.3) is satisfied for a constant vector C and a smooth function ψ. Then, by considering (5.4), we obtain

$$3e_1(H)e_1 + (k_1^2 + k_2^2 + k_3^2)N = \psi(N + C) \tag{5.5}$$

which yields $\langle e_a, C\rangle = 0,\ a = 2, 3$. By applying e_2 and e_3 to this equation, we get $\langle\widetilde{\nabla}_{e_2} e_2, C\rangle = \langle\widetilde{\nabla}_{e_3} e_3, C\rangle = 0$. By considering (5.2), we obtain $C = 0$. Therefore, (5.5) becomes $3e_1(H)e_1 + (k_1^2 + k_2^2 + k_3^2)N = \psi N$ which implies $e_1(H) = 0$. Hence, we obtain that H is constant.

Proof of the sufficient condition follows from (5.4). $\square$

By a similar way, we also obtain

THEOREM 5.2. *Let M be a hypersurface in $\mathbb{E}^4_1$ given by* (3.3) *for a smooth function f_1. Then, M has pointwise 1-type Gauss map if and only if it has constant mean curvature.*

PROOF. Let M be a hypersurface given by (3.3). Then, similar to hypersurfaces given by (3.2), we have

$$\begin{aligned} &\nabla_{e_1} e_i = 0, \qquad i = 1,2,3,\\ &\nabla_{e_2} e_1 = A e_2, \quad \nabla_{e_2} e_2 = \varepsilon A e_1, \quad \nabla_{e_2} e_3 = 0,\\ &\nabla_{e_3} e_1 = B e_3, \quad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = -B e_1 \end{aligned}$$

for smooth functions

$$A = \frac{1}{s\sqrt{f'(s)^2+1}}, \quad B = \frac{f'(s)}{f(s)\sqrt{f'(s)^2+1}},$$

where

$$e_1 = \frac{1}{\sqrt{f'(s)^2+1}}\partial_s, \quad e_2 = \frac{1}{s}\partial_t, \quad e_3 = \frac{1}{f(s)}\partial_u.$$

Furthermore, the shape operator of M becomes

$$S = \begin{pmatrix} -\frac{f''(s)}{(f'(s)^2+1)^{3/2}} & 0 & 0 \\ 0 & -\frac{f'(s)}{s\sqrt{f'(s)^2+1}} & 0 \\ 0 & 0 & \frac{1}{f(s)\sqrt{f'(s)^2+1}}. \end{pmatrix} \tag{5.6}$$

By a further computation, we see that (5.4) is satisfied for k_1, k_2, k_3 given in (5.6).

In order to prove the necessary condition, we use exactly the same way that we did in the proof of Theorem 5.1 and we obtain that if (5.4) is satisfied, then C must be zero. Furthermore, similar to Theorem 5.1, the proof of the sufficient condition follows from (5.4). □

REMARK 5.1. The author would like to announce that he has recently obtained analogous results for the hypersurface families given in cases (1)–(6) of Theorem 3.3.

Now, we want to consider Gauss map of hypersurface family given by (3.1).

Let M be a hypersurface given by (3.1) for a constant a and a smooth function ϕ. We consider the local orthonormal frame field $\{e_1, e_2, e_3\}$ of the tangent bundle of M such that

$$e_1 = \frac{1}{\sqrt{-\varepsilon(2\phi'(s)+1)}}\partial_s, \quad e_2 = \frac{1}{s}\partial_t, \quad e_3 = \frac{1}{2a+s}\partial_u$$

and the unit normal vector field of M is

$$N = \frac{1}{\sqrt{-\varepsilon(2\phi'(s)+1)}}\left(\frac{t^2+u^2}{2} - \phi'(s), t, u, \frac{t^2+u^2}{2} - \phi'(s) - 1\right),$$

where we put $\varepsilon = \langle e_1, e_1 \rangle = \pm 1$. We want to note the equation

$$(1,0,0,1) = -\frac{\varepsilon}{\sqrt{-\varepsilon\,(2\phi'(s)+1)}}(e_1 - N). \tag{5.7}$$

By a direct computation, we obtain that the Levi-Civita connection satisfies (5.2) for some smooth functions

$$A = \frac{1}{s\sqrt{-\varepsilon\,(2\phi'(s)+1)}}, \quad B = \frac{1}{(2a+s)\sqrt{-\varepsilon\,(2\phi'(s)+1)}}.$$

Furthermore, the shape operator of M becomes

$$S = \begin{pmatrix} -\frac{\varepsilon\phi''(s)}{(-\varepsilon(2\phi'(s)+1))^{3/2}} & 0 & 0 \\ 0 & -\frac{1}{s\sqrt{-\varepsilon(2\phi'(s)+1)}} & 0 \\ 0 & 0 & -\frac{1}{(2a+s)\sqrt{-\varepsilon(2\phi'(s)+1)}} \end{pmatrix}$$

We also put

$$s_1 = \operatorname{trace} S \quad \text{and} \quad s_2 = \operatorname{trace} S^2. \tag{5.8}$$

By a further computation, we have

$$\Delta N = e_1(s_1)e_1 + s_2 N \tag{5.9}$$

(see [**5**, Lemma 3.2]). We first obtain the following lemma.

Lemma 5.1. *Let M be a hypersurface given by* (3.1) *with pointwise 1-type Gauss map. Then,* (2.3) *is satisfied for the function*

$$\psi = \psi(s) = e_1(s_1) + s_2 \tag{5.10}$$

and a constant vector with the form

$$C = C_1(s)(e_1 - N), \tag{5.11}$$

where C_1 is an appropriately chosen function and s_1, s_2 are functions given by (5.8).

Proof. If M has pointwise 1-type Gauss map, then (2.3) is satisfied for some C, ψ. From (2.3) and (5.9), we have

$$e_1(s_1)e_1 + s_2 N = \psi(N + C) \tag{5.12}$$

from which we obtain

$$C = C_1 e_1 + C_4 N. \tag{5.13}$$

Since C is constant, we have $e_a(C) = 0,\ a = 2, 3$. By combining this equation with (5.2) and (5.13), we obtain

$$e_2(C_1)e_1 + e_2(C_4)N + \frac{C_1 + C_4}{s\sqrt{-\varepsilon(2\phi'(s)+1)}}e_2 = 0,$$
$$e_3(C_1)e_1 + e_3(C_4)N + \frac{C_1 + C_4}{(2a+s)\sqrt{-\varepsilon(2\phi'(s)+1)}}e_3 = 0$$

which yields $C_1 + C_4$=0 and $e_2(C_1) = e_3(C_1) = 0$. Therefore, we obtained (5.11). Now, (5.12) becomes $e_1(s_1)e_1 + s_2 N = \psi(N + C_1(s)(e_1 - N))$ from which we have

$e_1(s_1) = \psi C_1(s)$ and $s_2 = \psi(1 - C_1(s))$. By combining these equations, we get (5.10). $\square$

THEOREM 5.3. *Let M be a hypersurface given by* (3.1). *Then M has pointwise 1-type Gauss map if and only if the differential equation*

$$s_1' = c(e_1(s_1) + s_2) \tag{5.14}$$

is satisfied for a constant c.

PROOF. Let M has pointwise 1-type Gauss map. Then, (2.3) is satisfied for the constant vector C and smooth function ψ given in Lemma 5.1. Since C is proportional to $e_1 - N$, (5.7) implies $C = (-\varepsilon c, 0, 0, -\varepsilon c)$ for a constant c. Thus, (5.7) implies

$$C = \frac{c}{\sqrt{-\varepsilon\,(2\phi'(s) + 1)}}(e_1 - N).$$

Therefore, we have $C_1 = \frac{c}{\sqrt{-\varepsilon(2\phi'(s)+1)}}$. Hence, $e_1(s_1) = \psi C_1(s)$ becomes

$$e_1(s_1) = \frac{c(e_1(s_1) + s_2)}{\sqrt{-\varepsilon\,(2\phi'(s) + 1)}}$$

which gives (5.14).

The proof of the sufficient condition follows from a direct computation. $\square$

REMARK 5.2. We announce that he have recently generalized the result obtained in the previous theorem by considering hypersurfaces given by (4.2) with pointwise 1-type Gauss map.

5.2. Hyperbolic Gauss map of hypersurfaces in $\mathbb{H}^4$. In this subsection, we consider a family of hypersurfaces in $\mathbb{H}^4$ considering their mean curvature and hyperbolic Gauss map N. Let M be a hypersurface in $\mathbb{H}^4$ given by (3.5) for a smooth function A. We consider the local orthonormal frame field $\{e_1, e_2, e_3\}$ of the tangent bundle of M such that

$$e_1 = \frac{s}{\sqrt{(A(s) - sA'(s))^2 + 1}}\partial_s, \quad e_2 = \frac{1}{A(s)}\partial_t, \quad e_3 = \frac{1}{s}\partial_u.$$

Note that we have

$$\begin{aligned} N = \frac{1}{\sqrt{(A - sA')^2 + 1}}\Big(&2aAA' - \frac{a}{s}(A^2 + 1) + asu^2 + \frac{s}{4a}, su, sA'\cos t, \\ &sA'\sin t, 2aAA' - \frac{a}{s}(A^2 + 1) + asu^2 - \frac{s}{4a}\Big). \end{aligned}$$

By a direct computation, we obtain that the principle directions of M are

$$\begin{aligned} k_1 &= \frac{A(3s^2A'^2 + 1) - 3sA^2A' - s(sA'' + s^2A'^3 + A') + A^3}{((A - sA')^2 + 1)^{3/2}}, \\ k_2 &= \frac{-sAA' + A^2 + 1}{A\sqrt{(A - sA')^2 + 1}}, \quad k_3 = \frac{A - sA'}{\sqrt{(A - sA')^2 + 1}}. \end{aligned} \tag{5.15}$$

Furthermore the Laplacian of the Gauss map N is

$$\Delta N = e_1(k_1 + k_2 + k_3)e_1 + (k_1^2 + k_2^2 + k_3^2)N + (k_1 + k_2 + k_3)x. \tag{5.16}$$

By combining (5.15) and (5.16), we obtain the following theorem.

THEOREM 5.4. *Let M be a hypersurface in $\mathbb{H}^4$ given by* (3.5) *for a smooth function A. Then, the following statements are equivalent to each other*

(1) *M has pointwise* 1*-type hyperbolic Gauss map of the first kind, i.e., N satisfies* (2.3) *for $C = 0$;*
(2) *M is minimal;*
(3) *$A = A(s)$ satisfies*

$$-s^2AA'' - 3s^3AA'^3 + \left(9s^2A^2 + s^2\right)A'^2 + \left(-9sA^3 - 5sA\right)A' + 3A^4 + 4A^2 + 1 = 0.$$

Now, let M be a hypersurface given in (3.6) for a smooth function A. Note that the principle curvatures of M are

$$\begin{aligned} k_1 &= \frac{A(3s^2A'^2 + 1) - 3sA^2A' - s(sA'' + s^2A'^3 + A') + A^3}{((A - sA')^2 + 1)^{3/2}}, \\ k_2 = k_3 &= \frac{A - sA'}{\sqrt{(A - sA')^2 + 1}}. \end{aligned} \tag{5.17}$$

Also the Laplacian of the Gauss map N takes the form given in (5.16).

By combining (5.16) and (5.17), we obtain the following theorem.

THEOREM 5.5. *Let M be a hypersurface in $\mathbb{H}^4$ given by* (3.6) *for a smooth function A. Then the following statements are equivalent to each other*

(1) *M has pointwise* 1*-type hyperbolic Gauss map of the first kind, i.e., N satisfies* (2.3) *for $C = 0$;*
(2) *M is minimal;*
(3) *$A = A(s)$ satisfies*

$$-s^2A'' - 3s^3A'^3 + 9s^2AA'^2 + \left(-9sA^2 - 3s\right)A' + 3A^3 + 3A = 0.$$

Acknowledgements. This work was presented at the XIX Geometrical seminar held in Zlatibor, Serbia in 2016. The author would like to express his deep gratitude for the warm hospitality of the local committee of the conference. The author also wants to note that he is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) (Project Name: 'Y_EUCL2TIP' Project Number: 114F199).

References

1. A. Arvanitoyeorgos, F. Defever, G. Kaimakamis, V. Papantoniou, *Biharmonic Lorentz hypersurfaces in E_1^4*, Pac. J. Math. **229** (2007), 293–305.
2. R. Caddeo, S. Montaldo, C. Oniciuc, P. Piu, *Surfaces in the three-dimensional space forms with divergence-free stress-bienergy tensor*, Ann. Mat. Pura Appl. **193** (2014), 529–550.
3. B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type, 2nd Edition*, World Scientific, Hackensack, NJ, 2014.
4. B.-Y. Chen, P. Piccini, *Submanifolds with Finite Type Gauss Map*, Bull. Austral. Math. Soc. **35** (1987), 161–186.

5. U. Dursun, *Hypersurfaces with pointwise 1-type Gauss map in Lorentz-Minkowski space*, Proc. Est. Acad. Sci. **58**(3) (2009), 146–161.
6. M. Choi, U. H. Ki, Y. J. Suh, *Classification of Rotation Surfaces in Pseudo-Euclidean Space*, J. Korean Math. Soc. **35** (1998), 315–330.
7. Y. Fu, N. C. Turgay, *Complete classification of biconservative hypersurfaces with diagonalizable shape operator in the Minkowski* 4*-space*, Int. J. Math. **27**(5) (2016), article number: 1650041,.
8. G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas*, Chinese Ann. Math., Ser. A **7** (1986), 389–402.
9. ———, *The conservation law for* 2*-harmonic maps between Riemannian manifolds*, Acta Math. Sinica 30 (1987), 220–225,.
10. D. Kumari, *On biconservative Lorentz hypersurfaces with non-diagonalizable shape operator*, preprint, see (arXiv:1610.03005)
11. M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pac. J. Math. **118** (1985), 165–197.
12. N. C. Turgay, *A classification of biharmonic hypersurfaces in the Minkowski spaces of arbitrary dimension*, Hacet J. Math. Stat. **45** (2016), 1125–1134.
13. N. C. Turgay, A. Upadhyay, *On biconservative hypersurfaces in 4-dimensional Riemannian space forms*, submitted, (see arXiv:1702.05469).
14. A. Upadhyay, N. C. Turgay, *A Classification of Biconservative Hypersurfaces in a Pseudo-Euclidean Space*, J. Math. Anal. Appl. **444** (2016), 1703–1720.

Istanbul Technical University
Faculty of Science and Letters
Department of Mathematics
Istanbul
Turkey
turgayn@itu.edu.tr