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EINSTEIN–HILBERT TYPE ACTION ON SPACETIMES

Vladimir Rovenski

Abstract. The mixed gravitational field equations have been recently introduced for codimension one foliated manifolds, e.g. stably causal and globally hyperbolic spacetimes. These Euler–Lagrange equations for the total mixed scalar curvature (as analog of Einstein–Hilbert action) involve a new kind of Ricci curvature (called the mixed Ricci curvature). In the work, we derive Euler–Lagrange equations of the action for any spacetime, in fact, for a pseudo-Riemannian manifold endowed with a non-degenerate distribution. The obtained equations are presented in the classical form of Einstein field equation with the new Ricci type curvature instead of Ricci curvature.

Introduction

Distributions on a manifold M (i.e., subbundles of TM) appear in various situations, e.g. as fields of tangent planes of foliations or kernels of differential forms, and many models in physics are foliated, e.g., twisted or warped products.

Let (M,g) be a pseudo-Riemannian manifold endowed with non-degenerate distribution $\mathcal{D}\subset TM$. The *mixed scalar curvature*, $S_{\rm mix}$ (that is an averaged sectional curvature of planes that non-trivially intersect $\mathcal{D}$ and its orthogonal complement), see (1.1) and [**11**], is one of the simplest curvature invariants of a manifold endowed with a distribution. Note that from a mathematical point of view, a *spacetime* is described as a d-dimensional, time-orientable manifold, equipped with a Lorentzian metric, there also exists a timelike unit vector field N, whose orthogonal distribution is not necessarily integrable. Recall [**3, 4**] that stably causal and, in particular, globally hyperbolic spacetimes are naturally endowed with a codimension-one foliation (that is level hypersurfaces of a time-function). If $\mathcal{D}$ is spanned by N such that $g(N,N)=\varepsilon_N\in\{-1,1\}$, then $S_{\rm mix}=\varepsilon_N\,{\rm Ric}_{N,N}$, where ${\rm Ric}_{N,N}$ is the Ricci curvature in the N-direction.

The *mixed Einstein–Hilbert action* has been introduced in [**1**] as analog of the Einstein–Hilbert action, with the scalar curvature replaced by $S_{\rm mix}$:

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$$J_{\mathcal{D},\Omega}:\ g\mapsto\int_\Omega\Big\{\frac{1}{2\mathfrak{a}}\,(S_{\rm mix}(g)-2\,\Lambda)+\mathcal{L}(g)\Big\}\,d\operatorname{vol}_g\,. \tag{0.1}$$

Here Λ is the cosmological constant, $\mathcal{L}$-Lagrangian describing the matter contents, and $\mathfrak{a}$-the coupling constant. The integral is taken over M if it converges; otherwise, one integrates over an arbitrarily large, relatively compact domain Ω containing supports of variations g_t with $g_0=g$. The physical meaning of (0.1) has been discussed in [**1**] for the case of a globally hyperbolic spacetime $(M^4,\,g)$ when $\mathcal{D}=\operatorname{Span}(N)$ and hence $S_{\rm mix}=g(N,N)\operatorname{Ric}_{N,N}$, and the following have been obtained: the Euler–Lagrange equations, see (3.12), called the *mixed gravitational field equations*; sufficient conditions for existence of solutions for empty space, the conservation law analogous to conservation law of stress-energy tensor in relativity; equations of motion of a particle on isoparametric foliations in the "mixed gravitational field"; the linearized mixed field equations about the Minkowski metric; the value of coupling constant $\mathfrak{a}$ in the weak field and low velocity limit.

In this paper, we explore (0.1) for any spacetime, the obtained mixed gravitational field equations generalize the result of [**1**]. In fact, we work in arbitrary number of dimensions of a pseudo-Riemannian manifold endowed with a distribution, and also generalize certain results of [**2, 8**], where the particular case of variations of metric (called adapted variations) has been examined. In this setting, we present the Euler–Lagrange equations for (0.1) in the classical form of the Einstein field equations:

$$\operatorname{Ric}_{\mathcal{D}}-(1/2)\,\operatorname{Scal}_{\mathcal{D}}\cdot g+\Lambda\,g=\mathfrak{a}\,\Theta, \tag{0.2}$$

where Θ is the stress-energy tensor, while the Ricci tensor and scalar curvature are replaced by a new Ricci type curvature and its trace. Consequently, (0.2) contains the new *mixed Einstein tensor* $\mathrm{G}_{\mathcal{D}}:=\operatorname{Ric}_{\mathcal{D}}-(1/2)\,\operatorname{Scal}_{\mathcal{D}}\cdot g$, whose properties should be further investigated. The following equations with $\operatorname{Ric}_{\mathcal{D}}$ seem to be interesting to solve w.r.t. g (this issue will be addressed in further work): (i) (0.2) in vacuum, i.e., for $\Lambda=\Theta=0$; (ii) $\operatorname{Ric}_{\mathcal{D}}(g)=\lambda\,g$, i.e., the "mixed Einstein metrics"; (iii) the mixed Ricci equation: $\operatorname{Ric}_{\mathcal{D}}=r$ for a given $(0,2)$-tensor r; (iv) the mixed Ricci flow: $\partial_t g_t=-2\operatorname{Ric}_{\mathcal{D}}(g_t)$; (v) the Yamabe-type problem: $\operatorname{Scal}_{\mathcal{D}}(g)=\text{const}$.

The paper has an introduction and three sections. Section 1 reviews necessary definitions of tensors. Section 2 starts with variation formulas for quantities on a manifold with a distribution. Using them, we deduce Euler–Lagrange equations of (0.1), which generalize results in [**2, 8**] and use them to build the *mixed Ricci tensor* $\operatorname{Ric}_{\mathcal{D}}$ obeying (0.2). In Proposition 2.2 we observe that the replacement of $S_{\rm mix}$ by $\operatorname{Scal}_{\mathcal{D}}$ in (0.1) leads to the same Euler–Lagrange equations (0.2). In Section 3 we explore $\operatorname{Ric}_{\mathcal{D}}$ for spacetimes, see (3.6); for integrable $\mathcal{D}$ we compare the tensor, see (3.10), with its initial prototype defined in [**1**] for globally hyperbolic spacetimes.

1. Preliminaries

The section reviews necessary definitions of tensors, some of them were introduced in [**8, 11**]. We consider a connected manifold M^{n+p} with a pseudo-Riemannian metric g and a non-degenerate distribution $\mathcal{D}$ of rank $\dim\mathcal{D}_x=n$ for every $x \in M$. A distribution (i.e., a subbundle) $\mathcal{D} \subset TM$ is *non-degenerate*, if $\mathcal{D}_x \subset T_xM$ is a non-degenerate subspace of g_x for every $x \in M$; in this case, its g-orthogonal complement $\mathcal{D}^\perp$ is also non-degenerate. A pair $(\mathcal{D}, \mathcal{D}^\perp)$ of complementary orthogonal distributions is called an *almost product structure* on (M, g), [**5**].

Let $\mathfrak{X}_M$ (resp., $\mathfrak{X}_\mathcal{D}$) be the module over $C^\infty(M)$ of all vector fields on M (resp. on $\mathcal{D}$). The "musical" isomorphisms $\sharp$ and $\flat$ are used for rank one tensors, e.g. if $\omega \in T^1_0(M)$ is a 1-form and $X \in \mathfrak{X}_M$ then $\omega(X) = g(\omega^\sharp, X) = X^\flat(\omega^\sharp)$. For $(0,2)$-tensors A and B we have $\langle A, B\rangle = \operatorname{Tr}_g(A^\sharp B^\sharp) = \langle A^\sharp, B^\sharp\rangle$. We get $X = X^\top + X^\perp$, where $X^\top$ is the $\mathcal{D}$-component of $X \in \mathfrak{X}_M$ w.r.t. g. Thus, $g = g^\top + g^\perp$, where

$$g^\perp(X,Y) = g(X^\perp, Y^\perp), \quad g^\top(X,Y) = g(X^\top, Y^\top), \qquad (X, Y \in \mathfrak{X}_M).$$

The following convention is adopted for the range of indices:

$$a, b, c, \ldots \in \{1, \ldots, n\}, \quad i, j, k, \ldots \in \{1, \ldots, p\}, \quad n + p > 2.$$

It will be convenient to use orthogonal frames with certain nice properties. It is not hard to show that the local adapted orthonormal frame $\{E_a, \mathcal{E}_i\}$, where $\{E_a\} \subset \mathcal{D}$, and $\varepsilon_i = g(\mathcal{E}_i, \mathcal{E}_i) \in \{-1, 1\}$, $\varepsilon_a = g(E_a, E_a) \in \{-1, 1\}$, always exists on M.

We will define several tensors for one of distributions (say, $\mathcal{D}$; similar tensors for $\mathcal{D}^\perp$ will be denoted using $^\perp$ notation). Let ∇ be the Levi-Civita connection of g. The integrability tensor and the second fundamental form of $\mathcal{D}$, are given by

$$T(X,Y) = (1/2)\,[X^\top,\, Y^\top]^\perp, \quad h(X,Y) = (1/2)\,(\nabla_{X^\top}(Y^\top) + \nabla_{Y^\top}X^\top)^\perp,$$

respectively. We use inner products of tensors, e.g.

$$\langle h, h\rangle = \sum\nolimits_{a,b} \varepsilon_a \varepsilon_b\, g(h(E_a, E_b), h(E_a, E_b)),$$
$$\langle T, T\rangle = \sum\nolimits_{a,b} \varepsilon_a \varepsilon_b\, g(T(E_a, E_b), T(E_a, E_b)).$$

The mean curvature vector of $\mathcal{D}$ is $H = \operatorname{Tr}_g h = \sum_a \varepsilon_a h(E_a, E_a)$. A distribution $\mathcal{D}$ is called *totally umbilical*, *harmonic*, or *totally geodesic*, if $h = \frac{1}{n} H g_{|\mathcal{D}}$, $H = 0$, or $h = 0$, resp. The Weingarten operator A_Z of $\mathcal{D}$ w.r.t. $Z \in \mathfrak{X}_{\mathcal{D}^\perp}$, and the operator $T^\sharp_Z$ are defined by

$$g(A_Z(X), Y) = g(h(X,Y), Z), \quad g(T^\sharp_Z(X), Y) = g(T(X,Y), Z).$$

We will use the following convention for indices of various tensors: $T^\sharp_i := T^\sharp_{\mathcal{E}_i}$, $A_i := A_{\mathcal{E}_i}$, etc. Define a self-adjoint $(1,1)$-tensor $\mathcal{T} = \sum_i \varepsilon_i (T^\sharp_i)^2$ and a self-adjoint $(1,1)$-tensor with zero trace

$$\mathcal{K} := \sum\nolimits_i \varepsilon_i\, [T^\sharp_i, A_i] = \sum\nolimits_i \varepsilon_i\, (T^\sharp_i A_i - A_i\, T^\sharp_i),$$

which vanishes when $\mathcal{D}$ is either integrable or totally umbilical. Define $(1,2)$-tensors

$$\alpha(X,Y) = (A_{X^\perp}(Y^\top) + A_{Y^\perp}(X^\top))/2,$$
$$\theta(X,Y) = (T^\sharp_{X^\perp}(Y^\top) + T^\sharp_{Y^\perp}(X^\top))/2,$$
$$\delta_Z(X,Y) = (g(\nabla_{X^\top} Z,\, Y^\perp) + g(\nabla_{Y^\top} Z,\, X^\perp))/2.$$

Note that α, θ and δ_Z are symmetric and vanish for $(X,Y) \in (\mathcal{D}^\perp \times \mathcal{D}^\perp) \cup (\mathcal{D} \times \mathcal{D})$.

For any $(0,2)$-tensors P, Q and S on TM, define a tensor $\Upsilon_{P,Q}$ by

$$\langle \Upsilon_{P,Q}, S\rangle = \sum\nolimits_{\nu,\mu} \varepsilon_\nu\varepsilon_\mu [S(P(e_\nu, e_\mu), Q(e_\nu, e_\mu)) + S(Q(e_\nu, e_\mu), P(e_\nu, e_\mu))],$$

where $\{e_\nu\}$ is a local orthonormal basis of TM and $\varepsilon_\nu = g(e_\nu, e_\nu) \in \{-1, 1\}$.

Notice the properties $\Upsilon_{P,Q} = \Upsilon_{Q,P}$ and $\Upsilon_{P,Q_1+Q_2} = \Upsilon_{P,Q_1} + \Upsilon_{P,Q_2}$.

The curvature tensor is given by $R_{X,Y}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. The function

$$S_{\rm mix} = \sum\nolimits_{a,i} \varepsilon_a\varepsilon_i\, g(R_{a,i}\, E_a,\, \mathcal{E}_i) \tag{1.1}$$

is called the *mixed scalar curvature* of $\mathcal{D}$ (or $\mathcal{D}^\perp$) w.r.t. g. The following formula, see [**11**], has many applications and is used in the proof of Theorem 2.1 below:

$$\begin{aligned} S_{\rm mix} = {\rm div}(H + H^\perp) + g(H, H) - \langle h, h\rangle + \langle T, T\rangle \\ + g(H^\perp, H^\perp) - \langle h^\perp, h^\perp\rangle + \langle T^\perp, T^\perp\rangle. \end{aligned} \tag{1.2}$$

The projection of the gradient of $f \in C^1(M)$ onto $\mathcal{D}$ is $\nabla^\top f := (\nabla f)^\top$. The $\mathcal{D}$-Laplacian of $f \in C^2(M)$ is $\Delta^\top f = {\rm div}^\top(\nabla^\top f)$. The $\mathcal{D}^\perp$-*divergence* of $X \in \mathfrak{X}_M$ is defined by ${\rm div}^\perp X = \sum_i \varepsilon_i\, g(\nabla_i X, \mathcal{E}_i)$. Thus, ${\rm div}\, X = {\rm div}^\perp X + {\rm div}^\top X$. For $X \in \mathfrak{X}_{\mathcal{D}^\perp}$ we have ${\rm div}^\perp X = {\rm div}\, X + g(X, H)$. Similarly, for a $(1,2)$-tensor P, define a $(0,2)$-tensor ${\rm div}\ P = {\rm div}^\top P + {\rm div}^\perp P$, where

$$({\rm div}^\perp P)(X, Y) = \sum\nolimits_i \varepsilon_i\, g((\nabla_i\, P)(X, Y), \mathcal{E}_i), \quad X, Y \in \mathfrak{X}_M.$$

For a $\mathcal{D}^\perp$-valued $(0,2)$-tensor P, using $\langle P,\, H\rangle(X, Y) := g(P(X, Y),\, H)$ we obtain

$${\rm div}^\perp P = {\rm div}\, P + \langle P,\, H\rangle\,. \tag{1.3}$$

2. Euler–Lagrange equations for variations of metric

The section contains the Euler–Lagrange equations of the variational principle $\delta J_{\mathcal{D},\Omega}(g) = 0$ on a relatively compact domain Ω of a manifold M endowed with a distribution $\mathcal{D}$. We consider variations $\{g_t\}_{\,|t|<\varepsilon}$ of metric $g_0 = g$ on M such that the induced infinitesimal variations, presented by a symmetric $(0,2)$-tensor $B_t \equiv \partial g_t/\partial t$, are supported in $\Omega \subset M$. We adopt the notations $\partial_t \equiv \partial/\partial t,\ B \equiv \partial_t g_{t\,|\,t=0}$. For a $(0,2)$-tensor C we have $\langle C, B\rangle = \langle {\rm Sym}(C), B\rangle$.

Given $\mathcal{D}$, its g_t-orthogonal complement $\mathcal{D}_t^\perp$ depends on t; hence, the g_t-projections onto $\mathcal{D}$ (denoted by $^\top$) and onto $\mathcal{D}_0^\perp = \mathcal{D}^\perp$ (denoted by $^\perp$) also depend on t. If (g_t) preserve orthogonality of $\mathcal{D}$ and $\mathcal{D}^\perp$ then projections $^\top$ and $^\perp$ do not depend on t, and we obtain *adapted variations*, see [**2, 8**]. Among adapted variations, those preserving metric on $\mathcal{D}$ will be called $g^\perp$-*variations*.

Recall [**10**] that ∇^t (the Levi-Civita connection of g_t) is evolved as

$$\begin{aligned} 2g_t(\partial_t(\nabla^t_X Y), Z) = (\nabla^t_X B)(Y, Z) + (\nabla^t_Y B)(X, Z) \\ - (\nabla^t_Z B)(X, Y), \quad X, Y, Z \in \mathfrak{X}_M. \end{aligned} \tag{2.1}$$

Using (2.1), one may choose a nice evolution of a local orthonormal frame.

LEMMA 2.1. *Let a local $(\mathcal{D},\,\mathcal{D}^\perp)$-adapted and orthonormal for $t=0$ frame $\{E_a(t),\,\mathcal{E}_i(t)\}$ be evolved by $\{g_t\}_{|t|<\varepsilon}$ according to*

$$\partial_t E_a = -(1/2)\,B_t^\sharp(E_a)^\top, \qquad \partial_t \mathcal{E}_i = -(1/2)\,B_t^\sharp(\mathcal{E}_i)^\perp - B_t^\sharp(\mathcal{E}_i)^\top.$$

Then $\{E_a(t),\mathcal{E}_i(t)\}$ is a g_t-orthonormal frame and $\{E_a(t)\}\subset\mathcal{D}$, $\{\mathcal{E}_i(t)\}\subset\mathcal{D}_t^\perp$.

Using (2.1), Lemma 2.1 and decomposition of tensor fields w.r.t. $g=g^\top+g^\perp$, we obtain variation formulas (for detailed proof the reader is referred to [**9**]).

PROPOSITION 2.1. *For any variation g_t of metric on M we have at $t=0$*

$$\begin{aligned}
\partial_t\langle h^\perp,h^\perp\rangle &= \big\langle \operatorname{div} h^\perp - 4\Upsilon_{\alpha^\perp,\theta} + \mathcal{K}^\flat - \tfrac12\Upsilon_{h^\perp,h^\perp}, B\big\rangle - \operatorname{div}(\langle h^\perp,B\rangle),\\
\partial_t g_t(H^\perp,H^\perp) &= \langle(\operatorname{div} H^\perp)g^\perp + 4\langle\theta,H^\perp\rangle - H^{\perp\flat}\otimes H^{\perp\flat},\ B\rangle - \operatorname{div}((\operatorname{Tr}_{\mathcal{D}^\perp} B^\sharp)H^\perp),\\
\partial_t\langle h,h\rangle &= \big\langle\operatorname{div} h + \mathcal{K}^\flat - 2\operatorname{div}\alpha - 2\Upsilon_{\alpha,\alpha^\perp+\theta^\perp} - \tfrac12\Upsilon_{h,h}, B\big\rangle + \operatorname{div}(\langle 2\alpha - h, B\rangle),\\
\partial_t g_t(H,H) &= \big\langle 2\langle\theta^\perp-\alpha^\perp,H\rangle + 2\operatorname{Sym}(H^\flat\otimes H^{\perp\flat})\\
&\quad - 2\delta_H + (\operatorname{div} H)g^\top - H^\flat\otimes H^\flat, B\big\rangle + \operatorname{div}(2(B^\sharp H)^\top - (\operatorname{Tr}_{\mathcal{D}} B^\sharp)H),\\
\partial_t\langle T^\perp,T^\perp\rangle &= \big\langle 2\mathcal{T}^{\perp\flat} + 2\Upsilon_{\theta^\perp,\theta-\alpha} - 2\operatorname{div}\theta^\perp + \tfrac12\Upsilon_{T^\perp,T^\perp}, B\big\rangle + \operatorname{div}(2\langle\theta^\perp,B\rangle),\\
\partial_t\langle T,T\rangle &= \big\langle\tfrac12\Upsilon_{T,T} + 2\mathcal{T}^\flat, B\big\rangle,
\end{aligned}$$

REMARK 2.1. Formulas of Proposition 2.1 can be presented in three-component form, in which $\mathcal{D}\times\mathcal{D}$-and $\mathcal{D}^\perp\times\mathcal{D}^\perp$-components are dual, i.e., of the same form with interchanged $\mathcal{D}$ and $\mathcal{D}^\perp$. For example, for $g^\top$-variations we obtain the following:

$$\begin{aligned}
\partial_t\langle h,h\rangle &= \langle\operatorname{div} h + \mathcal{K}^\flat,\ B\rangle - \operatorname{div}(\langle h,B\rangle),\\
\partial_t g_t(H,H) &= \langle\,(\operatorname{div} H)\,g^\top,\ B\rangle - \operatorname{div}((\operatorname{Tr}_{\mathcal{D}} B^\sharp)H),\\
\partial_t\langle h^\perp,h^\perp\rangle &= -\langle\Upsilon_{h^\perp,h^\perp},\ B\rangle/2,\\
\partial_t g_t(H^\perp,H^\perp) &= -\langle H^{\perp\flat}\otimes H^{\perp\flat},\ B\rangle,\\
\partial_t\langle T,T\rangle &= 2\,\langle\,\mathcal{T}^\flat,\ B\,\rangle,\\
\partial_t\langle T^\perp,T^\perp\rangle &= \langle\Upsilon_{T^\perp,T^\perp},B\rangle/2.
\end{aligned}$$

Assume that the Lagrangian can be written as a sum of two terms $\mathcal{L}_{gr}+\mathcal{L}$, where $\mathcal{L}_{gr}=\frac{1}{2\mathfrak{a}}\,(S_{\rm mix}-2\,\Lambda)$ is the gravity Lagrangian, see (0.1), and the matter Lagrangian $\mathcal{L}$ depends only on g and not on its derivatives. Set $J: g\mapsto\int_\Omega\mathcal{L}(g)\,d\operatorname{vol}_g$ on M. Variation of the metric g with $B=\partial_t g_{t\,|t=0}$ produces the stress-energy tensor Θ such that

$$\frac{d}{dt}\,J(g_t)_{\,|\,t=0}=\frac12\int_\Omega\langle\Theta,B\rangle\,d\operatorname{vol}_g,$$

and which governs the interaction of the “gravitational field" with the given field representing the matter. Note that, see [**1**],

$$\Theta_{\mu\nu}=-2\,\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}+g_{\mu\nu}\mathcal{L},\quad \Big(2\,\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}+g^{\mu\nu}\mathcal{L}\Big)B_{\mu\nu}=\Theta^{\mu\nu}B_{\mu\nu}=\langle\Theta,B\rangle.$$

The product $TM\times TM$ is the sum of three subbundles, $\mathcal{D}^\perp\times\mathcal{D}^\perp$, $\mathcal{D}\times\mathcal{D}$ and

$$V:=(\mathcal{D}\times\mathcal{D}^\perp)+(\mathcal{D}^\perp\times\mathcal{D}).$$

DEFINITION 2.1. The symmetric $(0,2)$-tensor $\operatorname{Ric}_{\mathcal D}$ defined by

$$
\begin{aligned}
\operatorname{Ric}_{\mathcal{D}\,|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} &= \operatorname{div} h^\perp + \mathcal{K}^{\perp\flat} - 2\mathcal{T}^{\perp\flat} + H^\flat\otimes H^\flat - \frac12\Upsilon_{h,h} - \frac12\Upsilon_{T,T} + \eta\, g^\perp,\\
\operatorname{Ric}_{\mathcal{D}\,|\,V} &= 2\,\delta_H + 2\operatorname{div}(\theta^\perp-\alpha) - 4\langle\theta, H^\perp\rangle - 2\langle\theta^\perp-\alpha^\perp, H\rangle\\
&\quad - 4\Upsilon_{\alpha^\perp,\theta} - 2\Upsilon_{\alpha,\alpha^\perp} - 2\Upsilon_{\theta^\perp,\theta} - 2\operatorname{Sym}(H^\flat\otimes H^{\perp\flat}),\\
\operatorname{Ric}_{\mathcal{D}|\,\mathcal{D}\times\mathcal{D}} &= \operatorname{div} h + \mathcal{K}^\flat - 2\mathcal{T}^\flat + H^{\perp\flat}\otimes H^{\perp\flat} - \frac12\Upsilon_{h^\perp,h^\perp} - \frac12\Upsilon_{T^\perp,T^\perp} + \mu\, g^\top,
\end{aligned}
\tag{2.2}
$$

where

$$
\eta = -\frac{n-1}{p+n-2}\operatorname{div}(H^\perp - H),\quad \mu = \frac{p-1}{p+n-2}\operatorname{div}(H^\perp - H),
\tag{2.3}
$$

is referred to as the *mixed Ricci curvature*. The trace of $\operatorname{Ric}_{\mathcal D}$ is then

$$
\operatorname{Scal}_{\mathcal D} = S_{\rm mix} + \frac{p-n}{n+p-2}\operatorname{div}(H^\perp - H).
\tag{2.4}
$$

PROPOSITION 2.2. *Euler–Lagrange equations for* (0.1) *are the same as for the action*

$$
\hat J_{\mathcal{D},\Omega}:\ g\mapsto \int_\Omega \Big\{\frac{1}{2\mathfrak{a}}\,(\operatorname{Scal}_{\mathcal D}(g) - 2\,\Lambda) + \mathcal{L}(g)\Big\}\, d\operatorname{vol}_g\ .
\tag{2.5}
$$

PROOF. For any X_t supported on Ω and any g_t we have

$$
\int_\Omega \operatorname{div} X_t\, d\operatorname{vol}_{g_t} = \int_{\partial\Omega} g(X_t,\nu)\, dA_{g_t},
$$

ν being outward-pointing normal unit vector field to $\partial\Omega$. If $\partial_t g$ and $\partial_t X$ are supported in Ω then the rhs integral does not depend on t: $\frac{d}{dt}\int_\Omega \operatorname{div} X_t\, d\operatorname{vol}_{g_t} = 0$. Thus, for $X_t = \frac{p-n}{n+p-2}\,(H^\perp - H)$ and by (2.4), if we replace $S_{\rm mix}$ by $\operatorname{Scal}_{\mathcal D}$ in (0.1) then get the same Euler–Lagrange equations. □

THEOREM 2.1 (Euler–Lagrange equations). *A metric g on M with nondegenerate $\mathcal D$ is critical for* (0.1)*, or equivalently, for* (2.5)*, if and only if g is a solution to* (0.2)*, where* $\operatorname{Ric}_{\mathcal D}$ *is given by* (2.2).

PROOF. We derive for the gravitational part $J_{{\rm mix},\mathcal{D},\Omega}: g\mapsto \int_\Omega S_{\rm mix}(g)\, d\operatorname{vol}_g$ of (0.1). By (1.2),

$$
\frac{d}{dt}\,J_{{\rm mix},\mathcal{D},\Omega}(g_t) = \frac{d}{dt}\int_\Omega Q(g_t)\, d\operatorname{vol}_{g_t}\,,
$$

where $Q(g) := S_{\rm mix}(g) - \operatorname{div}(H + H^\perp)$ is given using (1.2) as

$$
Q(g) = g(H,H) - \langle h,h\rangle_g + g(H^\perp,H^\perp) - \langle h^\perp,h^\perp\rangle_g + \langle T,T\rangle_g + \langle T^\perp,T^\perp\rangle_g\,.
\tag{2.6}
$$

Recall, see [**10**], that the volume form is evolved as

$$
\partial_t\left(d\operatorname{vol}_{g_t}\right)_{|\,t=0} = \frac12\,(\operatorname{Tr}_g B)\, d\operatorname{vol}_g,
\tag{2.7}
$$

and $\operatorname{Tr}_g B = \langle g, B\rangle$, where $B = \{\partial_t g_t\}_{\,|\,t=0}$. Applying Proposition 2.1 to (2.6), using (1.3), (2.7) and removing integrals of divergences of vector fields compactly supported in Ω, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\,J_{\mathrm{mix},\mathcal{D},\Omega}(g_t)_{|\,t=0} = \\
&= \int_\Omega \big(\partial_t Q(g_t)_{\,|\,t=0} + \frac12\,(\operatorname{Tr}_g B)(S_{\mathrm{mix}}(g) - \operatorname{div}(H+H^\perp))\big)\,d\operatorname{vol}_g \\
&= \int_\Omega \big\langle\, 4\,\langle\theta, H^\perp\rangle - H^{\perp\flat}\otimes H^{\perp\flat} - \operatorname{div} h^\perp + 4\Upsilon_{\alpha^\perp,\theta} - \mathcal{K}^\flat + \frac12\,\Upsilon_{h^\perp,h^\perp} \\
&- H^\flat\otimes H^\flat + 2\,\langle\theta^\perp - \alpha^\perp,\, H\rangle + 2\,\operatorname{Sym}(H^\flat\otimes H^{\perp\flat}) - 2\,\delta_H + 2\,\operatorname{div}(\alpha-\theta^\perp) \\
&+ 2\,\Upsilon_{\alpha,\alpha^\perp+\theta^\perp} + \frac12\Upsilon_{h,h} - \operatorname{div} h - \mathcal{K}^\flat + 2\,\mathcal{T}^{\perp\flat} + 2\,\Upsilon_{\theta^\perp,\theta-\alpha} + \frac12\,\Upsilon_{T^\perp,T^\perp} \\
&+ \frac12\,\Upsilon_{T,T} + \mathcal{T}^\flat + \frac12 S_{\mathrm{mix}}\cdot g + \frac12\operatorname{div}(H^\perp - H)(\,g^\perp - g^\top),\ B\big\rangle\,d\operatorname{vol}_g\,.
\end{aligned}
\tag{2.8}
$$

If g is critical for $J_{\mathrm{mix},\mathcal{D},\Omega}$ w.r.t. all variations (g_t) then the integral in (2.8) is zero for arbitrary symmetric $(0,2)$-tensor B. We can further decompose the resulting Euler–Lagrange equations into three independent parts: its V-, $\mathcal{D}\times\mathcal{D}$- and $\mathcal{D}^\perp\times\mathcal{D}^\perp$-components. In this way, for the action (0.1) we obtain (2.2). The $\mathcal{D}^\perp\times\mathcal{D}^\perp$-component $(2.2)_3$ is dual to the $\mathcal{D}\times\mathcal{D}$-component $(2.2)_1$. Substituting (2.2) with arbitrary $(\mu,\,\eta)$ into (0.2), and comparing the result with (2.2), we find (μ,η) in (2.3) as solution of a linear system

$$(p-2)\,\eta + n\,\mu = \operatorname{div}(H^\perp - H),\quad (n-2)\,\mu + p\,\eta = -\operatorname{div}(H^\perp - H). \qquad \square$$

Remark 2.2. Components $(2.2)_1$ and $(2.2)_3$ can be obtained using adapted variations of metric, see also [**8**]. Component $(2.2)_2$ is not symmetric under the change $\mathcal{D}^\perp \leftrightarrow \mathcal{D}$, because $S_{\mathrm{mix}}(g_t)$ (when $\mathcal{D}$ is fixed) is different from the mixed scalar curvature for g_t when $\mathcal{D}^\perp$ is fixed while $\mathcal{D}_t$ varies. In order to organize better the div-terms in $(2.2)_2$, we introduce the symmetric $(0,2)$-tensor:

$$\operatorname{Ric}^\top(X,Y) = \sum\nolimits_a \varepsilon_a\big(g(R_{E_a,X^\top}E_a, Y^\perp) + g(R_{E_a,Y^\top}E_a, X^\perp)\big).$$

Using the Codazzi equation of [**5**, Theorem 2.4], V-component $(2.2)_2$ can be written as

$$
\begin{aligned}
\operatorname{Ric}_{\mathcal{D}\,|\,V} &= \operatorname{Ric}^\top + 2\operatorname{div}(\theta^\perp + \theta) + 2\langle\alpha-\theta, H^\perp\rangle \\
&\quad + 2\langle\alpha^\perp - \theta^\perp, H\rangle - 2\,\operatorname{Sym}(H^\flat\otimes H^{\perp\flat}) - 2\Upsilon_{\alpha,\alpha^\perp} + 2\Upsilon_{\theta,\theta^\perp}.
\end{aligned}
\tag{2.9}
$$

As an immediate consequence of (2.9), we get: if $\mathcal{D}$ and $\mathcal{D}^\perp$ are integrable then

$$\operatorname{Ric}_{\mathcal{D}\,|\,V} = \operatorname{Ric}^\top + 2\langle\alpha, H^\perp\rangle + 2\langle\alpha^\perp, H\rangle - 2\,\operatorname{Sym}(H^\flat\otimes H^{\perp\flat}) - 2\Upsilon_{\alpha,\alpha^\perp}.$$

Corollary 2.1. *A metric g on M with non-degenerate $\mathcal{D}$ is critical for the action* (0.1) *w.r.t. all* adapted *variations of g if and only if g solves* (0.2)*, where* $\operatorname{Ric}_{\mathcal{D}}$ *is built from* $(2.2)_1$ *and* $(2.2)_3$ *(that is* $\operatorname{Ric}_{\mathcal{D}\,|\,V} = 0$*).*

Next, we will examine (0.2) for generalized product metrics, having many applications. Let $M = M_1 \times M_2$ be a product of pseudo-Riemannian manifolds $(M_i\,,\,g_i)$ $(i = 1, 2)$, and let $\pi_i : M \to M_i$ and $d\pi_i : TM \to TM_i$ be the canonical projections. Given positive twisting functions $\phi, \psi \in C^\infty(M)$, a *double-twisted product* $M_1 \times_{(\phi,\psi)} M_2$ is $M_1 \times M_2$ with the metric $\bar g = \phi\, g^\top + \psi\, g^\perp$, where $g^\top = \pi_1^* g_1$ and $g^\perp = \pi_2^* g_2$. If either ϕ or ψ is constant, then we have a twisted product. The leaves $M_1 \times \{y\}$ (tangent to $\mathcal{D}$) and the fibers $\{x\} \times M_2$ (tangent to $\mathcal{D}^\perp$) are totally umbilical in $(M, \bar g)$ and this property characterizes double-twisted products, see [**6**]. Recall that $\mathrm{Hess}_\psi(X, Y) = \bar g(\bar\nabla_X(\bar\nabla\psi), Y)$ for all $X, Y \in \mathfrak{X}_M$.

The next proposition describes $\mathrm{Ric}_{\mathcal{D}}$ for double-twisted products, and shows that the stress-tensor in (0.2) should be of a particular structure. The result can be also used for finding special solutions of the mixed Ricci equation and mixed Einstein metrics mentioned in the introduction.

PROPOSITION 2.3. *Let $g = g^\top + g^\perp$ be a direct product metric on $M = M_1 \times M_2$. Then a double-twisted product metric $\bar g = \psi\, g^\top + \phi\, g^\perp$ is critical for* (0.1) *if and only if $\bar g$ solves* (0.2) *with*

$$\begin{aligned} \mathrm{Ric}_{\mathcal{D}\,|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} &= (F_1 + \Lambda)\, g^\perp + F_2 (\nabla^\perp\phi)^\flat \otimes (\nabla^\perp\phi)^\flat, \\ \mathrm{Ric}_{\mathcal{D}\,|\,V} &= F_3 (\nabla^\top\phi)^\flat \otimes (\nabla^\perp\psi)^\flat + F_4\, (\mathrm{Hess}_\psi)_{|V}, \\ \mathrm{Ric}_{\mathcal{D}\,|\,\mathcal{D}\times\mathcal{D}} &= (F_5 + \Lambda)\, g^\top + F_6 (\nabla^\top\psi)^\flat \otimes (\nabla^\top\psi)^\flat, \end{aligned} \tag{2.10}$$

where (F_5, F_6) are dual under the change $(\mathcal{D}^\perp, \psi) \leftrightarrow (\mathcal{D}, \phi)$ to (F_1, F_2), resp., and

$$\begin{aligned} F_1 &= \frac{n(n-1)}{4}\, \psi^{-2} g(\nabla^\perp\psi, \nabla^\perp\psi) + \Big(1 - \frac{p}{4}\Big)(p-1)\psi^{-1}\phi^{-1} g(\nabla^\top\phi, \nabla^\top\phi) \\ &\quad - (p-1)\psi^{-1}\Delta^\top\phi + \Big(1 - \frac{n}{2}\Big)(p-1)\psi^{-2} g(\nabla^\top\psi, \nabla^\top\phi), \\ F_2 &= -\frac{n(n-1)}{2}\, \phi^{-2}\psi^{-2}, \\ F_3 &= -(n-1)\psi^{-1}\Big(\frac{p-2}{4}\,\phi^{-1} - \frac{1}{2}\,\psi^{-1}\Big), \quad F_4 = \frac{1-n}{2}\psi^{-1}. \end{aligned}$$

Note that constant ϕ, ψ are the only solutions of (2.10) with $\Lambda = 0 = \Theta$.

3. The mixed Ricci tensor on spacetimes

Let $n = 1$. In this case, $\mathcal{D}$ is spanned by a nonsingular vector field N, i.e., the distribution tangent to one-dimensional foliation of (M^{p+1}, g) by N-curves. An example of such foliations is provided by a circle action $S^1 \times M \to M$ without fixed points. Now, let $g(N, N) = \varepsilon_N \in \{-1, 1\}$, then $g = g^\top + g^\perp$, where

$$g^\top = \varepsilon_N N^\flat \otimes N^\flat. \tag{3.1}$$

For physically relevant applications, one should take $p = 3$ and $\varepsilon_N = -1$, while $g_{|\,\mathcal{D}^\perp} > 0$. Note that $S_{\mathrm{mix}} = \varepsilon_N\, \mathrm{Ric}_{N,N}$, and the action (0.1) reduces itself to

$$J_{N,\Omega} :\ g \mapsto \int_\Omega \Big\{ \frac{1}{2\mathfrak{a}}\, (\varepsilon_N\, \mathrm{Ric}_{N,N}(g) - 2\,\Lambda) + \mathcal{L}(g) \Big\}\, d\,\mathrm{vol}_g\,. \tag{3.2}$$

For a foliation by N-curves we have $T=0$, while $\langle T^\perp, T^\perp\rangle = \varepsilon_N \langle T^{\perp\sharp}_N, T^{\perp\sharp}_N\rangle$. It follows easily that $h = H\, g^\top$ and $\langle h, h\rangle = g(H,H)$, where $H=\varepsilon_N\nabla_N N$ is the curvature of N-lines, and

$$H^\perp=\varepsilon_N\tau_1 N,\quad h^\perp = h^\perp_{sc} N,\quad g(H^\perp,H^\perp)=\tau_1^2,\quad \langle h^\perp, h^\perp\rangle=\tau_2, \tag{3.3}$$

where $h^\perp_{sc}=\varepsilon_N\langle h^\perp, N\rangle$ is the scalar second fundamental form of $\mathcal{D}^\perp$ and τ_i is the trace of the ith power of $A^\perp_N$. By (1.3), we obtain

$$\operatorname{div} h^\perp = \nabla_N\, h^\perp_{sc} - \tau_1 h^\perp_{sc},\quad \operatorname{div} h = (\operatorname{div} H)\, g^\top. \tag{3.4}$$

Note that $\operatorname{div} N = -\tau_1$ and $\operatorname{div}(\tau_1 N) = N(\tau_1)-\tau_1^2$. Denote by $R_N : X\to R_{N,X}N$ the Jacobi operator of N. Recall the identities, see [**7, 8**],

$$\begin{aligned}\epsilon_N\big(R_N+(A^\perp_N)^2+(T^{\perp\sharp}_N{}^2)\big)^\flat &= N(h^\perp_{sc}) - H^\flat\otimes H^\flat + \operatorname{Def}_{\mathcal{D}}\, H,\\ \epsilon_N\operatorname{Ric}_{N,N} &= \operatorname{div} H + \epsilon_N(N(\tau_1)-\tau_2)+\langle T^\perp, T^\perp\rangle.\end{aligned} \tag{3.5}$$

The *$\mathcal{D}$-deformation tensor* of $Z\in\mathfrak{X}_M$ is the symmetric part of ∇Z restricted to $\mathcal{D}$:

$$2\,(\operatorname{Def}_{\mathcal{D}}\, Z)(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X),\quad (X,Y\in\mathfrak{X}_{\mathcal{D}}).$$

Proposition 3.1. *For $\mathcal{D}$ spanned by N, the symmetric tensor* $\operatorname{Ric}_{\mathcal{D}}$ *is given by*

$$\begin{aligned}\operatorname{Ric}_{\mathcal{D}\,|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} &= \nabla_N\, h^\perp_{sc} - \tau_1 h^\perp_{sc} - \varepsilon_N(2\,(T^{\perp\sharp}_N)^2 + [T^{\perp\sharp}_N, A^\perp_N])^\flat,\\ \operatorname{Ric}_{\mathcal{D}}(\,\cdot\,,N)_{\,|\,\mathcal{D}^\perp} &= \operatorname{div}^\perp(T^{\perp\sharp}_N) + 2\,(T^{\perp\sharp}_N H)^\flat\\ \operatorname{Ric}_{\mathcal{D}}(N,N) &= \varepsilon_N(N(\tau_1)-\tau_2) - \langle T^\perp, T^\perp\rangle,\end{aligned} \tag{3.6}$$

and its trace is

$$\operatorname{Scal}_{\mathcal{D}} = \varepsilon_N\operatorname{Ric}_{N,N} + \operatorname{div}(\varepsilon_N\tau_1 N - H).$$

Due to (3.5), *an equivalent form of* $\operatorname{Ric}_{\mathcal{D}}$ *just involving the curvature tensor is*

$$\begin{aligned}\operatorname{Ric}_{\mathcal{D}\,|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} &= \varepsilon_N(R_N + (A^\perp_N)^2 - (T^{\perp\sharp}_N)^2 + [T^{\perp\sharp}_N, A^\perp_N])^\flat + H^\flat\otimes H^\flat\\ &\quad - \tau_1 h^\perp_{sc} - \operatorname{Def}_{\mathcal{D}}\, H - \frac{1}{2}\,(\varepsilon_N\operatorname{Ric}_{N,N} + \operatorname{div}(\varepsilon_N\tau_1 N - H))\, g^\perp,\\ \operatorname{Ric}_{\mathcal{D}}(\,\cdot\,,N)_{\,|\,\mathcal{D}^\perp} &= \operatorname{div}^\perp(T^{\perp\sharp}_N) + 2\,(T^{\perp\sharp}_N H)^\flat,\\ \operatorname{Ric}_{\mathcal{D}}(N,N) &= \varepsilon_N\operatorname{Ric}_{N,N} - 4\,\langle T^\perp, T^\perp\rangle - \operatorname{div}(\varepsilon_N\,\tau_1 N + H).\end{aligned} \tag{3.7}$$

Proof. By (2.3), we have $\eta=0$ and $\mu=\varepsilon_N(N(\tau_1)-\tau_1^2)-\operatorname{div} H$. Substituting values (3.3), (3.4) and

$$\Upsilon_{h,h} = 2\,H^\flat\otimes H^\flat,\quad \mathcal{T}=\varepsilon_N(T^{\perp\sharp}_N)^2,\quad \Upsilon_{T,T}=0$$

into $(2.2)_1$ yields $(3.6)_1$. Substituting values (3.3), (3.4) and

$$\begin{aligned}&h = H\, g^\top,\quad H^{\perp\flat}\otimes H^{\perp\flat} = \varepsilon_N\,\tau_1^2\, g^\top,\quad \mathcal{K}=0=\mathcal{T},\\ &\Upsilon_{h^\perp,h^\perp} = 2\,\varepsilon_N\,\tau_2\, g^\top,\quad \Upsilon_{T^\perp,T^\perp} = 2\,\langle T^\perp, T^\perp\rangle\, g^\top\end{aligned}$$

into $(2.2)_3$ yields $(3.6)_3$. Note also that $\theta=0$; hence,

$$\begin{aligned}
2\,(\operatorname{div}\alpha)(X,N) &= g(\nabla_N H-\tau_1 H, X),\\
2\,\langle\theta^\perp-\alpha^\perp, H\rangle(X,N) &= -g(T_N^{\perp\sharp}(H)+A_N^\perp(H), X),\\
2\ \mathrm{Sym}(H^\flat\otimes H^{\perp\flat})(X,N) &= g(\tau_1 H,\, X),\\
2\,\delta_H(X,N) &= g(\nabla_N H, X),\\
2\Upsilon_{\alpha,\alpha^\perp}(X,N) &= g(A_N^\perp(H), X)
\end{aligned}$$

for all X orthogonal to N, and the V-component $(2.2)_2$ is

$$\mathrm{Ric}_{\,\mathcal{D}\,|\,V}=\operatorname{div}\theta^\perp-\langle\theta^\perp,\,H\rangle. \tag{3.8}$$

The lhs of (3.8) vanishes on all pairs of vectors except those from V. We have

$$\begin{aligned}
2\ \operatorname{div}\theta^\perp(X,N) &= \sum\nolimits_i \varepsilon_i\, g((\nabla_i\, T_N^{\perp\sharp})(X),\mathcal{E}_i)+\varepsilon_N g(\nabla_N(T_N^{\perp\sharp}(X)),N)\\
&= \operatorname{div}^\perp(T_N^{\perp\sharp})(X)+g(T_N^{\perp\sharp}(H),X)
\end{aligned}$$

for $X\in\mathfrak{X}_{\mathcal{D}^\perp}$ Hence, (3.8) is written as $(3.6)_2$. Using (3.5) in (3.6) yields (3.7). The above and (3.1) yield (0.2). □

Theorem 2.1 and Proposition 3.1 yield the following.

Corollary 3.1 (The mixed gravitational field equations). *A metric g on M with $\mathcal{D}$ spanned by a unit vector field N is critical for* (3.2) *if and only if g is a solution of* (0.2) *with* $\mathrm{Ric}_{\mathcal{D}}$ *given in* (3.6). *In this case,* (0.2) *splits into the system*

$$\begin{aligned}
&\nabla_N\, h_{sc}^\perp-\tau_1 h_{sc}^\perp-\varepsilon_N\big(2(T_N^{\perp\sharp})^2+[A_N^\perp,\,T_N^{\perp\sharp}]\big)^\flat\\
&\quad+\langle T^\perp,T^\perp\rangle-\frac12\big(\varepsilon_N(2\,N(\tau_1)-\tau_1^2-\tau_2)-2\Lambda\big)\,g^\perp=\mathfrak{a}\,\Theta_{|\,\mathcal{D}^\perp\times\mathcal{D}^\perp},\\
&\qquad\qquad\operatorname{div}^\perp(T_N^{\perp\sharp})+2\,(T_N^{\perp\sharp}H)^\flat=\mathfrak{a}\,\Theta_{\,\bullet,N\,|\,\mathcal{D}^\perp},\\
&\qquad\qquad\frac12\,\varepsilon_N(\tau_1^2-\tau_2)-\frac32\,\langle T^\perp,T^\perp\rangle+\Lambda=\mathfrak{a}\,\Theta_{N,N}.
\end{aligned} \tag{3.9}$$

Example 3.1. Let a unit vector field N generates a geodesic Riemannian flow (i.e., $h^\perp=0=h$) on a pseudo-Riemannian manifold (M^{p+1},g). In this case,

$$\mathrm{Ric}_{\,\mathcal{D}}(X,N)=-\,\mathrm{Ric}_{X,N}\,.$$

As an example, consider the Hopf fibration $S^{2m+1}\to CP^m$, for which N is tangent to the fibers $\{S^1\}$. We have $T^\perp(X,Y)=g(JX,Y)N$ for the standard almost complex structure J on $\mathbb{R}^{2m+2}$ and metric $g>0$, and hence $T_N^{\perp\sharp}(X)=J(X)$ and $\langle T^\perp,T^\perp\rangle=2m$. It follows that $(\nabla_Z\,T_N^{\sharp})(X)=(\nabla_Z\,J)(X)=0$, and $(3.9)_2$ is valid with $\Theta_{|\,\mathcal{D}^\perp\times\mathcal{D}}=0$. Moreover, $(3.9)_1$ and $(3.9)_3$ are satisfied with

$$\mathfrak{a}\,\Theta_{|\,\mathcal{D}^\perp\times\mathcal{D}^\perp}=(2-m+\Lambda)\,g^\perp,\quad \mathfrak{a}\,\Theta_{\,N,N}=\Lambda-3.$$

The reader can find more examples in [**8, 9**], and for spacetimes, in [**1**].

COROLLARY 3.2. *A metric g on M with non-degenerate $\mathcal{D} = \operatorname{Span}(N)$ and* integrable $\mathcal{D}^\perp$ *is critical for* (3.2) *if and only if g solves* (0.2)*, where the nonzero components of* $\operatorname{Ric}_{\mathcal{D}}$ *are given by*

$$\operatorname{Ric}_{\mathcal{D}\,|\mathcal{D}^\perp\times\mathcal{D}^\perp} = \nabla_N\, h^\perp_{sc} - \tau_1 h^\perp_{sc}, \quad \operatorname{Ric}_{\mathcal{D}}(N,N) = \varepsilon_N(N(\tau_1) - \tau_2). \tag{3.10}$$

In this case, (0.2) *splits into the system*

$$\begin{aligned} \nabla_N\, h^\perp_{sc} - \tau_1 h^\perp_{sc} - \frac{1}{2}\,(\varepsilon_N(2\,N(\tau_1) - \tau_1^2 - \tau_2) - 2\Lambda)\, g^\perp &= \mathfrak{a}\,\Theta_{|\,\mathcal{D}^\perp\times\mathcal{D}^\perp},\\ \frac{1}{2}\,\varepsilon_N(\tau_1^2 - \tau_2) + \Lambda &= \mathfrak{a}\,\Theta_{N,N}. \end{aligned} \tag{3.11}$$

PROOF. Since $(3.9)_2$ is valid for integrable $\mathcal{D}^\perp$, in this case, variations g_t constant on $\mathcal{D}$ yield the same Euler–Lagrange equations as adapted $g^\perp$-variations. □

Action (3.2) has been studied in [**1**] for a globally hyperbolic spacetime (M^4, g), which is naturally foliated, see [**3**]. There, $\mathcal{D} = \operatorname{Span}(N)$ with $\varepsilon_N = -1$, and $\mathcal{D}^\perp$ is integrable (i.e., $T^\perp = 0$). Although Euler–Lagrange equations for (3.2) coincide with (3.11), they are written in [**1**] in terms of a non-symmetric tensor $\operatorname{Ric}^1_{\mathcal{D}}$, which is different from (3.10):

$$\begin{aligned} &\operatorname{Ric}^1_{\mathcal{D}|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} = \nabla_N\, h^\perp_{sc} - \tau_1 h^\perp_{sc},\\ &\operatorname{Ric}^1_{\mathcal{D}}(X,N) = -\operatorname{Ric}^1_{\mathcal{D}}(N,X) = \operatorname{div}(A^\perp_N(X)), \quad X\in\mathfrak{X}_{\mathcal{D}^\perp},\\ &\operatorname{Ric}^1_{\mathcal{D}}(N,N) = \operatorname{div} H, \end{aligned}$$

with the trace $\operatorname{Scal}^1_{\mathcal{D}} = \varepsilon_N \operatorname{Ric}_{N,N} - \varepsilon_N(\tau_1^2 - \tau_2)$. Consequently, the mixed gravitational field equations given in [**1**] are slightly different from (0.2):

$$\operatorname{Ric}^1_{\mathcal{D}} - \frac{1}{2}\operatorname{Scal}^1_{\mathcal{D}}\cdot g + \varepsilon_N \operatorname{Ric}_{N,N}\big(\frac{1}{2}\, g - \varepsilon_N N^\flat\otimes N^\flat\big) + \Lambda\, g = \mathfrak{a}\,\Theta. \tag{3.12}$$

Since we actually use the symmetric part of $\operatorname{Ric}^1_{\mathcal{D}}$ in (3.12), its left hand side vanishes when evaluated on (X, N). Thus, comparing $\operatorname{Ric}_{\mathcal{D}}$ with $\operatorname{Ric}^1_{\mathcal{D}}$, we find

$$\begin{aligned} \operatorname{Sym}(\operatorname{Ric}^1_{\mathcal{D}}) - \operatorname{Ric}_{\mathcal{D}} &= (\operatorname{div} H - \varepsilon_N(N(\tau_1) - \tau_2))\, g^\top,\\ \operatorname{Scal}^1_{\mathcal{D}} - \operatorname{Scal}_{\mathcal{D}} &= \operatorname{div} H - \varepsilon_N(N(\tau_1) - \tau_2). \end{aligned}$$

Certainly, (3.12) reduces to (3.11) when evaluated on $\mathcal{D}^\perp\times\mathcal{D}^\perp$ and (N, N), respectively. This corresponds to the following values of (η, μ) in (2.2) when $n = 1$:

$$\eta = -\operatorname{div}(H^\perp - H) + \varepsilon_N \operatorname{Ric}_{N,N}, \quad \mu = \operatorname{div}(H^\perp - H) - \varepsilon_N \operatorname{Ric}_{N,N}\,.$$

Comparing (3.12) with (0.2) we conclude that (2.3) is the best choice of (η, μ).

References

1. E. Barletta, S. Dragomir, V. Rovenski, M. Soret, *Mixed gravitational field equations on globally hyperbolic spacetimes*, Classical Quantum Gravity **30** (2013), 085015.
2. E. Barletta, S. Dragomir, V. Rovenski, *The mixed Einstein–Hilbert action and extrinsic geometry of foliated manifolds*, Balkan J. Geom. Appl. **22**(1) (2017), 1–17.
3. A. N. Bernal, M. Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes*, Commun. Math. Phys. **257** (2005), 43–50.

4. A. Fathi, A. Siconolfi, *On smooth time functions*, Math. Proc. Cambridge Philos. Soc. **152**(2) (2012), 303–339.
5. A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16**(7) (1967), 715–737.
6. R. Ponge, H. Reckziegel, *Twisted products in pseudo-Riemannian geometry*, Geom. Dedicata **48** (1993), 15–25.
7. V. Rovenski, P. Walczak, *Topics in extrinsic geometry of codimension-one foliations*, SpringerBriefs Math., Springer-Verlag, 2011.
8. V. Rovenski, T. Zawadzki, *The Einstein–Hilbert type action on foliated pseudo-Riemannian manifolds*, arXiv:1604.00985, 2016.
9. ———, *Variations of the total mixed scalar curvature of a distribution*, Ann. Glob. Anal. Geom. (2018), 1–36.
10. P. Topping, *Lectures on the Ricci flow*, LMS Lecture Notes 325, Cambridge Univ. Press, 2006.
11. P. Walczak, *An integral formula for a Riemannian manifold with two orthogonal complementary distributions*. Colloq. Math. **58** (1990), 243–252.

Department of Mathematics
University of Haifa
Haifa
Israel
vrovenski@univ.haifa.ac.il