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ISOTHERMIC SURFACES OBTAINED FROM HARMONIC MAPS IN S^6

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Abstract. The harmonicity of a smooth map from a Riemann surface into the 6-dimensional sphere S^6 amounts to the closeness of a certain 1-form that can be written in terms of the nearly Kähler structure of S^6. We will prove that the immersions F in $\mathbb{R}^7$ obtained from superconformal harmonic maps in $S^3 \subset S^6$ by integration of the corresponding closed 1-forms are isothermic. The isothermic surfaces so obtained include a certain class of constant mean curvature surfaces in $\mathbb{R}^3$ that can be deformed isometrically through isothermic surfaces into non-spherical pseudo-umbilical surfaces in $\mathbb{R}^7$.

1. Introduction

It is a well-known fact that any non-conformal harmonic map φ from a simply-connected Riemann surface Σ into the round 2-sphere S^2 is the Gauss map of a constant Gauss curvature surface, $F : \Sigma \to \mathbb{R}^3$, and of two parallel constant mean curvature surfaces, $F^{\pm} = F \pm \varphi : \Sigma \to \mathbb{R}^3$; the surface F integrates the closed 1-form $\omega = \varphi \times *d\varphi$, where $\times$ denotes the standard cross product of $\mathbb{R}^3$.

Again, the harmonicity of a smooth map $\varphi : \Sigma \to S^6$ amounts to the closeness of the differential 1-form $\omega = \varphi \times *d\varphi$, where $\times$ stands now for the 7-dimensional cross product. This means that we can integrate on simply-connected domains in order to obtain a map $F : \Sigma \to \mathbb{R}^7$. If φ is a conformal harmonic immersion, then F is a conformal immersion; and, in contrast with the 3-dimensional case, where F is necessarily a totally umbilical surface, F can exhibit a wide variety of geometrical behaviors in the 7-dimensional case [**3**].

Recall [**4**] that a surface in $\mathbb{R}^n$ is *isothermic* if, away from umbilic points, it admits *conformal curvature line* (CCL) coordinates, that is, conformal coordinates with respect to which each second fundamental form is diagonal. In this short note, we prove that the immersions F in $\mathbb{R}^7$ obtained from *superconformal* [**1, 3**] harmonic maps in $S^3 = S^6 \cap W$ by integration of the corresponding 1-forms ω, with W a 4-dimensional subspace of $\mathbb{R}^7$, are isothermic. We will also see that the

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isothermic surfaces so obtained include a certain class of constant mean curvature surfaces in $\mathbb{R}^3$ that can be deformed isometrically through isothermic surfaces into non-spherical pseudo-umbilical surfaces in $\mathbb{R}^7$.

2. Harmonic maps from Riemann surfaces into S^6

Let $\cdot$ be the standard inner product on $\mathbb{R}^7$ and $\mathbf{e}_1, \dots, \mathbf{e}_7$ be the canonical basis of $\mathbb{R}^7$. Fix the 7-dimensional cross product $\times$ defined by the multiplication table

(2.1)

$\times$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	0	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$-\mathbf{e}_3$	0	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	0	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	0	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	0	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	0	$-\mathbf{e}_1$
$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	0

The cross product $\times$ satisfies the following identities, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^7$:

(P1) $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{y} = 0$;
(P2) $(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{x} \times \mathbf{y}) = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) - (\mathbf{x} \cdot \mathbf{y})^2$;
(P3) $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$;
(P4) $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$;
(P5) $(\mathbf{x} \times \mathbf{y}) \times (\mathbf{x} \times \mathbf{z}) = ((\mathbf{x} \times \mathbf{y}) \times \mathbf{z}) \times \mathbf{x} + ((\mathbf{y} \times \mathbf{z}) \times \mathbf{x}) \times \mathbf{x} + ((\mathbf{z} \times \mathbf{x}) \times \mathbf{x}) \times \mathbf{y}$;
(P6) $\mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = -(\mathbf{x} \cdot \mathbf{x})\mathbf{y} + (\mathbf{x} \cdot \mathbf{y})\mathbf{x}$;
(P7) $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) + (\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = 2(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z} - (\mathbf{y} \cdot \mathbf{z})\mathbf{x}$.

Extend the inner product $\cdot$ and the cross product $\times$ by complex bilinearity to $\mathbb{C}^7 = \mathbb{R}^7 \otimes \mathbb{C}$. We also denote these complex bilinear extensions by $\cdot$ and $\times$, respectively.

The standard nearly Kähler structure J on the 6-dimensional unit sphere S^6 can be written in terms of the cross product $\times$ as follows: for each $\mathbf{x} \in S^6$, $J\mathbf{u} = \mathbf{x} \times \mathbf{u}$, for all $\mathbf{u} \in T_{\mathbf{x}} S^6$. Let Σ be a Riemann surface with local conformal coordinate $z = x + iy$, and let $\varphi : \Sigma \to S^{n-1}$ be a harmonic map, that is, $\triangle\varphi \perp T_\varphi S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n | \, \varphi \cdot \mathbf{u} = 0\}$. For $n = 7$, this means that $\varphi \times \triangle\varphi = 0$, which is equivalent to the closeness of the one form $\omega = \varphi \times *d\varphi = J * d\varphi$.

3. Isothermic surfaces from harmonic maps into S^6

If $\varphi : \Sigma \to S^6$ is a harmonic immersion and Σ is simply-connected, we can integrate to obtain an immersion $F : \Sigma \to \mathbb{R}^7$ such that $dF = \varphi \times *d\varphi$. In local conformal coordinates $z = x + iy$ of Σ, this can be written in the form

$$F_z = i\varphi \times \varphi_z, \quad F_{\bar{z}} = -i\varphi \times \varphi_{\bar{z}}.$$

Making use of the properties for the cross product, we obtain the following formulae for the first and second fundamental forms of the immersion F in terms of φ and its derivatives.

PROPOSITION 3.1. [**3**] *Let $\mathbf{I}_F$ and $\mathbf{II}_F$ be the first and the second fundamental forms of $F : \Sigma \to \mathbb{R}^7$, respectively. Let N be a vector field of the normal bundle $TF^\perp$. Then, with respect to the local conformal coordinates $z = x + iy$ of Σ, we have*

$$\mathbf{I}_F = \begin{pmatrix} |\varphi_y|^2 & -\varphi_x \cdot \varphi_y \\ -\varphi_x \cdot \varphi_y & |\varphi_x|^2 \end{pmatrix} \tag{3.1}$$

$$\mathbf{II}_F^N := \mathbf{II}_F \cdot N = \begin{pmatrix} (\varphi_x \times \varphi_y) \cdot N + (\varphi \times \varphi_{xy}) \cdot N & (\varphi \times \varphi_{yy}) \cdot N \\ (\varphi \times \varphi_{yy}) \cdot N & (\varphi_x \times \varphi_y) \cdot N - (\varphi \times \varphi_{xy}) \cdot N \end{pmatrix}. \tag{3.2}$$

If φ is conformal, F is also conformal. Let $e^{2\alpha}$ be the common conformal factor of φ and F. One can check [**3**] that the mean curvature vector of F is given by

$$\mathbf{h}_F = \frac{1}{2} \mathrm{tr}\, \mathbf{I}_F^{-1} \mathbf{II}_F = \frac{e^{-2\alpha}}{2} \big\{ \mathbf{II}_F(F_x, F_x) + \mathbf{II}_F(F_y, F_y) \big\} = e^{-2\alpha} \varphi_x \times \varphi_y. \tag{3.3}$$

Next we establish our main result. Recall that a harmonic map in S^3 is *superconformal* if it has finite isotropy $r = 3$ [**1, 2, 3**].

THEOREM 3.1. *If $\varphi : \Sigma \to S^6 \cap W$ is a superconformal harmonic immersion, where W is a 4-dimensional subspace of $\mathbb{R}^7$, then F is isothermic.*

PROOF. Let $z = x + iy$ be local conformal coordinates on Σ and consider the harmonic sequence $\{\varphi_j\}_{j \in \mathbb{Z}}$ associated to φ (see [**1, 2, 3**]). For each j, $\varphi_{-j} = \overline{\varphi}_j$ and there exists a local meromorphic section f_j of φ_j (with respect to the Koszul-Malgrange holomorphic structure) such that [**2**]:

$$\frac{\partial f_j}{\partial z} = f_{j+1} + \frac{\partial}{\partial z} \log |f_j|^2 f_j; \quad \frac{\partial f_{j+1}}{\partial \bar{z}} = -\frac{|f_{j+1}|^2}{|f_j|^2} f_j; \quad |f_j||f_{-j}| = 1 \text{ (if } f_j \neq 0).$$

Since φ is superconformal in $S^3 = S^6 \cap W$, we have: the harmonic line bundles $\varphi_{-1}, \varphi, \varphi_1, \varphi_2$ are mutually orthogonal; φ_2 is real, that is, $\varphi_{-2} = \varphi_2$; and

$$f_{-2} = \frac{P_{\varphi_2}(\varphi_{\bar{z}\bar{z}})}{|P_{\varphi_2}(\varphi_{zz})|^2}, \quad f_2 = P_{\varphi_2}(\varphi_{zz}),$$

where P_{φ_2} denotes the orthogonal projection onto φ_2. Both f_{-2} and f_2 are local meromorphic sections of φ_2. Hence there exists a meromorphic function g on Σ such that $f_{-2} = g f_2$. Equivalently,

$$\frac{P_{\varphi_2}(\varphi_{\bar{z}\bar{z}})}{|P_{\varphi_2}(\varphi_{zz})|^2} = g P_{\varphi_2}(\varphi_{zz}), \quad \frac{P_{\varphi_2}(\varphi_{zz})}{|P_{\varphi_2}(\varphi_{zz})|^2} = \bar{g} P_{\varphi_2}(\varphi_{\bar{z}\bar{z}}). \tag{3.4}$$

Locally, away from the isolated zeros of $P_{\varphi_2}(\varphi_{zz})$, we can consider the holomorphic non-vanishing function $q = \sqrt{g}$ and rewrite (3.4) as

$$\overline{q} P_{\varphi_2}(\varphi_{\bar{z}\bar{z}}) = q P_{\varphi_2}(\varphi_{zz}).$$

Define new local conformal coordinates $w = u + iv$ by $\frac{dw}{dz} = \frac{e^{\pi i/4}}{\sqrt{q}}$. With respect to these coordinates, we have $P_{\varphi_2}(\varphi_{\bar{w}\bar{w}}) = -P_{\varphi_2}(\varphi_{ww})$. Then,

$$F^\perp_{\bar{w}\bar{w}} = -i\varphi \times P_{\varphi_2}(\varphi_{\bar{w}\bar{w}}) = i\varphi \times P_{\varphi_2}(\varphi_{ww}) = F^\perp_{ww},$$

which implies $F_{uv}^{\perp} = 0$, where $\perp$ denotes the component in the normal bundle. Then $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ diagonalize $\mathbf{II}_F^N$ for any normal vector field N to F.

In view of (3.2), the umbilic points of $\mathbf{II}_F^N$, for each normal section N, are precisely the points where $P_{\varphi_2}(\varphi_{zz}) = 0$. $\square$

4. Isometric deformations of CMC into pseudo-umbilical surfaces

A 4-dimensional subspace W of $\mathbb{R}^7$ is *coassociative* if $V = W^{\perp}$ is closed with respect to $\times$. It can be shown [**3**] that, if W is a coassociative 4-space, then $V \times W = W$ and $W \times W = V$.

If φ is superconformal in some 3-dimensional sphere $S^3 = S^6 \cap W$, where W is a coassociative 4-space, then, up to translation, F is a constant mean curvature surface in the 3-space $V = W^{\perp}$. This is a consequence of the following result.

Theorem 4.1. [**3**] *Let $\varphi : \Sigma \to S^6$ be a conformal harmonic immersion. Then $F : \Sigma \to \mathbb{R}^7$ has a parallel mean curvature vector field and it is not pseudo-umbilical if, and only if, φ is superconformal in some* 3*-dimensional sphere $S^3 = S^6 \cap W$, where W is a coassociative* 4*-space.*

Example 4.1. Let $W = \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ and $\varphi : \mathbb{C} \to S^3 = S^6 \cap W$ be defined by $\varphi(x, y) = \frac{1}{\sqrt{2}}\big(\cos x\, \mathbf{e}_4 + \sin x\, \mathbf{e}_5 + \cos y\, \mathbf{e}_6 + \sin y\, \mathbf{e}_7\big)$, which is a superconformal harmonic map and parameterizes a Clifford torus. Taking into account multiplication table (2.1), one can check that the associated surface $F : \mathbb{C} \to V \subset \mathbb{R}^7$ is the cylinder given by

$$F(x, y) = \frac{1}{2}\big(- (x + y)\, \mathbf{e}_1 - \cos(x - y)\, \mathbf{e}_2 + \sin(x - y)\, \mathbf{e}_3\big). \tag{4.1}$$

Remark 4.1. Since the Gauss map of a CMC surface without umbilical points is a non-conformal harmonic map, we see that theorem 4.1 also gives a procedure to obtain non-conformal harmonic maps into S^2 from superconformal harmonic maps into S^3: starting with a superconformal harmonic map $\varphi : \Sigma \to S^3 = S^6 \cap W$, where W is a coassociative 4-space, the Gauss map of F is precisely the mean curvature vector field $\mathbf{h}_F : \Sigma \to S^2$ given by (3.3); this is a non-conformal harmonic map.

Given a 4-dimensional subspace W of $\mathbb{R}^7$, an orthogonal direct sum decomposition $W = W_1 \oplus W_2$, with $\dim W_1 = \dim W_2 = 2$, is said to be $\times$-*compatible* if $W_1 \times W_1 \perp W_2 \times W_2$. For example, $W = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ admits $\times$-compatible decompositions.

Recall that an immersion F is said to be *pseudo-umbilical* if $\mathbf{II}_F \cdot \mathbf{h}_F = \lambda \mathbf{I}_F$ for some smooth function λ on Σ.

Theorem 4.2. [**3**] *If φ is a superconformal harmonic map in $S^3 = S^6 \cap W$, for some* 4*-space W admitting a $\times$-compatible decomposition, then F is pseudo-umbilical with non-parallel mean curvature vector field.*

For a general dimension, a pseudo-umbilical submanifold M^n of $\mathbb{R}^m$ has mean curvature vector field parallel in the normal bundle if, and only if, M^n is either a minimal submanifold of $\mathbb{R}^m$ or a minimal submanifold of a hypersphere of $\mathbb{R}^m$ [**6**]. Hence, the pseudo-umbilical surfaces of theorem 4.2 are neither minimal in $\mathbb{R}^7$ nor minimal in hyperspheres of $\mathbb{R}^7$. By exploiting the notion of $\mathbf{III}$-*deformation*, Vlachos [**5**] established a method that gives examples of full pseudo-umbilical surfaces in $\mathbb{R}^4$ in the same conditions. Next we apply theorem 4.2 in order to obtain an example of a full pseudo-umbilical surface in $\mathbb{R}^5$.

Example 4.2. Let $W = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $\varphi : \mathbb{C} \to S^3 = S^6 \cap W$ be the Clifford torus $\varphi(x, y) = \frac{1}{\sqrt{2}}\big(\cos x\, \mathbf{e}_1 + \sin x\, \mathbf{e}_2 + \cos y\, \mathbf{e}_3 + \sin y\, \mathbf{e}_4\big)$. The corresponding immersion $F : \mathbb{C} \to \mathbb{R}^7$ is given by

$$F(x, y) = \frac{1}{2}(\cos x \sin y\, \mathbf{e}_1 + \sin x \sin y\, \mathbf{e}_2 - y\, \mathbf{e_3} \\ + \sin x \cos y\, \mathbf{e}_5 - \cos x \cos y\, \mathbf{e}_6 + x\, \mathbf{e}_7). \tag{4.2}$$

By theorem 4.2, since W admits $\times$-compatible decompositions and φ is superconformal in $S^3 = S^6 \cap W$, the immersion F is pseudo-umbilical with a non-parallel mean curvature vector field. It can be shown (by straightforward computation of the derivatives) that F is full in some 5-dimensional subspace of $\mathbb{R}^7$.

Taking theorem 4.1 and theorem 4.2 into account, we also can obtain examples of isometric deformations of CMC surfaces into pseudo-umbilical surfaces as follows. Let φ be a superconformal harmonic map from Σ into $S^3 = S^6 \cap W$, where W is a coassociative 4-space. Let $g : [0, a] \to SO(7)$ be a smooth map with $g(0)$ the identity of $SO(7)$ and $g(a)W = \tilde{W}$, where W is a coassociative 4-space and $\tilde{W}$ is a 4-space admitting $\times$-compatible decompositions. Then we have a one-parameter family of congruent superconformal harmonic maps: for each $\lambda \in [0, a]$, $\varphi_\lambda = g(\lambda)\varphi : \Sigma \to g(\lambda)W$. We can integrate in order to obtain a smooth one-parameter family of conformal immersions F_λ (with $\mathbf{I}_{F_0} = \mathbf{I}_{F_\lambda}$ for all λ, because $\{\varphi_\lambda\}$ is a one-parameter family of congruent harmonic maps), where F_0 is a CMC surface and F_a is a pseudo-umbilical surface with non-parallel mean curvature vector field.

Example 4.3. Let $g : [0, \frac{\pi}{2}] \to SO(7)$ be the smooth map defined as follows: for each $\lambda \in [0, \frac{\pi}{2}]$, $g(\lambda) = [\mathbf{v}_1(\lambda), \dots, \mathbf{v}_7(\lambda)]$ is the matrix (written with respect to the canonical basis of $\mathbb{R}^7$) whose columns are the vectors

$$\begin{aligned}
&\mathbf{v}_1(\lambda) = \cos\lambda\, \mathbf{e}_1 + \sin\lambda\, \mathbf{e}_7, && \mathbf{v}_2(\lambda) = \cos\lambda\, \mathbf{e}_2 - \sin\lambda\, \mathbf{e}_5,\\
&\mathbf{v}_3(\lambda) = \cos\lambda\, \mathbf{e}_3 - \sin\lambda\, \mathbf{e}_6, && \mathbf{v}_4(\lambda) = \sin^2\lambda\, \mathbf{e}_1 + \cos\lambda\, \mathbf{e}_4 - \sin\lambda\cos\lambda\, \mathbf{e}_7,\\
&\mathbf{v}_5(\lambda) = \cos\lambda\, \mathbf{e}_5 + \sin\lambda\, \mathbf{e}_2, && \mathbf{v}_6(\lambda) = \cos\lambda\, \mathbf{e}_6 + \sin\lambda\, \mathbf{e}_3,\\
&\mathbf{v}_7(\lambda) = -\sin\lambda\cos\lambda\, \mathbf{e}_1 + \sin\lambda\, \mathbf{e}_4 + \cos^2\lambda\, \mathbf{e}_7.
\end{aligned}$$

It is clear that $g(0)$ is the identity of $SO(7)$ and $g(\frac{\pi}{2})W = \tilde{W}$, with

$$W = \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}, \quad \tilde{W} = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

Hence, after integration, we obtain the following isometric deformation of the cylinder (4.1) into the pseudo-umbilical surface (4.2):

$$F_\lambda(x,y) = \frac{1}{2}\big(-\cos x \sin y \mathbf{v}_4(\lambda) \times \mathbf{v}_6(\lambda) + \cos x \cos y \mathbf{v}_4(\lambda) \times \mathbf{v}_7(\lambda) \\ - \sin x \sin y \mathbf{v}_5(\lambda) \times \mathbf{v}_6(\lambda) + \sin x \cos y \mathbf{v}_5(\lambda) \times \mathbf{v}_7(\lambda) \\ + x \mathbf{v}_6(\lambda) \times \mathbf{v}_7(\lambda) - y \mathbf{v}_4(\lambda) \times \mathbf{v}_5(\lambda)\big).$$

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