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TOP DENSE HYPERBOLIC BALL PACKINGS AND COVERINGS FOR COMPLETE COXETER ORTHOSCHEME GROUPS

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Abstract. In n-dimensional hyperbolic space $\mathbf{H}^n$ $(n \geqslant 2)$, there are three types of spheres (balls): the sphere, horosphere and hypersphere. If $n = 2, 3$ we know a universal upper bound of the ball packing densities, where each ball's volume is related to the volume of the corresponding Dirichlet–Voronoi (D-V) cell. E.g., in $\mathbf{H}^3$ a densest (not unique) horoball packing is derived from the $\{3, 3, 6\}$ Coxeter tiling consisting of ideal regular simplices $T^{\infty}_{\mathrm{reg}}$ with dihedral angles $\frac{\pi}{3}$. The density of this packing is $\delta_3^{\infty} \approx 0.85328$ and this provides a very rough upper bound for the ball packing densities as well. However, there are no "essential" results regarding the "classical" ball packings with congruent balls, and for ball coverings either.

The goal of this paper is to find the extremal ball arrangements in $\mathbf{H}^3$ with "classical balls". We consider only periodic congruent ball arrangements (for simplicity) related to the generalized, so-called *complete Coxeter orthoschemes* and their extended groups. In Theorems 1.1 and 1.2 we formulate also conjectures for the densest ball packing with density $0.77147\ldots$ and the loosest ball covering with density $1.36893\ldots$, respectively. Both are related with the extended Coxeter group $(5, 3, 5)$ and the so-called hyperbolic football manifold. These facts can have important relations with fullerenes in crystallography.

1. Introduction

Ball packing problems concern the arrangements of non-overlapping equal balls packed in space. Usually, space is the classical three-dimensional Euclidean space $\mathbf{E}^3$. However, ball packing problems can be generalized to the other 3-dimensional Thurston geometries (see e.g., [**12**]).

In an n-dimensional space of constant curvature $\mathbf{E}^n$, $\mathbf{H}^n$, $\mathbf{S}^n$ $(n \geqslant 2)$ let $d_n(r)$ be the density of $n + 1$ congruent balls of radius r mutually touching one another with respect to the simplex spanned by the centres of the balls. Fejes Tóth and Coxeter [**5**] conjectured that in an n-dimensional space of constant curvature the density of packing balls of radius r cannot exceed $d_n(r)$. This conjecture has been proved by Rogers [**15**] in Euclidean n-space. The 2-dimensional case has been solved

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by Fejes Tóth [**5**]. In a three-dimensional space of constant curvature the problem has been investigated (and solved it in some sense) by Böröczky and Florian [**2**] and it has been studied by Böröczky [**1**] for n-dimensional spaces of constant curvature ($n \geqslant 4$). The upper bound $d_n(\infty)$ for $\mathbf{H}^n$ ($n = 2, 3$) is attained for a regular horoball packing, that is, a packing by horoballs inscribed in the cells of a regular honeycomb of $\overline{\mathbf{H}}^n$ (i.e., $\mathbf{H}^n$ is closed by its ideal points, or ends).

In $\overline{\mathbf{H}}^3$ there is exactly one horoball packing with horoballs in the same type whose Dirichlet–Voronoi cells give rise to a regular honeycomb described by the Schläfli symbol $\{6,\ 3,\ 3\}$. Its dual $\{3,3,6\}$ consists of ideal regular simplices T^{∞}_{reg} with dihedral angles $\frac{\pi}{3}$ building up a 6-cycle around each edge of the tessellation. The density of this packing is $\delta_3^{\infty} \approx 0.8532885328$ that attends at other horoball packings as well [**9**]. We have considered some new aspects related to the horoball and hyperball packings in [**16, 17, 18, 11, 10**].

However, there are no "essential" results regarding the "classical ball packings and coverings" with congruent balls. What are the extremal ball arrangements in $\mathbf{H}^n$ and what are their densities?

The goal of this paper is to study the above problems in $\mathbf{H}^3$ with "classical balls". We consider periodic congruent ball packings and coverings (for simplicity and for good constructions, only) related to the generalized, so-called complete Coxeter orthoschemes and their extended groups. We formulate two theorems and conjectures for the densest ball packing with density $0.77147\ldots$, and for the loosest ball covering with density $1.36893\ldots$, respectively.

Definition 1.1. For a given packing of $\mathbf{H}^n$ ($n \geqslant 2$) each ball's volume is related to the volume of its D-V cell, (i.e., the closed domain whose any point lies not farther from the given ball's centre than from any other centre of the ball system), then take the infimum of these ratios for all balls; this is called the density of the given packing. Then the supremum of these infima is taken for all ball packings of $\mathbf{H}^n$ to get the maximum packing density and the densest packing of $\mathbf{H}^n$ (if such a packing exists at all).

Theorem 1.1. *Among the extended complete Coxeter orthoscheme groups the group* $(5,3,5)$*, extended by a half-turn, provides the ball centre orbit of* A_3 *for the densest ball packing by its football D-V-cell* $\{5,6,6\}$ *and inscribed ball, (see Fig. 1 and 3). The density of this ball packing is* $0.77147\ldots$ *with radius* $r = 0.95142\cdots = A_3F_{03}$.

This is the conjectured maximal density for all ball packings of $\mathbf{H}^3$.

Definition 1.2. For a given covering of $\mathbf{H}^n$ each ball's volume is related to the volume of its D-V cell, than take the supremum of these ratios for all balls (this is called the density of the given covering); then the infimum of these suprema is taken for all ball coverings of $\mathbf{H}^n$ to get the minimal covering density and the loosest covering of $\mathbf{H}^n$ (if such a covering exists at all).

Theorem 1.2. *Among the above groups the same (with half-turn extended* $(5,3,5)$ *group) provides the same (congruent) ball centre orbit of* A_3 *for the loosest ball covering, again by its football D-V-cell* $\{5,6,6\}$ *but with its circumscribed* *ball, see Figs.* 1 *and* 3. *The density of this ball covering is* $1.36893\dots$ *with radius* $R = 1.12484\dots = A_3F_{12}$.

This is the conjectured minimal density for all ball coverings of $\mathbf{H}^3$.

Our numerical results are collected in Tables, where we give only the relevant data (because of the page restriction of this paper).

Our systematic computations will give the Proof of Theorems 1.1-1.2.

We will use the well-known Beltrami-Cayley-Klein model of $\mathbf{H}^3$ with the classical projective metric calculus (see e.g., [**12**]).

2. The projective model and complete orthoschemes

We use for $\mathbf{H}^3$ (and analogously for $\mathbf{H}^n$, $n > 3$) the projective model in the Lorentz space $\mathbf{E}^{1,3}$ whose real vector space $\mathbf{V}^4$ equipped with the bilinear form of signature $(1,3)$, $\langle \mathbf{x},\ \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$, where the non-zero vectors $\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbf{V}^4$ and $\mathbf{y} = (y^0, y^1, y^2, y^3) \in \mathbf{V}^4$, are determined up to real factors, for representing points of $\mathcal{P}^n(\mathbf{R})$. Then $\mathbf{H}^3$ can be interpreted as the interior of the quadric $Q = \{(\mathbf{x}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{x}\rangle = 0\} =: \partial\mathbf{H}^3$ in the real projective space $\mathcal{P}^3(\mathbf{V}^4, \boldsymbol{V}_4)$ (here $\boldsymbol{V}_4$ is the dual space of $\mathbf{V}^4$). Namely, for an interior point $Y(\mathbf{y})$ holds $\langle \mathbf{y},\ \mathbf{y}\rangle < 0$.

The points of the boundary $\partial\mathbf{H}^3$ in $\mathcal{P}^3$ are called points at infinity, or at the absolute of $\mathbf{H}^3$. The points lying outside $\partial\mathbf{H}^3$ are said to be outer points of $\mathbf{H}^3$ relative to Q. Let $(\mathbf{x}) \in \mathcal{P}^3$, a point $(\mathbf{y}) \in \mathcal{P}^3$ is said to be conjugate to $(\mathbf{x})$ if $\langle \mathbf{x},\ \mathbf{y}\rangle = 0$ holds. The set of all points which are conjugate to $(\mathbf{x})$ form a (polar) hyperplane $pol(\mathbf{x}) := \{(\mathbf{y}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{y}\rangle = 0\}$. Thus the quadric Q induces a bijection (linear polarity $\mathbf{V}^4 \to \boldsymbol{V}_4$) from the points of $\mathcal{P}^3$ onto their polar hyperplanes.

The point $X(\mathbf{x})$ and the hyperplane $\alpha(\boldsymbol{a})$ are incident if $\mathbf{x}\boldsymbol{a} = 0$ ($\mathbf{x} \in \mathbf{V}^4 \setminus \{\mathbf{0}\}$, $\boldsymbol{a} \in \boldsymbol{V}_4 \setminus \{\mathbf{0}\}$). The constant $k = \sqrt{-1/K}$ is the natural length unit in $\mathbf{H}^3$, where K denotes the constant negative sectional curvature. In the following we may assume that $k = 1$.

2.1. Characteristic orthoschemes and their volumes. *An orthoscheme* $\mathcal{O}$ *in* $\mathbf{H}^n$ $n \geqslant 2$ *in classical sense* is a simplex bounded by $n+1$ hyperplanes $H_0, \dots, H_n$ such that $H_i \perp H_j$, for $j \neq i-1, i, i+1$. Or, equivalently, the $n+1$ vertices of $\mathcal{O}$ can be labelled by $A_0, A_1, \dots, A_n$ in such a way that $\operatorname{span}(A_0, \dots, A_i) \perp \operatorname{span}(A_i, \dots, A_n)$ for $0 < i < n-1$.

Geometrically, complete orthoschemes of degree $m = 0, 1, 2$ can be described as follows:

(1) For $m = 0$, they coincide with the class of classical orthoschemes introduced by Schläfli. The initial and final vertices, A_0 and A_n of the orthogonal edge-path A_iA_{i+1}, $i = 0, \dots, n-1$, are called principal vertices of the orthoscheme (see Remark 4.1).

(2) A complete orthoscheme of degree $m = 1$ can be constructed from an orthoscheme with one outer principal vertex, one of A_0 or A_n, which is simply truncated by its polar plane (see Fig. 1-2).

(3) A complete orthoscheme of degree $m = 2$ can be constructed from an orthoscheme with two outer principal vertices, A_0 and A_n, which is doubly truncated by their polar planes $pol(A_0)$ and $pol(A_n)$ (see Figs. 1 and 2).

For the *complete Coxeter orthoschemes* $\mathcal{O} \subset \mathbf{H}^n$, we adopt the usual conventions and sometimes even use them in the Coxeter case: If two nodes are related by the weight $\cos\frac{\pi}{p}$, then they are joined by a $(p-2)$-fold line for $p = 3,\ 4$ and by a single line marked by p for $p \geqslant 5$. In the hyperbolic case if two bounding hyperplanes of S are parallel, then the corresponding nodes are joined by a line marked ∞. If they are divergent then their nodes are joined by a dotted line.

In the following, we concentrate only on dimensions 3 and on hyperbolic Coxeter–Schläfli symbol of the complete orthoscheme tiling $\mathcal{P}$ generated by reflections in the planes of a complete orthoscheme $\mathcal{O}$. To every scheme there is a corresponding symmetric 4×4 matrix (b^{ij}) where $b^{ii} = 1$ and, for $i \neq j \in \{0, 1, 2, 3\}$, b^{ij} equals $-\cos\alpha_{ij}$ with all angles α_{ij} between the faces i, j of $\mathcal{O}$.

For example, (b^{ij}) below is the so called Coxeter–Schläfli matrix with parameters $(u; v; w)$, i.e., $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$ to be discussed yet for hyperbolicity. Now only $3 \leqslant u, v, w$ come into account (see [**6, 7**]). Then

$$(b^{ij}) = \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle := \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & 0 & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{v} & 0 \\ 0 & -\cos\frac{\pi}{v} & 1 & -\cos\frac{\pi}{w} \\ 0 & 0 & -\cos\frac{\pi}{w} & 1 \end{pmatrix}. \tag{2.1}$$

This 3-dimensional complete (truncated or frustum) orthoscheme $\mathcal{O} = W_{uvw}$ and its reflection group $\mathbf{G}_{uvw}$ is depicted in Fig. 2, and by the symmetric Coxeter–Schläfli matrix (b^{ij}) in formula (2.1), furthermore by its inverse matrix (a_{ij}) in formula (2.2).

$$\begin{aligned} (a_{ij}) &= (b^{ij})^{-1} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle := \\ &= \frac{1}{B} \begin{pmatrix} \sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} & \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} \\ \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \sin^2\frac{\pi}{w} & \cos\frac{\pi}{v} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{v} & \sin^2\frac{\pi}{u} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} & \sin^2\frac{\pi}{u} - \cos^2\frac{\pi}{v} \end{pmatrix}, \end{aligned} \tag{2.2}$$

where

$$B = \det(b^{ij}) = \sin^2\frac{\pi}{u}\sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} < 0, \ \text{ i.e., } \ \sin\frac{\pi}{u}\sin\frac{\pi}{w} - \cos\frac{\pi}{v} < 0.$$

In what follows, we use the above orthoscheme whose volume is derived by the next Theorem of Kellerhals ([**8**], by the ideas of Lobachevsky):

THEOREM 2.1 (Kellerhals). *The volume of a three-dimensional hyperbolic complete orthoscheme $\mathcal{O} = W_{uvw} \subset \mathbf{H}^3$ is expressed with the essential angles $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$, $(0 \leqslant \alpha_{ij} \leqslant \frac{\pi}{2})$ (Fig. 1.a, b) in the following form:*

$$\begin{aligned} \mathrm{Vol}(\mathcal{O}) = \frac{1}{4}\{&\mathcal{L}(\alpha_{01} + \theta) - \mathcal{L}(\alpha_{01} - \theta) + \mathcal{L}(\frac{\pi}{2} + \alpha_{12} - \theta) \\ &+ \mathcal{L}(\frac{\pi}{2} - \alpha_{12} - \theta) + \mathcal{L}(\alpha_{23} + \theta) - \mathcal{L}(\alpha_{23} - \theta) + 2\mathcal{L}(\frac{\pi}{2} - \theta)\}, \end{aligned}$$

where $\theta \in [0, \frac{\pi}{2})$ *is defined by*

$$\tan(\theta) = \frac{\sqrt{\cos^2 \alpha_{12} - \sin^2 \alpha_{01} \sin^2 \alpha_{23}}}{\cos \alpha_{01} \cos \alpha_{23}},$$

and where $\mathcal{L}(x) := -\int_0^x \log|2\sin t|\, dt$ *denotes the Lobachevsky function (in J. Milnor's interpretation).*

The volume $\mathrm{Vol}(B(R))$ of a *ball* $B(R)$ of radius R can be computed by the classical formula of J. Bolyai:

$$\begin{aligned} \mathrm{Vol}(B(R)) &= 2\pi(\cosh(R)\sinh(R) - R) = \pi(\sinh(2R) - 2R) \\ &= \frac{4}{3}\pi R^3\Big(1 + \frac{1}{5}R^2 + \frac{2}{105}R^4 + \cdots\Big). \end{aligned} \tag{2.3}$$

3. Essential points in a complete (truncated) orthoscheme

Let $A_0(\mathbf{a}_0)$, $A_1(\mathbf{a}_1)$, $A_2(\mathbf{a}_2)$, $A_3(\mathbf{a}_3)$ be the vertices of the above complete orthoscheme W_{uvw} (see Fig. 1,2). The principal vertices A_0 and A_3 can be proper ($a_{ii} < 0$, $i \in \{0,3\}$), boundary ($a_{ii} = 0$) or outer points ($a_{ii} > 0$) We exploit the logical symmetry $0, 1 \leftrightarrow 3, 2$.

We distinguish the following main configurations of the principal vertices A_0 and A_3:

1. A_3 *is proper or boundary point* $\frac{\pi}{u} + \frac{\pi}{v} \geqslant \frac{\pi}{2}$.
- **1.i.** A_0 is proper or boundary point $\frac{\pi}{v} + \frac{\pi}{w} \geqslant \frac{\pi}{2}$.
- **1.s.i** $u = w$, $F_{03}F_{12}$ is half turn axis, $\mathbf{h}$ is the half turn changing $0 \leftrightarrow 3$, $1 \leftrightarrow 2$. Here a "half orthoscheme" $JQEB_{13}F_{12}B_{02}F_{03}A_2$ will be the fundamental domain of $\mathbf{G}_{u=w,v}$.
- **1.ii** A_0 is outer $\frac{\pi}{v} + \frac{\pi}{w} < \frac{\pi}{2}$, then $a_0(\boldsymbol{a}_0) = CLH$ is its polar plane.

2. A_3 *is outer point* $\frac{\pi}{u} + \frac{\pi}{v} < \frac{\pi}{2}$*, then* $a_3(\boldsymbol{a}_3) = JEQ$ *is its polar plane.*
- **2.i.** A_0 is proper or boundary point $\frac{\pi}{v} + \frac{\pi}{w} \geqslant \frac{\pi}{2}$.
- **2.ii** A_0 is also outer $\frac{\pi}{v} + \frac{\pi}{w} < \frac{\pi}{2}$, then $a_0(\boldsymbol{a}_0) = CLH$ is its polar plane.
- **2.s.ii** $u = w$, $F_{03}F_{12}$ is half turn axis, $\mathbf{h}$ is the half turn changing $0 \leftrightarrow 3$, $1 \leftrightarrow 2$. Here a "half orthoscheme" $JQEB_{13}F_{12}B_{02}F_{03}A_2$ will be the fundamental domain of $\mathbf{G}_{u=w,v}$.

We obtain with easy calculations the following inportant lemmas (see Fig. 1,2):

LEMMA 3.1. *Let* A_0 *be an outer principal vertex of the orthoscheme* W_{uvw} *and let* $a_0(\mathbf{a}_0) = CLH$ *be its polar plane where* $C = a_0 \cap A_0A_1$, $L = a_0 \cap A_0A_2$, $H = a_0 \cap A_0A_3$ *whose vectors are*

$$C(\mathbf{c}) = a_0 \cap A_0A_1;\ \mathbf{c} = \mathbf{a}_1 - \frac{a_{01}}{a_{00}}\mathbf{a}_0,\ \textit{with}$$

$$\langle \mathbf{c}, \mathbf{c} \rangle = \frac{(a_{11}a_{00} - a_{01}^2)}{a_{00}} = \langle \mathbf{c}, \mathbf{a}_1 \rangle = \frac{\sin^2 \frac{\pi}{w}}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{a_{11}}{a_{00}}$$

$$L(\mathbf{l}) = a_0 \cap A_0A_2;\ \mathbf{l} = \mathbf{a}_2 - \frac{a_{02}}{a_{00}}\mathbf{a}_0,\ \textit{with}$$

$$\langle \mathbf{l}, \mathbf{l} \rangle = \frac{(a_{22}a_{00} - a_{02}^2)}{a_{00}} = \langle \mathbf{l}, \mathbf{a}_2 \rangle = \frac{1}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{1}{Ba_{00}} \tag{3.1}$$

$$H(\mathbf{h}) = a_0 \cap A_0A_3; \ \mathbf{h} = \mathbf{a}_3 - \frac{a_{03}}{a_{00}}\mathbf{a}_0, \ \textit{with}$$

$$\langle \mathbf{h}, \mathbf{h} \rangle = \frac{(a_{33}a_{00} - a_{03}^2)}{a_{00}} = \langle \mathbf{h}, \mathbf{a}_3 \rangle = \frac{\sin^2 \frac{\pi}{v}}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{\sin^2 \frac{\pi}{v}}{Ba_{00}}$$

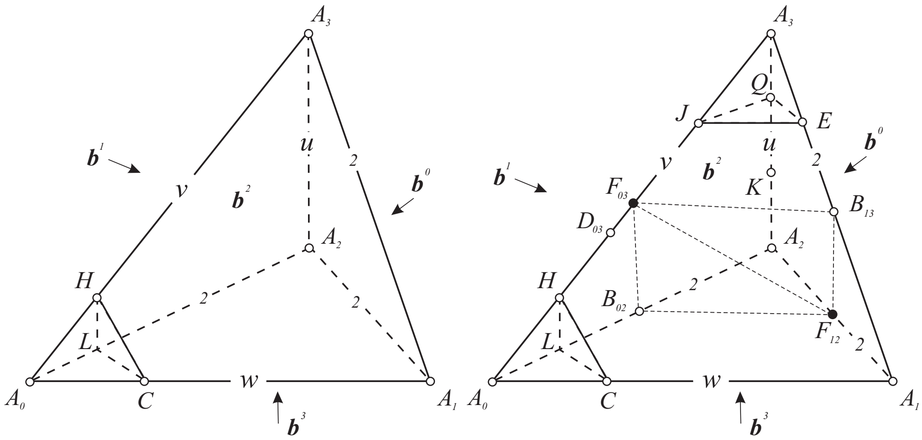

FIGURE 1. Simple and double truncated complete orthoschemes

LEMMA 3.2. *Let A_3 be an outer principal vertex of the orthoscheme W_{uvw} and let $a_3(\boldsymbol{a}_3) = JEQ$ be its polar plane where $J = a_3 \cap A_3A_0$, $E = a_3 \cap A_3A_1$, $Q = a_3 \cap A_3A_2$ whose vectors are*

$$J(\mathbf{j}) = a_3 \cap A_3A_0; \ \mathbf{j} = \mathbf{a}_0 - \frac{a_{03}}{a_{33}}\mathbf{a}_3, \ \textit{with}$$

$$\langle \mathbf{j}, \mathbf{j} \rangle = \frac{(a_{00}a_{33} - a_{03}^2)}{a_{33}} = \langle \mathbf{j}, \mathbf{a}_0 \rangle = \frac{\sin^2 \frac{\pi}{v}}{\sin^2 \frac{\pi}{u} - \cos^2 \frac{\pi}{v}} = \frac{\sin^2 \frac{\pi}{v}}{Ba_{33}}$$

$$E(\mathbf{e}) = a_3 \cap A_3A_1; \ \mathbf{e} = \mathbf{a}_1 - \frac{a_{13}}{a_{33}}\mathbf{a}_3, \ \textit{with}$$

$$\langle \mathbf{e}, \mathbf{e} \rangle = \frac{(a_{11}a_{33} - a_{13}^2)}{a_{33}} = \langle \mathbf{e}, \mathbf{a}_1 \rangle = \frac{1}{\sin^2 \frac{\pi}{u} - \cos^2 \frac{\pi}{v}} = \frac{1}{Ba_{33}} \tag{3.2}$$

$$Q(\mathbf{h}) = a_3 \cap A_3A_2; \ \mathbf{q} = \mathbf{a}_2 - \frac{a_{23}}{a_{33}}\mathbf{a}_3, \ \textit{with}$$

$$\langle \mathbf{q}, \mathbf{q} \rangle = \frac{(a_{22}a_{33} - a_{23}^2)}{a_{33}} = \langle \mathbf{q}, \mathbf{a}_2 \rangle = \frac{\sin^2 \frac{\pi}{u}}{\sin^2 \frac{\pi}{u} - \cos^2 \frac{\pi}{v}} = \frac{a_{22}}{a_{33}}.$$

LEMMA 3.3. *The midpoint $K(\mathbf{k})$ (see Fig. 1) of A_2Q can be determined by the following vector:*

$$\mathbf{k} = \frac{\mathbf{a}_2}{\sqrt{-a_{22}}} + \frac{\mathbf{q}}{\sqrt{-\langle \mathbf{q}, \mathbf{q} \rangle}} \quad \textit{with } \langle \mathbf{k}, \mathbf{k} \rangle = -2\Big(1 + \sqrt{\frac{1}{a_{33}}}\Big). \tag{3.3}$$

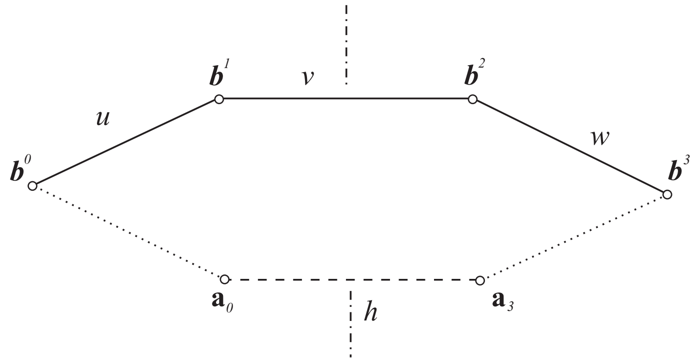

FIGURE 2.

Especially if $u = w$, the midpoints F_{03} of A_0A_3 and F_{12} of A_1A_2 can play important roles, since $F_{03}F_{12}$ will be the axis of half turn

$$\mathbf{h}: \ 0 \leftrightarrow 3, 1 \leftrightarrow 2, \ \text{i.e. } A_0 \leftrightarrow A_3, \ \boldsymbol{b}^0 \leftrightarrow \boldsymbol{b}^3, \ A_1 \leftrightarrow A_2, \ \boldsymbol{b}^1 \leftrightarrow \boldsymbol{b}^2.$$

(Here $a_{00} = a_{33}$ and $a_{11} = a_{22}$ hold, of course.)

LEMMA 3.4. *The midpoints $F_{03}(\mathbf{f}_{03})$ of $A_3A - 0$ and $F_{12}(\mathbf{f}_{12})$ of A_1A_2 (see Fig. 1) can be determined by the following vectors:*

$$\mathbf{f}_{03} = \mathbf{a}_0 + \mathbf{a}_3, \ \langle \mathbf{f}_{03}, \mathbf{f}_{03} \rangle = 2(a_{00} + a_{03}) < 0,$$
$$\mathbf{f}_{12} = \mathbf{a}_1 + \mathbf{a}_2, \ \langle \mathbf{f}_{12}, \mathbf{f}_{12} \rangle = 2(a_{11} + a_{12}) < 0,$$

independently of A_0 and A_3 both are either proper, boundary or outer points.

The perpendicular foot point $Y(\mathbf{y})$, dropped onto a plane $(\boldsymbol{u})$ from point $X(\mathbf{x})$ is given by

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}, \tag{3.4}$$

where $(\mathbf{u})$ is the pole of the plane $(\boldsymbol{u})$. In the considered cases the planes are the faces of the orthoscheme W_{uvw}, therefore the poles $\boldsymbol{b}_*^i = \mathbf{b}^i = b^{ij}\mathbf{a}_j$ play important roles in the computations that are derived by matrix (2.1):

$$\begin{aligned} \mathbf{b}^0 &= \mathbf{a}_0 - \cos\frac{\pi}{u}\mathbf{a}_1, \ \mathbf{b}^1 = -\cos\frac{\pi}{u}\mathbf{a}_0 + \mathbf{a}_1 - \cos\frac{\pi}{v}\mathbf{a}_2, \\ \mathbf{b}^2 &= -\cos\frac{\pi}{v}\mathbf{a}_1 + \mathbf{a}_2 - \cos\frac{\pi}{w}\mathbf{a}_3, \ \mathbf{b}^3 = -\cos\frac{\pi}{w}\mathbf{a}_2 + \mathbf{a}_3. \end{aligned} \tag{3.5}$$

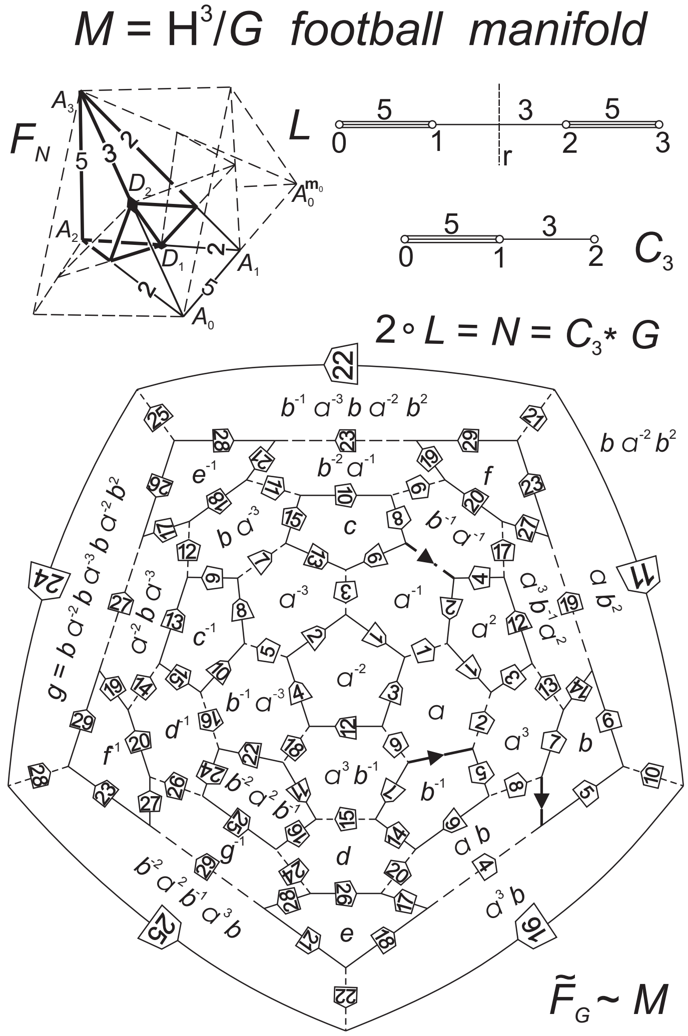

FIGURE 3.

E.g. the foot point $F^2_{12}(\mathbf{f}^2_{12})$ of $\mathbf{f}_{12} = \mathbf{a}_1 + \mathbf{a}_2$ onto the plane $\boldsymbol{b}^2$ (see Fig. 1) is

$$\mathbf{f}^2_{12} = \mathbf{f}_{12} + c\mathbf{b}^2 = (1 + \cos\frac{\pi}{v})\mathbf{a}_1 + \cos\frac{\pi}{w}\mathbf{a}_3, \text{ by } \mathbf{f}^2_{12}\boldsymbol{b}^2 = 0,$$

thus for coefficient $c = -1$. We similarly get the other foot points on the simplex side planes $\boldsymbol{b}^i$ $i \in \{0, 1, 2, 3\}$.

Dividing points of edges of W_{uvw} (or of its half domain) e.g., $D_{ij} \in A_iA_j$ $(i < j \in \{0, 1, 2, 3\})$ can also have special roles with more cumbersome calculations: $\mathbf{d}_{ij} = \mathbf{a}_i + d_{ij}\mathbf{a}_j$ will be interesting for appropriate coefficient d_{ij}. At least two locally minimal face distances from a D_{ij} occur for packing. Or at least two locally maximal vertex distances from a D_{ij} occur for covering. Maybe, we obtain different D_{ij}'s for locally maximal packing and locally minimal covering, respectively.

We leave out these lengthy discussions here. Only F_{03}, F_{12} and K will be mentioned in Sect. 4 at cases c, d and h, respectively.

Face points F^i on b^i or F_i on a_i can play similar roles (for at least three locally minimal face distances and at least three locally maximal vertex distances, respectively) for searching locally extremal densities, but we do not discuss details here, because they will not be relevant for the optimal packing and covering, respectively.

Extremal "incentre" I in the body of W_{uvw} for packing and "circumcentre" G in W_{uvw} for covering (for at least four locally minimal face distances and at least four locally maximal vertex distances, respectively) will not be discussed as well, because they provide only local extrema (far from our $(5, 3, 5)$ absolute optima (Table 1.s.i.a)), after lengthy but straightforward discussions.

4. Ball packings and coverings

For any fixed u, v, w, satisfying the hyperbolicity inequlities, we consider the competitive ball centres, as follow, for maximal packing density and minimal covering density. We indicate the centres where they occur, respectively. For the ball centre O we have roughly 8 competitive cases (Fig. 1):

a. A_3 is the ball centre, $|\operatorname{Stab}_{A_3}\mathbf{G}| = \frac{8uv}{4-(u-2)(v-2)}$,
b. A_2 is the ball centre, $|\operatorname{Stab}_{A_2}\mathbf{G}| = 4u$,
c. F_{03} is the ball centre, $|\operatorname{Stab}_{F_{03}}\mathbf{G}| = 4v,$ if $u = w$,
d. F_{12} is the ball centre $|\operatorname{Stab}_{F_{12}}\mathbf{G}| = 8,$ if $u = w$,
e. Q is the ball centre, $|\operatorname{Stab}_{Q}\mathbf{G}| = 4u$,
f. J is the ball centre, $|\operatorname{Stab}_{J}\mathbf{G}| = 4v$,
g. E is the ball centre, $|\operatorname{Stab}_{E}\mathbf{G}| = 8$,
h. the midpoint K of A_2Q is the ball centre, $|\operatorname{Stab}_{K}\mathbf{G}| = 2u$.

K gives (only) estimates for the points of A_2Q, in general. Competitive packing ball radii r for r^{opt} with maximal density

$$(4.1)\quad \delta^{opt} = \frac{\operatorname{Vol}(B(r^{opt}))}{|\operatorname{Stab}_O\mathbf{G}|\operatorname{Vol}(W_{uvw})} \text{ or } \delta^{opt} = \frac{\operatorname{Vol}(B(r^{opt}))}{\frac{1}{2}|\operatorname{Stab}_O\mathbf{G}|\operatorname{Vol}(W_{uvw})} \text{ if } u = w.$$

Competive covering ball radii R for R^{opt} with minimal density:

$$(4.2)\quad \Delta^{opt} = \frac{\operatorname{Vol}(B(R^{opt}))}{|\operatorname{Stab}_O\mathbf{G}|\operatorname{Vol}(W_{uvw})} \text{ or } \Delta^{opt} = \frac{\operatorname{Vol}(B(R^{opt}))}{\frac{1}{2}|\operatorname{Stab}_O\mathbf{G}|\operatorname{Vol}(W_{uvw})} \text{ if } u = w.$$

4.1. Case 1.i.a. In this case A_3 is proper $O = A_3$ is the ball centre. A_O is either proper or boundary point. In the latter case covering is not defined. The optimal radii can be derived by the *Beltrami–Cayley–Klein* model (see Section 2, 3):

$$
\begin{aligned}
&r = A_3A_2;\ \cosh(A_3A_2) = \frac{-a_{23}}{\sqrt{a_{22}a_{33}}} = \frac{\sin\frac{\pi}{u}\cos\frac{\pi}{w}}{\sqrt{\sin^2\frac{\pi}{u}-\cos^2\frac{\pi}{v}}} = \sqrt{1-\frac{1}{a_{33}}},\\
&R = A_3A_0;\ \cosh(A_3A_0) = \frac{-a_{03}}{\sqrt{a_{00}a_{33}}} = \sqrt{1-\frac{\sin^2\frac{\pi}{v}}{Ba_{00}a_{33}}},
\end{aligned}
\tag{4.3}
$$

The densities of the ball packings and coverings for given parameters u, v, w can be computed by formulas (4.1) and (4.2), also later on.

Table 1.i.a, Packing, $O = A_3,\ \frac{1}{u}+\frac{1}{v}>\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3,3,6)$	0.34657	0.04229	0.17861	0.17598
$(4,3,5)$	0.53064	0.03589	0.66207	0.38437
$(4,3,6)$	0.65847	0.10572	1.30405	0.25697
$(5,3,4)$	0.80846	0.03589	2.52145	0.58554
$(5,3,5)$	0.99639	0.09333	5.04848	0.45080
$(5,3,6)$	1.08394	0.17150	6.73795	0.32740
$(3,4,4)$	0.65848	0.07633	1.30405	0.35592
$(3,5,3)$	0.86830	0.03905	3.18663	**0.68003**

Table 1.i.a, Covering, $O = A_3,\ \frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}>\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	Δ^{opt}
$(4,3,5)$	1.22646	0.03589	10.41484	6.04641
$(5,3,4)$	1.22646	0.03589	10.41484	**2.41856**
$(5,3,5)$	1.90285	0.09333	58.62658	5.23495
$(3,5,3)$	1.38257	0.03905	16.16044	3.44864

4.2. Case 1.s.i.a. $O = A_3$ is the ball centre and $u = w$. The density is related to the "half orthoscheme".

$$
\begin{aligned}
&r = \min\{A_3A_2, A_3F_{03}\};\ \text{where}\ \cosh(A_3A_2)\ \text{is in (4.3)},\\
&\cosh(A_3F_{03}) = \sqrt{\frac{a_{03}}{2a_{33}}+\frac{1}{2}} = \cosh\Big(\frac{1}{2}A_3A_0\Big),\\
&R = A_3F_{12};\ \cosh(A_3F_{12}) = \frac{-(a_{13}+a_{23})}{\sqrt{2a_{33}(a_{12}+a_{22})}} = \sqrt{\frac{1}{2}+\frac{a_{03}-1}{2a_{33}}},
\end{aligned}
\tag{4.4}
$$

Table 1.s.i.a, Packing, $O = A_3,\ \frac{1}{u}+\frac{1}{v}>\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(5,3,5)$	0.95142	0.09333	4.31988	**0.77147**
$(3,5,3)$	0.69129	0.03905	1.52220	0.64967

Table 1.s.i.a, Covering, $O = A_3,\ \frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}>\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(5,3,5)$	1.12484	0.09333	7.66539	**1.36893**
$(3,5,3)$	0.89558	0.03905	3.53002	1.50661

4.3. Case 1.i.b. $O = A_2$ is the ball centre.

$$(4.5)\qquad \begin{gathered} r = A_2b^2 =: A_2A_2^2;\ \cosh(A_2b^2) = \sqrt{1-\frac{1}{a_{22}}} \\ R = \max\{A_2A_3, A_2A_0\};\ \cosh(A_2A_3) = \sqrt{1-\frac{1}{a_{33}}}, \\ \cosh(A_2A_0) = \frac{-a_{02}}{\sqrt{a_{00}a_{22}}} = \frac{\cot\frac{\pi}{u}\cos\frac{\pi}{v}}{\sqrt{\sin^2\frac{\pi}{w}-\cos^2\frac{\pi}{v}}} = \sqrt{1-\frac{1}{Ba_{00}a_{22}}}. \end{gathered}$$

Table 1.i.b, Packing, $O = A_2$, $\frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$

$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(4,3,5)$	0.38360	0.03589	0.24350	0.42409
$(5,3,4)$	0.45682	0.03589	0.41631	**0.58007**
$(5,3,5)$	0.58157	0.09333	0.88150	0.47227
$(4,4,3)$	0.48121	0.07633	0.48886	0.40029
$(3,5,3)$	0.34346	0.03905	0.17377	0.37082

Table 1.i.b, Covering, $O = A_2$, $\frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}>\frac{1}{2}$

$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	Δ^{opt}
$(5,3,4)$	0.84248	0.03589	2.88536	**4.02028**

4.4. Case 1.s.i.b. $O = A_2$ is the ball centre. $u = w$. The density is related to the "half orthoscheme".

$$(4.6)\qquad \begin{gathered} r = \min\{A_2b^2, A_2F_{12}\};\ \text{where}\ \cosh(A_2b^2)\ \text{in (4.5)}, \\ \cosh(A_2F_{12}) = \sqrt{\frac{a_{12}}{2a_{22}}+\frac{1}{2}} = \cosh\Big(\frac{1}{2}A_2A_1\Big) = \sqrt{\frac{1}{2}+\frac{\cos\frac{\pi}{v}}{2\sin^2\frac{\pi}{u}}}, \\ R = \max\{A_2A_3, A_2F_{03}\};\ \cosh(A_2A_3) = \frac{-a_{23}}{\sqrt{a_{22}a_{33}}} = \\ \cosh(A_2F_{03}) = \frac{-(a_{02}+a_{23})}{\sqrt{2a_{22}(a_{33}+a_{03})}} = \cot\frac{\pi}{u}\sqrt{\frac{\cos\frac{\pi}{v}+\sin^2\frac{\pi}{u}}{2(1-\cos\frac{\pi}{v})}}. \end{gathered}$$

Table 1.s.i.b, Packing, $O = A_2$, $\frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$

$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(5,3,5)$	0.45682	0.09333	0.41631	**0.44609**

Table 1.s.i.b, Covering, $O = A_2$, $\frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$

$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(5,3,5)$	0.99639	0.09333	5.04848	**5.40954**

4.5. Case 1.s.i.c. $O = F_{03}$ is the ball centre, $u = w$.

$$(4.7)\qquad \begin{gathered} r = F_{03}b^0 = F_{03}b^3;\ \text{where}\ \cosh(F_{03}b^0) = \sqrt{1+\frac{\cos\frac{\pi}{v}-\sin^2\frac{\pi}{u}}{2(1-\cos\frac{\pi}{v})}}, \\ R = \max\{F_{03}A_3, F_{03}A_2\} = F_{03}A_3;\ \cosh(F_{03}A_3) = \sqrt{\frac{a_{03}}{2a_{33}}+\frac{1}{2}}. \end{gathered}$$

Table 1.s.i.c, Packing, $O = F_{03}$, $u = w$, $\frac{1}{u} + \frac{1}{v} \geqslant \frac{1}{2}$, $\frac{1}{v} + \frac{1}{w} \geqslant \frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(4,4,4)$	0.56419	0.22899	0.80163	0.43759
$(3,5,3)$	0.38360	0.03905	0.24350	**0.62355**
$(3,6,3)$	0.61795	0.16916	1.06673	0.52551

Table 1.s.i.c, Covering, $O = F_{03}$, $u = w$, $\frac{1}{u} + \frac{1}{v} \geqslant \frac{1}{2}$, $\frac{1}{v} + \frac{1}{w} > \frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3,5,3)$	0.69129	0.03905	1.52220	**3.89804**

4.6. Case 1.s.i.d. $O = F_{12}$ is the ball centre and $u = w$.

$$
\begin{aligned}
&r = F_{12}b^1 = F_{12}b^2; \text{ where } \cosh(F_{12}b^2) = \sqrt{1 + \frac{\cos\frac{\pi}{v} - \sin^2\frac{\pi}{u}}{2}}, \\
&R = A_3F_{12};\ \cosh(A_3F_{12}) = \frac{-(a_{13}+a_{23})}{\sqrt{2a_{33}(a_{12}+a_{22})}} = \sqrt{\frac{1}{2} + \frac{a_{03}-1}{2a_{33}}}.
\end{aligned} \tag{4.8}
$$

We do not obtain relevant arrangements for optima.

4.7. Case 1.ii.a. In this case A_3 is proper $O = A_3$ is the ball centre. A_O is outer, $a_0 = CLH$ (see Fig. 1) is its polar plane.

$$
\begin{aligned}
&r = \min\{A_3A_2, A_3H\};\ \cosh(A_3A_2) \text{ is in (4.5)}, \\
&\cosh(A_3H) = \sqrt{1 - \frac{a_{03}^2}{a_{00}a_{33}}} = \frac{\sin\frac{\pi}{v}}{\sqrt{Ba_{00}a_{33}}}, \\
&R = A_3C;\ \cosh(A_3C) = \frac{a_{01}a_{03} - a_{00}a_{13}}{\sqrt{a_{00}a_{33}(a_{11}a_{00} - a_{01}^2)}} = \frac{\cot\frac{\pi}{w}\cos\frac{\pi}{v}}{\sqrt{a_{00}a_{33}B}} = \\
&= \sqrt{1 - \frac{1}{Ba_{11}a_{33}} - \frac{a_{03}^2}{a_{00}a_{33}}}.
\end{aligned} \tag{4.9}
$$

Table 1.ii.a, Packing, $O = A_3$, $\frac{1}{u} + \frac{1}{v} > \frac{1}{2}$, $\frac{1}{v} + \frac{1}{w} < \frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3,3,w\to\infty)$	$\mathrm{arsh}(\sqrt{2}/2) \approx$ ≈ 0.65848	0.15266	1.30405	**0.35592**
$(3,4,6)$	0.96242	0.19616	4.49014	**0.47687**
$(3,4,w\to\infty)$	$\log(1+\sqrt{2}) \approx$ ≈ 0.88137	0.25096	3.34793	0.27793
$(3,5,4)$	1.30631	0.21299	13.09457	0.51233
$(3,5,5)$	1.40036	0.26320	16.95557	**0.53684**
$(3,5,w\to\infty)$	$-\mathrm{arsh}\left(\frac{-1}{\sqrt{2-2\cos(\pi/5)}}\right) \approx$ ≈ 1.25850	0.33233	11.43025	0.28662
$(4,3,w\to\infty)$	$\log(1+\sqrt{2}) \approx$ ≈ 0.88137	0.25096	3.34793	**0.27793**
$(5,3,w\to\infty)$	$-\mathrm{arsh}\left(\frac{-1}{\sqrt{2-2\cos(\pi/5)}}\right) \approx$ ≈ 1.25850	0.33233	11.43025	**0.28662**

Table 1.ii.a, Covering, $O = A_3$, $\frac{1}{u}+\frac{1}{v}>\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$

$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3,3,8)$	1.30889	0.10721	13.18949	**5.12591**
$(3,4,5)$	1.53591	0.16596	24.17649	**3.03486**
$(3,5,4)$	1.93116	0.21299	62.56492	**2.44790**
$(3,5,5)$	2.08287	0.26320	88.11084	2.78974
$(4,3,8)$	1.81579	0.18790	47.88239	**5.30903**
$(5,3,8)$	2.37474	0.26094	166.52911	**5.31816**

4.8. Case 1.ii.b. $O = A_2$ is the ball centre.

$$\begin{gathered} r=\min\{A_2b^2, A_2L\};\ \text{where}\ \cosh(A_2b^2)=\sqrt{1-\frac{1}{a_{22}}}, \\ A_2a_0=A_2L;\ \cosh(A_2L)=\sqrt{1-\frac{a_{02}^2}{a_{00}a_{22}}}=\frac{1}{\sin u}\sqrt{\frac{B}{\sin^2\frac{\pi}{w}-\cos^2\frac{\pi}{v}}}, \\ R=\max\{A_2C, A_2A_3, A_2H\};\ \cosh(A_2C)=\frac{a_{01}a_{02}-a_{12}a_{00}}{\sqrt{a_{00}a_{22}(a_{11}a_{00}-a_{01}^2)}}= \\ =\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{u}\sin\frac{\pi}{w}\sqrt{a_{00}}}=\sqrt{1-\frac{1}{Ba_{11}a_{22}}-\frac{a_{02}^2a_{12}}{a_{00}a_{22}}}, \\ \cosh(A_2H)=\frac{a_{02}a_{03}-a_{23}a_{00}}{\sqrt{a_{00}a_{22}(a_{33}a_{00}-a_{03}^2)}}=\frac{\cot\frac{\pi}{w}}{\sin\frac{\pi}{v}\sqrt{a_{00}}}, \\ \cosh(A_2A_3)\ \text{is in (4.5)}. \end{gathered} \tag{4.10}$$

Table 1.ii.b, Packing, $O = A_2$, $\frac{1}{u}+\frac{1}{v}\geqslant\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$

$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3,4,6)$	0.60745	0.19616	1.01065	0.42934
$(3,4,7)$	0.64583	0.21218	1.22631	**0.48164**
$(3,3,w\to\infty)$	$\mathrm{arch}(2/\sqrt{3})\approx$ ≈ 0.54931	0.15266	0.73740	**0.40253**
$(3,5,5)$	0.67390	0.26320	1.40355	**0.44439**
$(3,5,w\to\infty)$	$\mathrm{arch}(2/\sqrt{3})\approx$ ≈ 0.54931	0.33233	0.73740	0.18491
$(4,3,w\to\infty)$	$\mathrm{arch}(\sqrt{6}/2)\approx$ ≈ 0.65848	0.25096	1.30405	**0.32477**
$(5,3,w\to\infty)$	$\mathrm{arch}\left(\frac{\sqrt{1+4\sin^2(\pi/5)}}{\sin(\pi/5)}\right)\approx$ ≈ 0.77173	0.33233	2.16804	**0.32619**
$(3,6,5)$	0.72182	0.35992	1.74787	**0.40469**
$(3,6,w\to\infty)$	$\mathrm{arch}(2/\sqrt{3})\approx$ ≈ 0.54931	0.42289	0.73740	0.14531
$(6,3,w\to\infty)$	$\mathrm{arch}(\sqrt{2})\approx$ ≈ 0.88137	0.42289	3.34793	**0.32987**
$(4,4,w\to\infty)$	$\mathrm{arch}(\sqrt{2})\approx$ ≈ 0.88137	0.45798	3.34793	**0.45689**

Table 1.ii.b, Covering, $O = A_2,\ \frac{1}{u}+\frac{1}{v}>\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}<\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3,3,8)$	1.16245	0.10721	8.60501	**6.68843**
$(3,4,5)$	1.14791	0.16596	8.23135	**4.13311**
$(3,5,4)$	1.30632	0.21299	13.09457	**5.12335**
$(4,3,9)$	1.55832	0.20295	25.59284	**7.88150**
$(5,3,9)$	1.80489	0.27783	46.67018	**8.39914**

4.9. Case 2.i.b. In this case A_3 is outer, $a_3 = JEQ$ is its polar plane, A_0 is proper or boundary point. $O = A_2$ is the ball centre (see Fig. 1).

$$
\begin{gathered}
r = \min\{A_2b^2, A_2a_3\};\ \cosh(A_2b^2) = \sqrt{1-\frac{1}{a_{22}}},\\
A_2a_3 = A_2Q;\ \cosh(A_2Q) = \frac{1}{\sqrt{a_{33}}} = \sqrt{1+\frac{\sin^2\frac{\pi}{u}\cos^2\frac{\pi}{w}}{\cos^2\frac{\pi}{v}-\sin^2\frac{\pi}{u}}},\\
R = \max\{A_2A_0, A_2J\};\ \cosh(A_2A_0) = \frac{-a_{02}}{\sqrt{a_{00}a_{22}}},\\
\cosh(A_2J) = \frac{a_{03}a_{23}-a_{02}a_{33}}{\sqrt{a_{33}a_{22}(a_{33}a_{00}-a_{03}^2)}} = \frac{\cot\frac{\pi}{u}\cot\frac{\pi}{v}}{\sqrt{a_{00}}}.
\end{gathered}
\tag{4.11}
$$

Table 2.i.b. Packing, $O = A_2,\ \frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}\geqslant\frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(7,3,3)$	0.70133	0.08856	1.59395	**0.64279**
$(7,3,4)$	0.81624	0.16297	2.60141	**0.57008**
$(7,3,5)$	0.87514	0.23326	3.27033	**0.50072**
$(5,4,3)$	0.69129	0.16596	1.52220	**0.45859**
$(4,5,3)$	0.69129	0.21299	1.52220	**0.44668**
$(7,3,6)$	0.90817	0.31781	3.69768	**0.41553**
$(5,4,4)$	0.86233	0.34084	3.11499	**0.45696**
$(4,6,3)$	0.65848	0.31717	1.30405	**0.25697**

Table 2.i.b. Covering, $O = A_2,\ \frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}>\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(8,3,3)$	1.12838	0.10721	7.75022	**2.25901**
$(7,3,4)$	1.36005	0.16297	15.19874	**3.33068**
$(7,3,5)$	1.88213	0.23326	55.88945	**8.55728**
$(5,4,3)$	1.28550	0.16596	12.34723	**3.71986**
$(4,5,3)$	1.61692	0.21299	29.64079	**8.69789**

The cases 2. i. e, f, g, h with $O = Q, J, E, K$, respectively, will be not relevant for optimal densities.

4.10. Case 2.ii.b. In this case A_3 is outer, $a_3 = JEQ$ is its polar plane, A_0 is outer with polar plane $a_0 = CLH$. $O = A_2$ is the ball centre (see Fig. 1).

$$
\begin{gathered}
r = \min\{A_2b^2, A_2Q, A_2L\};\ \cosh(A_2b^2) = \sqrt{1-\frac{1}{a_{22}}},\ \cosh(A_2Q) = \frac{1}{\sqrt{a_{33}}},\\
R = \max\{A_2C, A_2H, A_2J\};\ \cosh(A_2C) = \frac{a_{01}a_{02}-a_{12}a_{00}}{\sqrt{a_{00}a_{22}(a_{11}a_{00}-a_{01}^2)}},\\
\cosh(A_2H) = \frac{\cot\frac{\pi}{w}}{\sin\frac{\pi}{v}\sqrt{a_{00}}},\ \cosh(A_2J) = \frac{\cot\frac{\pi}{u}\cot\frac{\pi}{v}}{\sqrt{a_{00}}}.
\end{gathered}
\tag{4.12}
$$

Table 2.ii.b. Packing, $O = A_2$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(7,3,7)$	0.92836	0.38325	3.97899	**0.37080**
$(7,3,8)$	0.94156	0.41326	4.17152	**0.36051**
$(7,3,w\to\infty)$	$\text{arch}\left(\frac{\sqrt{5-4\cos^2(\pi/7)}}{\sqrt{1-\cos^2(\pi/7)}}\right)\approx 0.98513$	0.49195	4.85782	0.35267
$(5,4,5)$	0.91604	0.46190	3.80539	**0.41193**
$5,4,w\to\infty)$	$\text{arch}\left(\frac{(3-2\cos^2(\pi/5))}{\sqrt{2(1-\cos^2(\pi/5)}}\right)\approx 1.01789$	0.59404	5.42887	**0.45694**

Table 2.ii.b. Covering, $O = A_2$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(7,3,7)$	2.28239	0.38325	136.50395	**12.72062**
$(7,3,8)$	2.16470	0.41326	105.59967	9.12600
$(7,3,9)$	2.16413	0.43171	105.46662	**8.72492**
$(5,4,5)$	1.83500	0.46190	50.08711	**5.42186**
$(5,4,6)$	1.79568	0.50747	45.66776	**4.49957**

The cases 2. ii. e, f, g, h with $O = Q, J, E, K$, respectively are not relevant for optimal densities.

4.11. Case 2.s.ii.b. $O = A_2$ is the ball centre (see Fig. 1) and $u = w$. The density is related to the "half orthoscheme".
(4.13)

$$r=\min\{A_2b^2, A_2Q, A_2F_{12}\};\ \cosh(A_2b^2)=\sqrt{1-\frac{1}{a_{22}}},\ \cosh(A_2Q)=\frac{1}{\sqrt{a_{33}}},$$

$$\cosh(A_2F_{12})=\sqrt{\frac{1}{2}+\frac{\cos\frac{\pi}{v}}{2\sin^2\frac{\pi}{u}}},\ R=\max\{A_2J, A_2F_{03}\};$$

$$\cosh(A_2J)=\frac{\cot\frac{\pi}{u}\cot\frac{\pi}{v}}{\sqrt{a_{00}}},\ \cosh(A_2F_{03})=\cot\frac{\pi}{u}\sqrt{\frac{\cos\frac{\pi}{v}+\sin^2\frac{\pi}{u}}{2(1-\cos\frac{\pi}{v})}}.$$

Table 2.s.ii.b. Packing, $O = A_2$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$				
$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3,v\to\infty,3)$	$\text{arch}(\frac{\sqrt{42}}{6})\approx 0.39768$	0.44446	0.27191	**0.10196**
$(4,v\to\infty,4)$	$\text{arch}(\frac{\sqrt{6}}{2})\approx 0.65848$	0.63434	1.30405	**0.25697**
$(5,5,5)$	0.74746	0.57271	1.95548	**0.34144**
$(5,v\to\infty,3,5)$	$\text{arch}\left(\sqrt{2-\cos\left(\frac{\pi}{5}\right)}\right)\approx$ ≈ 0.55832	0.73015	0.77585	0.10626
$(6,4,6)$	0.78340	0.55557	2.27605	**0.34140**
$(6,v\to\infty,6)$	$\text{arch}(\frac{\sqrt{5}}{2})\approx 0.48121$	0.78465	0.48890	0.05192

Table 2.s.ii.b. Covering, $O = A_2$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$				
$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3,8,3)$	1.49263	0.32610	21.63099	**11.05541**
$(4,5,4)$	1.43911	0.43062	18.80212	**5.45785**
$(5,4,5)$	1.40493	0.46190	17.16523	**3.71622**
$(6,4,6)$	1.47083	0.55557	20.43783	**3.06559**

4.12. Case 2.s.ii.c. $O = F_{03}$ is the ball centre (see Fig. 1) and $u = w$.

(4.14)
$$r = \min\{F_{03}b^0, F_{03}J\};\ \cosh(F_{03}b^0) = \sqrt{1 + \frac{\cos\frac{\pi}{v} - \sin^2\frac{\pi}{u}}{2(1-\cos\frac{\pi}{v})}},$$
$$\cosh(F_{03}J) = \frac{\sin\frac{\pi}{v}}{\sqrt{2Ba_{33}(a_{33}+a_{03})}},\ R = \max\{F_{03}A_2, F_{03}Q\};$$
$$\cosh(F_{03}A_2) = \cot\frac{\pi}{u}\sqrt{\frac{\cos\frac{\pi}{v} + \sin^2\frac{\pi}{u}}{2(1-\cos\frac{\pi}{v})}},\ \cosh(F_{03}Q) = \frac{\cot\frac{\pi}{u}\cos\frac{\pi}{v}}{\sqrt{2Ba_{33}(a_{33}+a_{03})}}.$$

Table 2.s.ii.c. Packing, $O = F_{03}$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$

$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3,7,3)$	0.78871	0.27899	2.32647	**0.59564**
$(3,8,3)$	0.72041	0.32610	1.73696	0.33290
$(4,5,4)$	0.80846	0.43062	2.52145	0.58554
$(5,4,5)$	0.72146	0.46190	1.74508	0.47226
$(6,4,6)$	0.69217	0.55557	1.52838	0.34388

Table 2.s.ii.c. Covering, $O = F_{03}$, $\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$

$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3,8,3)$	1.26607	0.32610	11.68147	**2.23886**
$(4,5,4)$	1.22646	0.43062	10.41484	2.41856
$(5,4,5)$	1.28483	0.46190	12.32350	3.33500
$(5,5,5)$	1.51882	0.57271	23.14218	4.04082
$(6,4,6)$	1.43252	0.55557	18.47661	4.15712

4.13. Case 2.s.ii.d. $O = F_{12}$ is the ball centre (see Fig. 1) and $u = w$.

(4.15)
$$r = \min\{F_{12}b^1, F_{12}a_3\};\ \cosh(F_{12}b^1) = \sqrt{1 + \frac{\cos\frac{\pi}{v} - \sin^2\frac{\pi}{u}}{2}},$$
$$\cosh(F_{12}a_3) = \sqrt{1 + \frac{\cos^2\frac{\pi}{u}(\cos\frac{\pi}{v} + \sin^2\frac{\pi}{u})}{2(\cos^2\frac{\pi}{v} - \sin^2\frac{\pi}{u})}},\ R = \max\{F_{12}J, F_{12}Q\};$$
$$\cosh(F_{12}J) = \frac{\cos\frac{\pi}{u}(1+\cos\frac{\pi}{v})}{\sin\frac{\pi}{v}\sqrt{2a_{33}(\cos\frac{\pi}{v} + \sin^2\frac{\pi}{u})}},\ \cosh(F_{12}Q) = \sqrt{\frac{\cos\frac{\pi}{v} + \sin^2\frac{\pi}{u}}{2a_{33}\sin^2\frac{\pi}{u}}}.$$

This case is not relevant.

4.14. Case 2.s.ii.e. $O = Q$ is the ball centre (see Fig. 1) and $u = w$.

$$r = \min\{QA_2, QE, \frac{1}{2}QC\};\ \cosh(QA_2) = \frac{1}{\sqrt{a_{33}}},\ \cosh(QE) = \frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{u}},$$
(4.16)
$$\cosh\Big(\frac{1}{2}QC\Big) = \sqrt{\frac{1}{2} + \frac{a_{12}}{2a_{22}a_{33}}},\ R = \max\{QF_{03}, QF_{12}\};$$
$$\cosh(QF_{03}) = \frac{\cot\frac{\pi}{u}\cos\frac{\pi}{v}}{\sqrt{2Ba_{33}(a_{33}+a_{03})}},\ \cosh(QF_{12}) = \sqrt{\frac{\cos\frac{\pi}{v} + \sin^2\frac{\pi}{u}}{2a_{33}\sin^2\frac{\pi}{u}}}.$$

Table 2.s.ii.e. Packing, $O = Q,\ \frac{1}{u} + \frac{1}{v} < \frac{1}{2},\ \frac{1}{v} + \frac{1}{w} < \frac{1}{2}$

$(u,\ v,\ w)$	r^{opt}	$Vol(W_{uvw})$	$Vol(B(r^{opt}))$	δ^{opt}
$(3, v \to \infty, 3)$	$\text{arch}(\frac{2\sqrt{3}}{3}) \approx 0.54931$	0.44446	0.73740	0.27652
$(4, 8, 4)$	0.76429	0.56369	2.10109	**0.46592**
$(4, 9, 4)$	0.73883	0.57923	1.88365	0.40650
$(4, v \to \infty, 4)$	$\text{arch}(\frac{\sqrt{6}}{2}) \approx 0.65848$	0.63434	1.30405	0.25697
$(5, 5, 5)$	0.77537	0.57271	2.20130	0.38436
$(5, v \to \infty, 5)$	$\text{arch}\left(\frac{1}{2}\sqrt{2 - \cos\left(\frac{\pi}{5}\right)^2}\right) \approx \approx 0.55832$	0.73015	0.77585	0.10626
$(6, 4, 6)$	0.78340	0.55557	2.27605	0.34140

Table 2.s.ii.e. Covering, $O = Q,\ \frac{1}{u} + \frac{1}{v} < \frac{1}{2},\ \frac{1}{v} + \frac{1}{w} < \frac{1}{2}$

$(u,\ v,\ w)$	R^{opt}	$Vol(W_{uvw})$	$Vol(B(R^{opt}))$	Δ^{opt}
$(3, 8, 3)$	1.17362	0.32610	8.90096	4.54920
$(3, 9, 3)$	1.23555	0.35444	10.69512	5.02911
$(4, 5, 4)$	1.22646	0.43062	10.41484	3.02321
$(4, 8, 4)$	1.33092	0.56369	14.02565	3.11023
$(5, 4, 5)$	1.28483	0.46190	12.32350	2.66800
$(5, 5, 5)$	1.42880	0.57271	18.29491	3.19444
$(6, 4, 6)$	1.40674	0.55557	17.24885	**2.58725**
$(6, 5, 6)$	1.63628	0.64650	31.09468	4.00806
$(6, 6, 6)$	1.83634	0.69130	50.24429	6.05670

The cases 2. s. ii. f, g, h with $O = J, E, K$, respectively are not relevant for optimal densities.

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