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A NEW CURVATURELIKE TENSOR FIELD IN AN ALMOST CONTACT RIEMANNIAN MANIFOLD II

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Memories of Professor Mileva Prvanović

Abstract. In the last paper, we introduced a new curvaturlike tensor field in an almost contact Riemannian manifold and we showed some geometrical properties of this tensor field in a Kenmotsu and a Sasakian manifold. In this paper, we define another new curvaturelike tensor field, named $(\mathrm{CHR})_3$-curvature tensor in an almost contact Riemannian manifold which is called a contact holomorphic Riemannian curvature tensor of the second type. Then, using this tensor, we mainly research $(\mathrm{CHR})_3$-curvature tensor in a Sasakian manifold. Then we define the notion of the flatness of a $(\mathrm{CHR})_3$-curvature tensor and we show that a Sasakian manifold with a flat $(\mathrm{CHR})_3$-curvature tensor is flat. Next, we introduce the notion of $(\mathrm{CHR})_3$-η-Einstein in an almost contact Riemannian manifold. In particular, we show that Sasakian $(\mathrm{CHR})_3$-η-Einstein manifold is η-Einstain. Moreover, we define the notion of $(\mathrm{CHR})_3$-space form and consider this in a Sasakian manifold. Finally, we consider a conformal transformation of an almost contact Riemannian manifold and we get new invariant tensor fields (not the conformal curvature tensor) under this transformation. Finally, we prove that a conformally $(\mathrm{CHR})_3$-flat Sasakian manifold does not exist.

1. Almost contact Riemannian manifolds

A real $(2n+1)$-dimensional differentiable Riemannian manifold (M^{2n+1}, g) is said to be an almost contact Riemannian manifold if it has a $(1,1)$-tensor φ and a 1-form η which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any $Y, X \in TM^{2n+1}$, where ξ is defined by $g(\xi, X) = \eta(X)$. From $(1.1)_3$, the vector field ξ is unit and we say this vector field the *structure vector field* of

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the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold M^{2n+1}, we define a 2-form F as $F(X,Y) = g(\varphi X, Y)$ for any $X, Y \in TM^{2n+1}$, where TM^{2n+1} denotes the tangent bundle of M^{2n+1}. Then the 2-form F is skew-symmetric and we say that this tensor field is the *fundamental* 2-*form* of this almost contact Riemannian manifold.

In an almost contact Riemannian manifold, a section which is given by X and φX for a unit vector field X is called a φ-section of X. The sectional curvature $R(X, \varphi X, \varphi X, X)$ is said to be the *φ-holomorphic sectional curvature of* X, where R denotes the Riemannian curvature tensor with respect to g.

In a $(2n+1)$-dimensional almost contact Riemannian manifold M^{2n+1}, we define $\overset{*}{R}(X,Y,Z,W)$ which is called the *second Riemannian curvature tensor* as $\overset{*}{R}(X,Y,Z,W) = R(X,Y,\varphi Z,\varphi W)$ for any $X,Y,Z,W \in TM^{2n+1}$.

An almost contact Riemannian manifold $(M^{2n+1}, \varphi, g, \xi)$ is called a normal contact Riemannian or a Sasakian manifold if the structure vector field ξ and the fundamental 2-form F satisfies

$$\nabla_X \xi = \varphi X, \quad (\nabla_X F)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$$

for any $X,Y,Z \in TM^{2n+1}$.

In a Sasakian manifold, the Riemannian curvature tensor R and the Ricci tensor ρ with respect to g satisfy

$$\begin{aligned} R(X,Y,Z,\xi) &= \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \\ R(X,Y,\varphi Z,\varphi W) &= R(\varphi X,\varphi Y,Z,W) = R(X,Y,Z,W) + g(X,Z)g(Y,W) \\ &\quad - g(Y,Z)g(X,W) + F(X,W)F(Y,Z) - F(Y,W)F(X,Z), \\ R(\varphi X,\varphi Y,\varphi Z,\varphi W) &= R(X,Y,Z,W) + \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z) \\ &\quad + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W), \\ \rho(\varphi X,\varphi Y) &= \rho(X,Y) - 2n\eta(X)\eta(Y), \\ \rho(\varphi X,Y) + \rho(X,\varphi Y) &= 0, \qquad \rho(X,\xi) = 2n\eta(X) \end{aligned} \tag{1.3}$$

for any $X,Y,Z,W \in TM^{2n+1}$.

A Sasakian manifold is said to be a *Sasakian space form* if it has a constant φ-holomorpic sectional curvature. Then the curvature tensor field R satisfies [**2**]

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{c+3}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ & \quad + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \\ & \quad - F(Y,Z)F(X,W) + F(X,Z)F(Y,W) \\ & \quad - 2F(X,Y)F(Z,W)\}, \end{aligned} \tag{1.4}$$

where c is a constant holomorphic sectional curvature.

A Sasakian space form with 0 holomorphic sectional curvature is called to be *flat*.

An almost contact Riemannian manifold M^{2n+1} is said to be *η-Einstein* if the Ricci tensor ρ with respect to g satisfies $\rho(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ for certain differentiable functions a and b on M^{2n+1} which are called the *associated functions* of ρ and any $X,Y \in TM^{2n+1}$. In particular, in an η-Einstein Sasakian manifold, associated functions a and b satisfy the following relation

$$a + b = 2n, \quad \tau = 2n(a+1), \tag{1.5}$$

where τ is the scalar curvature with respect to g.

In a Sasakian manifold, the *C-Bochner curvature tensor* $CB(X,Y,Z,W)$ is defined by [**3**]

$$\begin{aligned} CB(X,Y,Z,W) &= R(X,Y,Z,W) + \frac{1}{2(n+2)}\{\rho(X,Z)g(\varphi Y,\varphi W) \\ &-\rho(Y,Z)g(\varphi X,\varphi W) + \rho(Y,W)g(\varphi X,\varphi Z) - \rho(X,W)g(\varphi Y,\varphi Z) \\ &-\tilde\rho(X,Z)F(Y,W) + \tilde\rho(Y,Z)F(X,W) - \tilde\rho(Y,W)F(X,Z) \\ &+\tilde\rho(X,W)F(Y,Z) - 2\tilde\rho(X,Y)F(Z,W) - 2\tilde\rho(Z,W)F(X,Y)\} \\ &-\frac{\tau+2n(2n+3)}{4(n+1)(n+2)}\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\} \\ &+\frac{\tau-3(2n+4)}{4(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+\frac{\tau+2n}{4(n+1)(n+2)}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta Z)g(X,W) - \eta(X)\eta(W)g(Y,Z)\}, \end{aligned} \tag{1.6}$$

where we put $\tilde\rho(X,Y) = \rho(X,\varphi Y)$.

2. A curvaturelike tensor field in an almost contact Riemannian manifold

In this section, we define a new curvaturelike tensor field in an almost contact Riemanian manifold.

In a differentiable manifold M, a $(0,4)$-type tensor field $T(X,Y,Z,W)$ is called *curvaturelike* if it satisfies

$$\begin{gathered} T(X,Y,Z,W) = -T(Y,X,Z,W), \quad T(X,Y,Z,W) = T(Z,W,X,Y), \\ T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0 \end{gathered}$$

for any $X,Y,Z,W \in TM$ [**4**].

In an almost contact Riemannian manifold (M^{2n+1},φ,ξ,g), we define a (0,4)-type tensor field $(\mathrm{CHR})_3(X,Y,Z,W)$ as

$$\begin{aligned} 16(\mathrm{CHR})_3(X,Y,Z,W) &= 3\{R(X,Y,Z,W) + R(\varphi X,\varphi Y,Z,W) \\ &+ R(\varphi Z,\varphi W,X,Y) + R(\varphi X,\varphi Y,\varphi Z,\varphi W)\} - R(\varphi W,\varphi Y,X,Z) \\ &- R(\varphi X,\varphi Z,W,Y) - R(\varphi Y,\varphi Z,X,W) - R(\varphi X,\varphi W,Y,Z) \\ &+ R(\varphi X,Z,\varphi W,Y) + R(X,\varphi Z,W,\varphi Y) + R(\varphi X,W,Y,\varphi Z) \\ &+ R(X,\varphi W,\varphi Y,Z) \\ &+ \eta(X)P(Z,W,Y) - \eta(Y)P(Z,W,X) \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&+ \eta(Z)P(X,Y,W) - \eta(W)P(X,Y,Z)\\
&+ \eta(X)\eta(W)Q(Y,Z) - \eta(X)\eta(Z)Q(Y,W)\\
&\qquad + \eta(Y)\eta(Z)Q(W,X) - \eta(Y)\eta(W)Q(Z,X),
\end{aligned}
$$

where we put

$$
\begin{aligned}
P(X,Y,Z) &= 3\{R(X,Y,Z,\xi) + R(\varphi X,\varphi Y,Z,\xi)\} + R(\varphi X,\varphi Z,Y,\xi)\\
&\quad + R(\varphi Z,\varphi Y,X,\xi) - R(X,\varphi Z,\varphi Y,\xi) - R(\varphi Z,Y,\varphi X,\xi),
\end{aligned}
\tag{2.2}
$$

$$
Q(X,Y) = 3R(\xi,X,Y,\xi) - R(\xi,\varphi X,\varphi Y,\xi)
\tag{2.3}
$$

for any $X,Y,Z,W \in TM^{2n+1}$. Then, we can easily check the above tensor field is curvaturelike. We call this tensor field a $(\mathrm{CHR})_3$-*curvature tensor* in an almost contact Riemannian manifold or a *contact holomorphic Riemannian curvature tensor of the second type*.

About the tensors $P(X,Y,Z)$ and $Q(X,Y)$, we can easily see

$$
\begin{aligned}
P(\varphi X,\varphi Y,Z) &= P(X,Y,Z) - \eta(X)Q(Y,Z) + \eta(Y)Q(X,Z),\\
P(X,\varphi Y,\varphi Z) &= 3\{R(X,\varphi Y,\varphi Z,\xi) - R(\varphi X,Y,\varphi Z,\xi)\}\\
&\quad - R(\varphi X,Z,\varphi Y,\xi) + R(Z,Y,X,\xi) - R(X,Z,Y,\xi)\\
&\quad + R(Z,\varphi Y,\varphi X,\xi) + \eta(Y)Q(Z,X)\\
&\quad - 2\{\eta(Z)S(X,Y) + \eta(Y)S(X,Z)\},\\
P(\varphi X,Y,\varphi Z) &= 3\{R(\varphi X,Y,\varphi Z,\xi) - R(X,\varphi Y,\varphi Z,\xi)\} + R(X,Z,Y,\xi)\\
&\quad - R(Z,\varphi Y,\varphi X,\xi) + R(\varphi X,Z,\varphi Y,\xi) - R(Z,Y,X,\xi)\\
&\quad - \eta(X)Q(Y,Z) + 2\{\eta(X)S(Y,Z) + \eta(Z)S(X,Y)\},\\
P(X,Y,\xi) &= 0,\quad P(\xi,X,Y) = Q(X,Y),\quad P(X,\xi,Y) = -Q(X,Y),
\end{aligned}
\tag{2.4}
$$

$$
Q(\varphi X,\varphi Y) = -Q(X,Y) + 2S(X,Y),\qquad Q(\xi,X) = 0,
\tag{2.5}
$$

where we put $S(X,Y) = R(\xi,X,Y,\xi) + R(\xi,\varphi X,\varphi Y,\xi)$. In particular, by virtue of (1.3), the tensor fields $P(X,Y,Z)$ and $Q(X,Y)$ in a Sasakian manifold are respectively satisfied

$$
\begin{aligned}
P(X,Y,Z) &= 2\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} = 2R(X,Y,Z,\xi),\\
Q(X,Y) &= 2\{g(X,Y) - \eta(X)\eta(Y)\} = 2g(\varphi X,\varphi Y)
\end{aligned}
$$

for any $X,Y,Z \in TM^{2n+1}$.

An almost contact Riemannian manifold is said to be $(\mathrm{CHR})_3$-*flat* if the $(\mathrm{CHR})_3$-curvature tensor vanishes, identically.

In an almost contact Riemannian manifold, the $(\mathrm{CHR})_3$-curvature tensor satisfies the following equations

$$
\begin{aligned}
&(\mathrm{CHR})_3(X,Y,Z,\xi) = 0,\\
&(\mathrm{CHR})_3(X,Y,\varphi Z,\varphi W) = (\mathrm{CHR})_3(\varphi X,\varphi Y,Z,W) = (\mathrm{CHR})_3(X,Y,Z,W),\\
&(\mathrm{CHR})_3(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X) - 2\eta(X)R(X,\varphi X,\varphi X,\xi)\\
&\qquad + \eta(X)^2R(\xi,\varphi X\varphi X,\xi)
\end{aligned}
$$

for any $X, Y, Z, W \in TM^{2n+1}$.

3. $(\mathrm{CHR})_3$-curvature tensor in a Sasakian manifold

In this section, we consider a $(\mathrm{CHR})_3$-flat Sasakian manifold and we prove this manifold is flat Sasakian.

By virtue of (1.3) and Bianchi identity [**6**], in a Sasakian manifold, the $(\mathrm{CHR})_3$-curvature tensor satisfies

$$\begin{aligned}(3.1)\quad (\mathrm{CHR})_3(X,Y,Z,W) &= R(X,Y,Z,W)\\ &+\frac{3}{4}\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\}\\ +\frac{1}{4}\{g(\varphi X,W)g(\varphi Y,Z) &- g(\varphi X,Z)g(\varphi Y,W)-2g(\varphi X,Y)g(\varphi Z,W)\\ &+\eta(X)\eta(Z)g(Y,W)-\eta(X)\eta(W)g(Y,Z)\\ &+\eta(Y)\eta(W)g(X,Z)-\eta(Y)\eta(Z)g(X,W)\}.\end{aligned}$$

Let us consider the $(\mathrm{CHR})_3$-flat Sasakian manifold, then we have from (3.1), the curvature tensor R is written as

$$\begin{aligned}R(X,Y,Z,W) &= \frac{3}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\\ -\frac{1}{4}\{g(\varphi X,W)g(\varphi Y,Z) &- g(\varphi X,Z)g(\varphi Y,W)-2g(\varphi X,Y)g(\varphi Z,W)\\ &+\eta(X)\eta(Z)g(Y,W)-\eta(X)\eta(W)g(Y,Z)\\ &+\eta(Y)\eta(W)g(X,Z)-\eta(Y)\eta(Z)g(X,W)\}.\end{aligned}$$

Comparing the above equation and (1.4), we obtain

THEOREM 3.1. *A* $(\mathrm{CHR})_3$*-flat Sasakian manifold is flat Sasakian.*

4. The properties of the $(\mathrm{CHR})_3$-curvature tensor.

We put $\rho(\mathrm{CHR})_3(X,Y)=\sum_{i=1}^{2n+1}(\mathrm{CHR})_3(e_i,X,Y,e_i)$ for a local orthonormal frame $(e_1,e_2,\dots,e_{2n+1})$ of an almost contact Riemannian manifold M^{2n+1}. Then we say $\rho(\mathrm{CHR})_3(X,Y)$ a $(\mathrm{CHR})_3$*-Ricci tensor.*

By virtue of (2.1), (2.2) and (2.3), the local representations of $(\mathrm{CHR})_3$-curvature tensor, the tensor fields P and Q are respectively written by

$$\begin{aligned}(4.1)\quad 16(\mathrm{CHR})_{3kjih} &= 3(R_{kjih}+\overset{*}{R}_{kjih}+\overset{*}{R}_{ihkj}+\overset{*}{R}_{kjml}\varphi_i{}^m\varphi_h{}^l)\\ &-\overset{*}{R}_{hjki}-\overset{*}{R}_{kihj}-\overset{*}{R}_{jikh}-\overset{*}{R}_{khji}\\ &+R_{tisj}\varphi_k{}^t\varphi_h{}^s+R_{kths}\varphi_i{}^t\varphi_j{}^s+R_{thjs}\varphi_k{}^t\varphi_i{}^s+R_{ktsi}\varphi_h{}^t\varphi_j{}^s\\ &+\eta_kP_{ihj}-\eta_jP_{kji}+\eta_iP_{kjh}-\eta_hP_{kji}\\ &+\eta_k\eta_hQ_{ji}-\eta_k\eta_iQ_{jh}+\eta_j\eta_iQ_{hk}-\eta_j\eta_hQ_{ik},\end{aligned}$$

$$(4.2)\qquad P_{jih}=3(R_{jihl}+\overset{*}{R}_{jihl})\xi^l+(\overset{*}{R}_{jhil}+\overset{*}{R}_{hijl})\xi^l$$

$$- (R_{jtsl}{\varphi_h}^t{\varphi_i}^s + R_{tisl}{\varphi_h}^t{\varphi_j}^s)\xi^l,$$

$$Q_{ih} = (3R_{mihl} - R_{mtsl}{\varphi_i}^t{\varphi_h}^s)\xi^m\xi^l, \tag{4.3}$$

where we put $\overset{*}{R}_{kjih} = R_{tsih}{\varphi_k}^t{\varphi_j}^s$ and the indices $\{k,j,\ldots,h\}$ run over the range $\{1,2,\ldots,2n+1\}$.

Moreover, equations (2.4) and (2.5) are respectively written as

$$\begin{aligned} P_{tsh}{\varphi_j}^t{\varphi_i}^s &= P_{jih} - \eta_j Q_{ih} + \eta_i Q_{jh}, \\ P_{jts}{\varphi_i}^t{\varphi_h}^s &= 3(R_{jtsl}{\varphi_i}^t{\varphi_h}^s\xi^l - R_{tisl}{\varphi_j}^t{\varphi_h}^s)\xi^l \\ &\quad - R_{thsl}{\varphi_j}^t{\varphi_i}^s\xi^l + R_{hijl}\xi^l - R_{jhil}\xi^l \\ &\quad + R_{htsl}{\varphi_i}^t{\varphi_j}^s\xi^l + \eta_i Q_{hj} - 2(\eta_i S_{hj} + \eta_h S_{ij}), \\ P_{tis}{\varphi_j}^t{\varphi_h}^s &= 3(R_{tisl}{\varphi_j}^t{\varphi_h}^s - R_{jtsl}{\varphi_i}^t{\varphi_h}^s)\xi^l + R_{jhil}\xi^l \\ &\quad - R_{htsl}{\varphi_i}^t{\varphi_j}^s\xi^l + R_{thsl}{\varphi_j}^t{\varphi_i}^s\xi^l - R_{hijl}\xi^l \\ &\quad - \eta_j Q_{ih} + 2(\eta_j S_{ih} + \eta_h S_{ji}), \\ P_{ihl}\xi^l = 0, \quad & P_{lih}\xi^l = Q_{ih}, \quad P_{ilh}\xi^l = -Q_{ih}, \end{aligned} \tag{4.4}$$

$$Q_{ts}{\varphi_i}^t{\varphi_h}^s = -Q_{ih} + 2S_{ih}, \quad Q_{lh}\xi^l = 0, \tag{4.5}$$

$$S_{ih} = \{R_{mihl} + R_{mtsl}{\varphi_i}^t{\varphi_h}^s\}\xi^m\xi^l. \tag{4.6}$$

Since, we know $\rho(\mathrm{CHR})_{3ji} = (\mathrm{CHR})_{3kjih}g^{kh}$, we have from (4.1), (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} 8\rho(\mathrm{CHR})_3(X,Y) &= \rho(X,Y) + \rho(\varphi X, \varphi Y) \\ &\quad - 3\{\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\} - \{\rho(X,\xi)\eta(Y) + \rho(Y,\xi)\eta(X)\} \\ &\quad + 3\{\overset{*}{\rho}(X,\xi)\eta(Y) + \overset{*}{\rho}(Y,\xi)\eta(X)\} + \rho(\xi,\xi)\eta(X)\eta(Y) \\ &\quad - \{R(\xi,X,Y,\xi) + R(\xi,\varphi X,\varphi Y,\xi)\}, \end{aligned} \tag{4.7}$$

where we put

$$\rho(X,Y) = \sum_{i=1}^{2n+1} R(e_i,X,Y,e_i), \quad \overset{*}{\rho}(X,Y) = \sum_{i=1}^{2n+1} \overset{*}{R}(e_i,X,Y,e_i).$$

In particular, the $(\mathrm{CHR})_3$-Ricci tensor in a Sasakian manifold is written by

$$\rho(\mathrm{CHR})_3(X,Y) = \rho(X,Y) - \frac{1}{2}\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\}. \tag{4.8}$$

From the above equation, we can easily have $\rho(\mathrm{CHR})_3(X,\xi) = 0$.

Next, we put $\tau(\mathrm{CHR})_3 = \sum_{i=1}^{2n+1} \rho(\mathrm{CHR})_3(e_i,e_i) = \rho(\mathrm{CHR})_{3ji}g^{ji}$ which is called the $(\mathrm{CHR})_3$-*scalar curvature* of $(\mathrm{CHR})_3$-curvature tensor. Then from (4.7), we have $4\tau(\mathrm{CHR})_3 = \tau - 2\rho(\xi,\xi) + 3\overset{*}{\tau}$, where we put $\overset{*}{\tau} = \sum_{i=1}^{2n+1} \overset{*}{\rho}(e_i,e_i)$. Then, in a Sasakian manifold, we have from (4.8)

$$\tau(\mathrm{CHR})_3 = \tau - n(3n+1). \tag{4.9}$$

Now, an almost contact Riemannian manifold M^{2n+1} is called $(\mathrm{CHR})_3$-η-*Einstein* if its $(\mathrm{CHR})_3$-Ricci tensor $\rho(\mathrm{CHR})_3(X,Y)$ has the form

$$\rho(\mathrm{CHR})_3(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{4.10}$$

for certain functions α and β which are called *associated functions* of $\rho(\mathrm{CHR})_3$. In particular, if our manifold is Sasakian, then we have from (4.8) and (4.10) the Ricci tensors is written as

$$\rho(X,Y) = \Big(\alpha + \frac{3n-1}{2}\Big)g(X,Y) + \Big(\beta + \frac{n+1}{2}\Big)\eta(X)\eta(Y),$$

that is, our manifold is η-Einstein.

Conversely, if our manifold M^{2n+1} is η-Einstein, then we can easily see that Sasakian manifolds are $(\mathrm{CHR})_3$-η-Einstein. Moreover, by virtue of (1.5), we obtain $\alpha+\beta = 0$ in a $(\mathrm{CHR})_3$-η-Einstein Sasakian manifold and the scalar curvature τ and the $(\mathrm{CHR})_3$-scalar curvature $\tau(\mathrm{CHR})_3$ are written as

$$\tau = n(2\alpha + 3n + 1), \quad \tau(\mathrm{CHR})_3 = 2n\alpha. \tag{4.11}$$

Thus we have

THEOREM 4.1. *A* $(2n+1)$*-dimensional Sasakian manifold* M^{2n+1} *is* $(\mathrm{CHR})_3$-η*-Einstein if and only if* M^{2n+1} *is* η*-Einstein and the scalar curvatures* τ *and* $\tau(\mathrm{CHR})_3$ *are respectively written by* (4.11) *which includes only the associated function* α.

COROLLARY 4.1. *In an* η-$(\mathrm{CHR})_3$*-Einstein Sasakian manifold* M^{2n+1}*, the scalar curvatures* τ *and* $\tau(\mathrm{CHR})_3$ *are constant if and only if one of its associated functions is constant.*

5. $(\mathrm{CHR})_3$-space form

In this section, we define a notion of a $(\mathrm{CHR})_3$-space form in an almost contact Riemannian manifold. Then we consider a Sasakian $(\mathrm{CHR})_3$-space form. And in this manifold, we determine the $(\mathrm{CHR})_3$-curvature tensor by the structure tensors.

DEFINITION 5.1. An almost contact Riemannian manifold is called a $(\mathrm{CHR})_3$-space form if its $(\mathrm{CHR})_3$-curvature tensor satisfies

$$(\mathrm{CHR})_3(X,\varphi X,\varphi X,X) = c\|X\|^4$$

for a certain constant c and any $X \in TM^{2n+1} - \{\xi\}$.

By virtue of (2.1), in an almost contact Riemannian manifold, we have

$$(\mathrm{CHR})_3(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X)$$

for any $X \in TM^{2n+1} - \{\xi\}$. This means that a Sasakian $(\mathrm{CHR})_3$-space form is a Sasakian space form. Thus we have from (1.4) and (3.1)

THEOREM 5.1. *In a* $(2n+1)$*-dimensional Sasakian* $(\mathrm{CHR})_3$*-space form, the* $(\mathrm{CHR})_3$*-curvatuee tensor,* $(\mathrm{CHR})_3$*-Ricci tensor and the* $(\mathrm{CHR})_3$*-scalar curvature are respectively given by*

$$(\mathrm{CHR})_3(X,Y,Z,W)=\frac{c}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)$$
$$-F(X,W)F(Y,Z)+F(X,Z)F(Y,W)-2F(X,Y)F(Z,W)$$
$$+\eta(X)\eta(Z)g(Y,W)-\eta(X)\eta(W)g(Y,Z)$$
$$+\eta(Y)\eta(W)g(X,Z)-\eta(Y)\eta(Z)g(X,W)\},$$

$$\rho(\mathrm{CHR})_3(X,Y)=\frac{cn}{2}\{g(X,Y)-\eta(X)\eta(Y)\},\quad \tau(\mathrm{CHR})_3=cn^2.$$

for any $X,Y,Z,W\in TM^{2n+1}$.

6. Conformal transformations of almost contact Riemannian manifolds

In an almost contact Riemannian manifold M^{2n+1}, we consider a following conformal transformation;

$$\bar g=e^{2f}g,\quad \bar g^{-1}=e^{-2f}g^{-1} \tag{6.1}$$

for a certain positive differentiable function f on M^{2n+1}. Then it is well known [**6**] that

$$\bar\nabla_X Y=\nabla_X Y+\theta(X)Y+\theta(Y)X-g(X,Y)U, \tag{6.2}$$

where $\bar\nabla$ (resp. ∇) means the covariant derivation with respect to $\bar g$ (resp. g), $\theta=df$ and $g(U,X)\overset{\text{def}}{=}\theta(X)$. Moreover, between the Riemannian curvature tensors $\bar R(X,Y,Z,W)$ with respect to $\bar g$ and $R(X,Y,Z,W)$ with respect to g, we have the following relation [**6**]

$$\begin{aligned} e^{-2f}\bar R(X,Y,Z,W)=R(X,Y,Z,W)+g(X,W)\sigma(Y,Z)+g(Y,Z)\sigma(X,W)\\ -g(X,Z)\sigma(Y,W)-g(Y,W)\sigma(X,Z),\end{aligned} \tag{6.3}$$

where we put

$$\sigma(X,Y)=(\nabla_X\theta)(Y)-\theta(X)\theta(Y)+\frac{1}{2}g(X,Y)\theta(U). \tag{6.4}$$

From (6.3) and (6.4), we obtain $\bar\rho(X,Y)=\rho(X,Y)+(2n-1)\sigma(X,Y)+\sigma g(X,Y)$, where $\bar\rho$ denotes the Ricci tensor with respect to $\bar g$ and

$$\sigma=\sum_{i=1}^{2n+1}\sigma(e_i,e_i)=\sigma_{ji}g^{ji}. \tag{6.5}$$

From the above two equations, we easily get $e^{2f}\bar\tau=\tau+4n\sigma$.

Similarly with (2.1), $(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$ with respect to $\bar g$ is given by

$$16(\overline{\mathrm{CHR}})_3(X,Y,Z,W)=3\{\bar R(X,Y,Z,W)+\bar R(\varphi X,\varphi Y,Z,W)$$
$$+\bar R(X,Y,\varphi Z,\varphi W)+\bar R(\varphi X,\varphi Y,\varphi Z,\varphi W)\}-\bar R(X,Z,\varphi W,\varphi Y)$$
$$-\bar R(\varphi X,\varphi Z,W,Y)-\bar R(X,W,\varphi Y,\varphi Z)-\bar R(\varphi X,\varphi W,Y,Z)$$
$$+\bar R(\varphi X,Z,\varphi W,Y)+\bar R(X,\varphi Z,W,\varphi Y)$$
$$+\bar R(\varphi X,W,Y,\varphi Z)+\bar R(X,\varphi W,\varphi Y,Z)$$

$$\begin{aligned}&+\eta(X)\bar{P}(Z,W,Y)-\eta(Y)\bar{P}(Z,W,X)\\&+\eta(Z)\bar{P}(X,Y,W)-\eta(W)\bar{P}(X,Y,Z)\\&+\eta(X)\eta(W)\bar{Q}(Y,Z)-\eta(X)\eta(Z)\bar{Q}(Y,W)\\&\qquad+\eta(Y)\eta(Z)\bar{Q}(W,X)-\eta(Y)\eta(W)\bar{Q}(Z,X),\end{aligned}$$

where we put

$$\begin{aligned}\bar{P}(X,Y,Z)&=3\{\bar{R}(X,Y,Z,\xi)+\bar{R}(\varphi X,\varphi Y,Z,\xi)\}+\bar{R}(\varphi X,\varphi Z,Y,\xi)\\&\quad+\bar{R}(\varphi Z,\varphi Y,X,\xi)-\bar{R}(X,\varphi Z,\varphi Y,\xi)-\bar{R}(\varphi Z,Y,\varphi X,\xi),\end{aligned}\tag{6.6}$$

$$\bar{Q}(X,Y)=3\bar{R}(\xi,X,Y,\xi)-\bar{R}(\xi,\varphi X,\varphi Y,\xi).\tag{6.7}$$

for any $X,Y,Z,W\in TM^{2n+1}$.

Let us consider the relation between the tensor fields $(\mathrm{CHR})_3(X,Y,Z,W)$ and $(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$.

By virtue of (6.3), (6.6) and (6.7) are respectively written as

$$\begin{aligned}e^{-2f}\bar{P}(X,Y,Z)&=P(X,Y,Z)+\eta(X)\{3\sigma(Y,Z)-\sigma(\varphi Y,\varphi Z)\}\\&-\eta(Y)\{3\sigma(X,Z)-\sigma(\varphi X,\varphi Z)\}+2\{g(Y,Z)\sigma(X,\xi)-g(X,Z)\sigma(Y,\xi)\\&+g(\varphi Y,Z)\sigma(\varphi X,\xi)-g(\varphi X,Z)\sigma(\varphi Y,\xi)-2g(\varphi X,Y)\sigma(\varphi Z,\xi)\}\\&+\eta(Z)\{\eta(Y)\sigma(X,\xi)-\eta(X)\sigma(Y,\xi)\},\end{aligned}\tag{6.8}$$

$$\begin{aligned}e^{-2f}\bar{Q}(X,Y)&=Q(X,Y)+3\sigma(X,Y)-\sigma(\varphi X,\varphi Y)+2g(X,Y)\sigma(\xi,\xi)\\&+\eta(X)\eta(Y)\sigma(\xi,\xi)-3\{\eta(X)\sigma(Y,\xi)+\eta(Y)\sigma(X,\xi)\}.\end{aligned}\tag{6.9}$$

To calculate $e^{-2f}(\overline{\mathrm{CHR}})_3(X,Y,Z,W)$, we separate the following three parts;

Part 1:

$$\begin{aligned}(\bar{\mathrm{I}})&\overset{\mathrm{put}}{=}3\{\bar{R}(X,Y,Z,W)+\bar{R}(\varphi X,\varphi Y,Z,W)+\bar{R}(X,Y,\varphi Z,\varphi W)\\&+\bar{R}(\varphi X,\varphi Y,\varphi Z,\varphi W)\}-\bar{R}(X,Z,\varphi W,\varphi Y)-\bar{R}(\varphi X,\varphi Z,W,Y)\\&-\bar{R}(X,W,\varphi Y,\varphi Z)-\bar{R}(\varphi X,\varphi W,Y,Z)+\bar{R}(\varphi X,Z,\varphi W,Y)\\&+\bar{R}(X,\varphi Z,W,\varphi Y)+\bar{R}(\varphi X,W,Y,\varphi Z)+\bar{R}(X,\varphi W,\varphi Y,Z).\end{aligned}\tag{6.10}$$

Part 2:

$$\begin{aligned}(\overline{\mathrm{II}})&\overset{\mathrm{put}}{=}\eta(X)\bar{P}(Z,W,Y)-\eta(Y)\bar{P}(Z,W,X)\\&+\eta(Z)\bar{P}(X,Y,W)-\eta(W)\bar{P}(X,Y,Z).\end{aligned}\tag{6.11}$$

Part 3:

$$\begin{aligned}(\overline{\mathrm{III}})&\overset{\mathrm{put}}{=}\eta(X)\eta(W)\bar{Q}(Y,Z)-\eta(X)\eta(Z)\bar{Q}(Y,W)x\\&+\eta(Y)\eta(Z)\bar{Q}(W,X)-\eta(Y)\eta(W)\bar{Q}(Z,X).\end{aligned}\tag{6.12}$$

By virtue of (6.3), (6.8) and (6.9), we obtain

$$\begin{aligned}e^{-2f}(\bar{\mathrm{I}})&=(\mathrm{I})+2g(X,W)\{\sigma(Y,Z)+\sigma(\varphi Y,\varphi Z)\}\\&+2g(Y,Z)\{\sigma(X,W)+\sigma(\varphi X,\varphi W)\}-2g(X,Z)\{\sigma(Y,W)+\sigma(\varphi Y,\varphi W)\}\end{aligned}$$

$$
\begin{aligned}
&- 2g(Y,W)\{\sigma(X,Z)+\sigma(\varphi X,\varphi Z)\}+2g(\varphi X,W)\{\sigma(\varphi Y,Z)-\sigma(Y,\varphi Z)\}\\
&+ 2g(\varphi Y,Z)\{\sigma(\varphi X,W)-\sigma(X,\varphi W)\}-2g(\varphi X,Z)\{\sigma(\varphi Y,W)-\sigma(Y,\varphi W)\}\\
&- 2g(\varphi Y,W)\{\sigma(\varphi X,Z)-\sigma(X,\varphi Z)\}-4g(\varphi X,Y)\{\sigma(\varphi Z,W)-\sigma(Z,\varphi W)\}\\
&- 4g(\varphi Z,W)\{\sigma(\varphi X,Y)-\sigma(X,\varphi Y)\}+\eta(X)\eta(W)\{\sigma(Y,Z)-3\sigma(\varphi Y,\varphi Z)\}\\
&- \eta(Y)\eta(W)\{\sigma(X,Z)-3\sigma(\varphi X,\varphi Z)\}-\eta(X)\eta(Z)\{\sigma(Y,W)-3\sigma(\varphi Y,\varphi W)\}\\
&+ \eta(Y)\eta(Z)\{\sigma(X,W)-3\sigma(\varphi X,\varphi W)\}.
\end{aligned}
$$

$$
\begin{aligned}
&e^{-2f}(\overline{\mathrm{II}}) = (\mathrm{II})+2\eta(X)\eta(Z)\{3\sigma(Y,W)-\sigma(\varphi Y,\varphi W)\}\\
&- 2\eta(X)\eta(W)\{3\sigma(Y,Z)-\sigma(\varphi Y,\varphi Z)\}-2\eta(Y)\eta(Z)\{3\sigma(X,W)-\sigma(\varphi X,\varphi W)\}\\
&+ 2\eta(Y)\eta(W)\{3\sigma(X,Z)-\sigma(\varphi X,\varphi Z)\}-2g(\varphi Y,W)\{\eta(X)\sigma(\varphi Z,\xi)\\
&- \eta(Z)\sigma(\varphi X,\xi)\}+2g(\varphi Y,Z)\{\eta(X)\sigma(\varphi W,\xi)-\eta(W)\sigma(\varphi X,\xi)\}\\
&+ 2g(\varphi X,W)\{\eta(Y)\sigma(\varphi Z,\xi)-\eta(Z)\sigma(\varphi Y,\xi)\}-2g(\varphi X,Z)\{\eta(Y)\sigma(\varphi W,\xi)\\
&- \eta(W)\sigma(\varphi Y,\xi)\}-4g(\varphi Z,W)\{\eta(X)\sigma(\varphi Y,\xi)-\eta(Y)\sigma(\varphi X,\xi)\}\\
&- 4g(\varphi X,Y)\{\eta(Z)\sigma(\varphi W,\xi)-\eta(W)\sigma(\varphi X,\xi)\}+2g(Y,W)\{\eta(X)\sigma(Z,\xi)\\
&+ \eta(Z)\sigma(X,\xi)\}-2g(Y,Z)\{\eta(X)\sigma(W,\xi)+\eta(W)\sigma(X,\xi)\}\\
&-2g(X,W)\{\eta(Y)\sigma(Z,\xi)+\eta(Z)\sigma(Y,\xi)\}+2g(X,Z)\{\eta(Y)\sigma(W,\xi)+\eta(W)\sigma(Y,\xi)\},
\end{aligned}
$$

$$
\begin{aligned}
&e^{-2f}(\overline{\mathrm{III}}) = (\mathrm{III})+\eta(X)\eta(W)\{3\sigma(Y,Z)-\sigma(\varphi Y,\varphi Z)\}\\
&- \eta(X)\eta(Z)\{3\sigma(Y,W)-\sigma(\varphi Y,\varphi W)\}+\eta(Y)\eta(Z)\{3\sigma(X,W)-\sigma(\varphi X,\varphi W)\}\\
&- \eta(Y)\eta(W)\{3\sigma(X,Z)-\sigma(\varphi X,\varphi Z)\}+2\sigma(\xi,\xi)\{\eta(X)\eta(W)g(Y,Z)\\
&- \eta(X)\eta(Z)g(Y,W)+\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z)\},
\end{aligned}
$$

where $(\bar{\mathrm{I}})$ denotes the geometric equation with respect to $\bar{g}$ of (I) with respect to g and etc.. By virtue of the above three equations, we have

$$
\begin{aligned}
(6.13)\quad &8e^{-2f}(\overline{\mathrm{CHR}})_3(X,Y,Z,W) = 8(\mathrm{CHR})_3(X,Y,Z,W)\\
&+ g(\varphi X,\varphi W)D(Y,Z)-g(\varphi X,\varphi Z)D(Y,W)+g(\varphi Y,\varphi Z)D(X,W)\\
&- g(\varphi Y,\varphi W)D(X,Z)+g(\varphi X,W)\{A(Y,Z)+B(Y,Z)\}\\
&+ g(\varphi Y,Z)\{A(X,W)+B(X,W)\}-g(\varphi X,Z)\{A(Y,W)+B(Y,W)\}\\
&- g(\varphi Y,W)\{A(X,Z)+B(X,Z)\}-2g(\varphi X,Y)\{A(Z,W)+B(Z,W)\}\\
&- 2g(\varphi Z,W)\{A(X,Y)+B(X,Y)\}+g(Y,W)C(X,Z)-g(Y,Z)C(X,W)\\
&+ g(X,Z)C(Y,W)-g(X,W)C(Y,Z)+\sigma(\xi,\xi)\{\eta(X)\eta(W)g(Y,Z)\\
&- \eta(X)\eta(Z)g(Y,W)+\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z)\},
\end{aligned}
$$

where we put

$$
\begin{aligned}
A(X,Y) &= \sigma(\varphi X,Y)-\sigma(X,\varphi Y),\\
B(X,Y) &= \eta(X)\sigma(\varphi Y,\xi)-\eta(Y)\sigma(\varphi X,\xi),\\
C(X,Y) &= \eta(X)\sigma(Y,\xi)+\eta(Y)\sigma(X,\xi),
\end{aligned}
$$

$$D(X,Y) = \sigma(X,Y) + \sigma(\varphi X, \varphi Y).$$

7. An invariant tensor field under a conformal transformation

The equation (6.13) is written locally by

$$\begin{aligned}
8e^{-2f}(\overline{\mathrm{CHR}})_{3kjih} &= 8(\mathrm{CHR})_{3kjih} + (g_{kh} - \eta_k\eta_h)D_{ji} \\
&+ (g_{ji} - \eta_j\eta_i)D_{kh} - (g_{ki} - \eta_k\eta_i)D_{jh} - (g_{jh} - \eta_j\eta_h)D_{ki} \\
&+ \varphi_{kh}(A_{ji} + B_{ji}) + \varphi_{ji}(A_{kh} + B_{kh}) \\
&- \varphi_{ki}(A_{jh} + B_{jh}) - \varphi_{jh}(A_{ki} + B_{ki}) - 2\varphi_{kj}(A_{ih} + B_{ih}) \\
&- 2\varphi_{ih}(A_{kj} + B_{kj}) + g_{jh}C_{ki} - g_{ji}C_{kh} + g_{ki}C_{jh} - g_{kh}C_{ji} \\
&+ \sigma_{ml}\xi^m\xi^l(\eta_k\eta_h g_{ji} - \eta_k\eta_i g_{jh} + \eta_j\eta_i g_{kh} - \eta_j\eta_h g_{ki}).
\end{aligned} \tag{7.1}$$

Using (7.1), we get

$$\begin{aligned}
4\rho(\overline{\mathrm{CHR}})_3(X,Y) &= 4\rho(\mathrm{CHR})_3(X,Y) + (n+2)D(X,Y), \\
-(n+2)C(X,Y) &+ \{\sigma - \sigma(\xi,\xi)\}g(X,Y) - \{\sigma - (n+3)\sigma(\xi,\xi)\}\eta(X)\eta(Y),
\end{aligned} \tag{7.2}$$

$$e^{2f}\tau(\overline{\mathrm{CHR}})_3 = \tau(\mathrm{CHR})_3 + (n+1)\{\sigma - \sigma(\xi,\xi)\}. \tag{7.3}$$

Since, we have from (7.2) and (7.3)

$$\begin{aligned}
D(X,Y) &= \frac{1}{n+2}\Big\{4\rho(\overline{\mathrm{CHR}})_3(X,Y) - \frac{\tau(\overline{\mathrm{CHR}})_3}{n+1}\bar{g}(X,Y)\Big\} \\
&- \frac{1}{n+2}\Big\{4\rho(\mathrm{CHR})_3(X,Y) - \frac{\tau(\mathrm{CHR})_3}{n+1}g(X,Y)\Big\} \\
&+ C(X,Y) + \frac{\sigma - (n+3)\sigma(\xi,\xi)}{n+2}\eta(X)\eta(Y).
\end{aligned} \tag{7.4}$$

The above equation and (7.1) give us

$$\begin{aligned}
&g(\varphi X, \varphi Y)D(Z,W) \\
&= \frac{e^{-2f}}{n+2}\Big\{4\rho(\overline{\mathrm{CHR}})_3(Z,W) - \frac{\tau(\overline{\mathrm{CHR}})_3}{n+1}\bar{g}(Z,W)\Big\}\bar{g}(\varphi X, \varphi Y) \\
&- \frac{1}{n+2}\Big\{4\rho(\mathrm{CHR})_3(Z,W) - \frac{\tau(\mathrm{CHR})_3}{n+1}g(Z,W)\Big\}g(\varphi X, \varphi Y) \\
&+ g(\varphi X, \varphi Y)C(Z,W) + \frac{\sigma - (n+3)\sigma(\xi,\xi)}{n+2}g(\varphi X, \varphi Y)\eta(Z)\eta(W).
\end{aligned} \tag{7.5}$$

By virtue of (7.5) and (6.13), we have

$$\begin{aligned}
&e^{-2f}\bar{T}(X,Y,Z,W) - T(X,Y,Z,W) \\
&= g(\varphi X, W)\{A(Y,Z) + B(Y,Z)\} + g(\varphi Y, Z)\{A(X,W) + B(X,W)\} \\
&- g(\varphi X, Z)\{A(Y,W) + B(Y,W)\} - g(\varphi Y, W)\{A(X,Z) + B(X,Z)\} \\
&- 2g(\varphi X, Y)\{A(Z,W) + B(Z,W)\} - 2g(\varphi Z, W)\{A(X,Y) + B(X,Y)\} \\
&+ \frac{\sigma - \sigma(\xi,\xi)}{n+2}\{g(X,W)\eta(Y)\eta(Z) + g(Y,Z)\eta(X)\eta(W)
\end{aligned} \tag{7.6}$$

$$- g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)\},$$

where we put

$$T(X,Y,Z,W) = 8(\mathrm{CHR})_3(X,Y,Z,W) - \frac{1}{n+2}\{E(Y,Z)g(\varphi X,\varphi W) + E(X,W)g(\varphi Y,\varphi Z) - E(Y,W)g(\varphi X,\varphi Z) - E(X,Z)g(\varphi Y,\varphi W)\}, \tag{7.7}$$

$$E(X,Y) = 4\rho(\mathrm{CHR})_3(X,Y) - \frac{\tau(\mathrm{CHR})_3}{n+1}g(X,Y). \tag{7.8}$$

Equation (7.6) is locally written by

$$\begin{aligned} e^{-2f}\bar{T}_{kjih} - T_{kjih} &= \varphi_{kh}(A_{ji}+B_{ji}) + \varphi_{ji}(A_{kh}+B_{kh}) \\ &- \varphi_{ki}(A_{jh}+B_{jh}) - \varphi_{jh}(A_{ki}+B_{ki}) - 2\varphi_{kj}(A_{ih}+B_{ih}) \\ &- 2\varphi_{ih}(A_{kj}+B_{kj}) + \frac{\sigma-\sigma(\xi,\xi)}{n+2}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h \\ &- g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \end{aligned} \tag{7.6$'$}$$

Contracting (7.6′) with φ^{kh}, we obtain

$$A_{ji} + B_{ji} = \frac{\varphi^{ts}}{2(n+1)}(e^{-2f}\bar{T}_{tjis} - T_{tjis}) - \frac{\sigma-\sigma(\xi,\xi)}{n+1}\varphi_{ji}. \tag{7.9}$$

Substituting (7.9) into (7.6′), we have

$$\begin{aligned} & e^{-2f}\Big\{\bar{T}_{kjih} - \frac{\bar{\varphi}^{ts}}{2(n+1)}(\bar{T}_{tjis}\bar{\varphi}_{kh} + \bar{T}_{tkhs}\bar{\varphi}_{ji} - \bar{T}_{tjhs}\bar{\varphi}_{ki} - \bar{T}_{tkis}\bar{\varphi}_{jh} \\ &- 2\bar{T}_{tkjs}\bar{\varphi}_{ih} - 2\bar{T}_{tihs}\bar{\varphi}_{kj}) + \frac{2\tau(\overline{\mathrm{CHR}})_3}{(n+1)^2}(\bar{\varphi}_{kh}\bar{\varphi}_{ji} - \bar{\varphi}_{ki}\bar{\varphi}_{jh} - 2\bar{\varphi}_{kj}\bar{\varphi}_{ih})\Big\} \\ &- \Big\{T_{kjih} - \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} \\ &- 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) + \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih})\Big\} \\ &= \frac{\sigma-\sigma(\xi,\xi)}{n+2}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i), \end{aligned} \tag{7.10}$$

where we put $\bar{\varphi}_{ji} = \varphi_j{}^l\bar{g}_{li}$ and $\bar{\varphi}^{ji} = \varphi_l{}^i\bar{g}^{lj}$. The right hand side of (7.10) is written by

$$\begin{aligned} & e^{-2f}\frac{\tau(\overline{\mathrm{CHR}})_3}{(n+1)(n+2)}(\bar{g}_{kh}\bar{\eta}_j\eta_i + \bar{g}_{ji}\bar{\eta}_k\eta_h - \bar{g}_{ki}\bar{\eta}_j\eta_h - \bar{g}_{jh}\bar{\eta}_k\eta_i) \\ &- \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i), \end{aligned} \tag{7.11}$$

where we put $\bar{\eta}_i = \xi^l\bar{g}_{li}$. Thus we have

Theorem 7.1. *In an almost contact Riemannian manifold, the tensor field $S(X,Y)Z$ is invariant under the conformal transformation* (6.1), *where the tensor field $S_{kji}{}^h$ is defined by*

$$S_{kji}{}^{l}g_{lh} = T_{kjih} - \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} - 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) + \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih}) - \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \tag{7.12}$$

In particular, in a Sasakian manifold, we have from (4.8) and (4.9)

$$E_{ji} = 4\rho_{ji} - \frac{1}{n+1}(\tau + 3n^2 + 3n - 2)g_{ji} - 2(n+1)\eta_j\eta_i. \tag{7.13}$$

By virtue of (3.1) and (7.13), equation (7.7) is given by

$$T_{kjih} = 8\{R_{kjih} + \frac{1}{4}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih})\} - \frac{4}{n+2}\{\rho_{ji}(g_{kh} - \eta_j\eta_h) + \rho_{kh}(g_{ji} - \eta_j\eta_i) - \rho_{jh}(g_{ki}\eta_k\eta_i) - \rho_{ki}(g_{jh} - \eta_j\eta_h)\} + \frac{2\{\tau - 2(3n+4)\}}{(n+1)(n+2)}(g_{kh}g_{ji} - g_{ki}g_{jh}) - \frac{\tau + n(3n+5)}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i). \tag{7.14}$$

Moreover, in a Sasakian manifold, using the Bianchi identity, we obtain

$$R_{tjis}\varphi^{ts} = -(2n-1)\varphi_{ji} - \tilde{\rho}_{ji}, \tag{7.15}$$

where we put $\tilde{\rho}_{ji} = \rho_{jl}\varphi_i{}^l$. Using this, we get

$$\varphi^{ts}T_{tjis} = \frac{2\{\tau - (6n^3 + 13n^2 + 3n - 2)\}}{(n+1)(n+2)}\varphi_{ji} - \frac{8(n+1)}{n+2}\tilde{\rho}_{ji}. \tag{7.16}$$

By virtue of (4.9), (7.14), (7.15) and (7.16), the tensor field $S(X,Y,Z,W)$ in a Sasakian manifold satisfies

$$\begin{aligned} \frac{1}{2}S(X,Y,Z,W) &= 4R(X,Y,Z,W) - \frac{2}{n+2}\{\rho(Y,Z)g(\varphi X,\varphi W) \\ &+ \rho(X,W)g(\varphi Z,\varphi Z) - \rho(Y,W)g(\varphi X,\varphi Z) - \rho(X,Z)g(\varphi Y,\varphi W)\} \\ &+ \frac{2}{n+2}\{\tilde{\rho}(Y,Z)\varphi(X,W) + \tilde{\rho}(X,W)\varphi(Y,Z) - \tilde{\rho}(Y,W)\varphi(X,Z) \\ &- \tilde{\rho}(X,Z)\varphi(Y,W) - 2\tilde{\rho}(X,Y)\varphi(Z,W) - 2\tilde{\rho}(Z,W)\varphi(X,W)\} \\ &+ \frac{\tau - 2(3n+4)}{(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+ \frac{\{\tau + 2n(2n+3)\}}{(n+1)(n+2)}\{\varphi(X,W)\varphi(Y,Z) - \varphi(X,Z)\varphi(Y,W) \\ &- 2\varphi(X,Y)\varphi(Z,W)\} - \frac{\tau + 2n}{(n+1)(n+2)}\{(g(X,W)\eta(Y)\eta(Z) \\ &+ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)\}. \end{aligned} \tag{7.17}$$

Thus comparing the above equation and (1.6), we have

Theorem 7.2. *In a Sasakian manifold, the tensor field* $\frac{1}{8}S(X,Y,Z,W)$ *is a C-Bochner curvature tensor.*

We call the tensor field $\frac{1}{8}S(X,Y,Z,W)$ the *conformal* $(\mathrm{CHR})_3$*-curvature tensor* or the *C-Bochner curvature tensor* in an almost contact Riemannian manifold. And an almost contact Riemannian manifold M^{2n+1} is called *conformally* $(\mathrm{CHR})_3$*-flat* if the tensor field $S(X,Y)Z$ vanishes for any $X,Y,Z \in TM^{2n+1}$, identically.

8. A conformally $(\mathrm{CHR})_3$-flat almost contact Riemannian manifold

Let an almost contact Riemannian manifold M^{2n+1} be conformally $(\mathrm{CHR})_3$-flat. Then we have from (7.12)

$$\begin{aligned}(8.1)\quad T_{kjih} = {} & \frac{\varphi^{ts}}{2(n+1)}(T_{tjis}\varphi_{kh} + T_{tkhs}\varphi_{ji} - T_{tjhs}\varphi_{ki} - T_{tkis}\varphi_{jh} \\ & - 2T_{tkjs}\varphi_{ih} - 2T_{tihs}\varphi_{kj}) - \frac{2\tau(\mathrm{CHR})_3}{(n+1)^2}(\varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih}) \\ & + \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i).\end{aligned}$$

By virtue of (7.7) and (7.8), we have

$$\begin{aligned}(8.2)\quad T_{kjih} = {} & 8(\mathrm{CHR})_{3kjih} - \frac{4}{n+2}\{(g_{kh} - \eta_k\eta_h)\rho(\mathrm{CHR})_{3ji} \\ & + (g_{ji} - \eta_j\eta_i)\rho(\mathrm{CHR})_{3kh} - (g_{ki} - \eta_k\eta_i)\rho(\mathrm{CHR})_{3jh} \\ & - (g_{jh} - \eta_j\eta_h)\rho(\mathrm{CHR})_{3ki}\} + \frac{2\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}g_{ji} - g_{ki}g_{jh}) \\ & - \frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{ji}\eta_k\eta_h + g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i - g_{ki}\eta_j\eta_h).\end{aligned}$$

Contracting (8.2) with φ^{kh}, we have

$$\begin{aligned}(8.3)\quad T_{tjis}\varphi^{ts} = {} & 8(\mathrm{CHR})_{3tjis}\varphi^{ts} + \frac{4}{n+2}\{\rho(\mathrm{CHR})_{3jl}\varphi_i{}^l - \rho(\mathrm{CHR})_{3il}\varphi_j{}^l\} \\ & + \frac{2\tau(\mathrm{CHR})_3}{(n+1)(n+2)}\varphi_{ji}.\end{aligned}$$

Substituting (8.3) into (8.1), we obtain

$$\begin{aligned}(8.4)\quad 4(\mathrm{CHR})_{3kjih} = {} & \frac{2}{n+2}\{(g_{kh} - \eta_k\eta_h)\rho(\mathrm{CHR})_{3ji} + (g_{ji} - \eta_j\eta_i)\rho(\mathrm{CHR})_{3kh} \\ & - (g_{ki} - \eta_k\eta_i)\rho(\mathrm{CHR})_{3jh} - (g_{jh} - \eta_j\eta_h)\rho(\mathrm{CHR})_{3ki}\} \\ & + \frac{2\varphi^{ts}}{n+1}\{(\mathrm{CHR})_{3tjis}\varphi_{kh} + (\mathrm{CHR})_{3tkhs}\varphi_{ji} - (\mathrm{CHR})_{3tjhs}\varphi_{ki} \\ & - (\mathrm{CHR})_{3tkis}\varphi_{jh} - 2(\mathrm{CHR})_{3tkjs}\varphi_{ih} - 2(\mathrm{CHR})_{3tihs}\varphi_{kj}\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(n+1)(n+2)}\{(\rho(\mathrm{CHR})_{3jl}\varphi_i{}^l-\rho(\mathrm{CHR})_{3il}\varphi_j{}^l)\varphi_{kh}\\
&+(\rho(\mathrm{CHR})_{3kl}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_k{}^l)\varphi_{ji}-(\rho(\mathrm{CHR})_{3jl}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_j{}^l)\varphi_{ki}\\
&-(\rho(\mathrm{CHR})_{3kl}\varphi_i{}^l-\rho(\mathrm{CHR})_{3il}\varphi_k{}^l)\varphi_{jh}-2(\rho(\mathrm{CHR})_{3kl}\varphi_j{}^l-\rho(\mathrm{CHR})_{3jl}\varphi_k{}^l)\varphi_{ih}\\
&-2(\rho(\mathrm{CHR})_{3il}\varphi_h{}^l-\rho(\mathrm{CHR})_{3hl}\varphi_i{}^l)\varphi_{kj}\}-\frac{\tau(\mathrm{CHR})_3}{(n+1)(n+2)}(g_{kh}g_{ji}-g_{ki}g_{jh}\\
&-\eta_k\eta_h g_{ji}-\eta_j\eta_i g_{kh}+\eta_k\eta_i g_{jh}+\eta_j\eta_h g_{ki}+\varphi_{kh}\varphi_{ji}-\varphi_{ki}\varphi_{jh}-2\varphi_{kj}\varphi_{ih}).
\end{aligned}
$$

Next, contraction of (8.4) with g^{kh} gives us

$$
\begin{aligned}
(8.5)\quad & 6(2n+3)\rho(\mathrm{CHR})_{3ji}=(2n+5)\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}\\
&+6(n+2)\varphi^{ts}\{(\mathrm{CHR})_{3tjls}\varphi_i{}^l+(\mathrm{CHR})_{3tils}\varphi_j{}^l\}-6\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l.
\end{aligned}
$$

Contracting the above equation with $\varphi_b{}^j\varphi_a{}^i$ and changing the indices, we have

$$
\begin{aligned}
(8.6)\quad & (n+2)\varphi^{ts}\{(\mathrm{CHR})_{3tils}\varphi_j{}^l+(\mathrm{CHR})_{3tjls}\varphi_i{}^l\}\\
&=(2n+3)\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l+\rho(\mathrm{CHR})_{3ji}+\rho(\mathrm{CHR})_3(\xi,\xi)\eta_j\eta_i\\
&-(n+2)\varphi^{ts}\xi^m\{\varphi_j{}^l\eta_i+\varphi_i{}^l\eta_j\}-\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}.
\end{aligned}
$$

Substituting (8.6) into (8.5), we get

$$
\begin{aligned}
&12(n+1)\{\rho(\mathrm{CHR})_{3ji}-\rho(\mathrm{CHR})_{3ml}\varphi_j{}^m\varphi_i{}^l\}\\
&=(2n-1)\{\rho(\mathrm{CHR})_{3jl}\xi^l\eta_i+\rho(\mathrm{CHR})_{3il}\xi^l\eta_j\}\\
&+6\rho(\mathrm{CHR})_3(\xi,\xi)\eta_j\eta_i-6(n+2)\varphi^{ts}(\mathrm{CHR})_{3tlms}\xi^m(\varphi_j{}^l\eta_i+\varphi_i{}^l\eta_j).
\end{aligned}
$$

From the above equation, we can easily get

$$
\begin{aligned}
(8.7)\quad & \rho(\mathrm{CHR})_3(X,Y)-\rho(\mathrm{CHR})_3(\varphi X,\varphi Y)+\rho(\mathrm{CHR})_3(\xi,\xi)\eta(X)\eta(Y)\\
&-\{\rho(\mathrm{CHR})_3(\xi,X)\eta(Y)+\rho(\mathrm{CHR})_3(\xi,Y)\eta(X)\}=0.
\end{aligned}
$$

Next, we have from (8.7)

$$
(8.8)\qquad \tau(\mathrm{CHR})_3=\frac{n+4}{3(n+2)}\rho(\mathrm{CHR})_3(\xi,\xi)+(\mathrm{CHR})_{3tmls}\varphi^{ts}\varphi^{ml}.
$$

Thus we have

THEOREM 8.1. *Let M^{2n+1} be a conformally $(\mathrm{CHR})_3$-flat almost contact Riemannian manifold. Then the $(\mathrm{CHR})_3$-curvature tensor, the $(\mathrm{CHR})_3$-Ricci tensor $\rho(\mathrm{CHR})_3$ and the $(\mathrm{CHR})_3$-scalar curvature $\tau(\mathrm{CHR})_3$ satisfy* (8.4), (8.7) *and* (8.8), *respectively.*

By virtue of (7.17), in a conformally $(\mathrm{CHR})_3$-flat Sasakian manifold, the Riemannian curvature tensor satisfies

$$
\begin{aligned}
(8.9)\quad & R(X,Y,Z,W)=\frac{1}{2(n+2)}\{\rho(Y,Z)g(\varphi X,\varphi W)+\rho(X,W)g(\varphi Y,\varphi Z)\\
&-\rho(Y,W)g(\varphi X,\varphi Z)-\rho(X,Z)g(\varphi Y,\varphi W)\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2(n+2)}\{\tilde\rho(Y,Z)F(X,W) + \tilde\rho(X,W)F(Y,Z) - \tilde\rho(Y,W)F(X,Z) \\
&- \tilde\rho(Y,W)F(X,Z) - \tilde\rho(X,Z)F(X,Y) - 2\tilde\rho(X,Y)F(Z,W) - 2\tilde\rho(Z,W)F(X,Y)\} \\
&- \frac{\tau - 2(3n+4)}{4(n+1)(n+2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\
&+ \frac{\tau + 2n(2n+3)}{4(n+1)(n+2)}\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\} \\
&+ \frac{\tau + 2n}{4(n+1)(n+2)}\{\eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W) \\
&- \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z)\}.
\end{aligned}
$$

Contracting (8.9) with g^{kh}, we get $g_{ji} = (2n+1)\eta_j\eta_i$. Thus we have

THEOREM 8.2. *There does not exist any conformally* (CHR)$_3$*-flat Sasakian manifold.*

From the above theorem, we can easily know

REMARK 8.1. There does not exist any C-Bochner flat Sasakian manifold.

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