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SECTIONAL CURVATURE IN 4-DIMENSIONAL MANIFOLDS OF NEUTRAL SIGNATURE

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Abstract. We consider the sectional curvature function on a 4-dimensional manifold admitting a metric g of neutral signature, $(+,+,-,-)$ together with a review of the situation for the other two signatures. The main results of the paper are: first, that if the sectional curvature function is not a constant function at any $m \in M$ (actually a slightly weaker assumption is made), the conformal class of g is always uniquely determined and in almost all cases g is uniquely determined on M, second, a study of the special cases when this latter uniqueness does not hold, third, the construction of the possible metrics in this latter case, fourth, some remarks on sectional curvature preserving vector fields and finally the complete solution when (M, g) is Ricci flat.

1. Introduction

The sectional curvature function has been studied for metrics of positive definite and Lorentz signature and a brief summary of this work which is relevant to the present article will be given at the beginning of Section 2. The purpose of this paper is to consider the case when g has neutral signature.

To settle notation, g, unless otherwise stated, is a smooth metric of *neutral signature* $(+,+,-,-)$ on a 4-dimensional, smooth, connected manifold M with Levi-Civita connection ∇, sometimes written, collectively, as (M, g). For $m \in M$, T_mM denotes the tangent space to M at m, B_m denotes the 6-dimensional vector space of 2-forms (here referred to as *bivectors*) at m and $*$ denotes the Hodge duality (linear) operator on B_m. Let $\sim$ denote the equivalence relation on nonzero bivectors at m given by $B_1 \sim B_2 \Leftrightarrow B_1 = kB_2$ for $k \in \mathbb{R}$ and let the resulting equivalence classes of *projective bivectors* be denoted by PB_m. Topologically, B_m and PB_m are $\mathbb{R}^6$ and $P\mathbb{R}^5$. The matrix rank of any nonzero member $F \in B_m$ is an even number and if it is 2, F is called *simple* and may be written in components as $F^{ab} = p^aq^b - q^ap^b$ for $p, q \in T_mM$ (sometimes abbreviated to $p \wedge q$). The 2-dimensional subspace (2-space) of T_mM spanned by p and q is uniquely determined by F and called the *blade* of F. The set of all simple members of B_m is denoted by SB_m and one easily passes to the subset (projective simple bivectors) of PB_m

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denoted by PSB_m which is in bijective correspondence with the collection of blades of members of SB_m. If $u, v \in T_mM$ the inner product $g(m)(u,v)$ is denoted by $u \cdot v$ and $0 \neq u \in T_m$ is called *spacelike*, respectively *timelike* or *null*, if $u \cdot u > 0$, (respectively < 0 or $= 0$). A 2-space W of T_mM is called *spacelike* if each nonzero member of W is spacelike, or each nonzero member of W is timelike, *timelike* if W contains exactly two, null 1-dimensional subspaces (*null directions*), *null* if V contains exactly one null direction and *totally null* if each nonzero member of V is null. Thus a totally null 2-space consists, apart from the zero vector, of null vectors any two of which are orthogonal. This classification of 2-spaces is mutually exclusive and exhaustive.

The set of all spacelike (respectively, timelike) 2-spaces at m is denoted by S_m (respectively, T_m) and the combined set of all null *and* totally null 2-spaces at m is denoted by N_m. A member $F \in SB_m$ is called *spacelike*, (respectively *timelike*, *null* or *totally null*) if its blade is of that type and F is then spacelike if $F^{ab}F_{ab} > 0$, timelike if $F^{ab}F_{ab} < 0$ and null or totally null if $F^{ab}F_{ab} = 0$. If one defines an orthonormal basis x, y, s, t of T_mM by $x \cdot x = y \cdot y = -s \cdot s = -t \cdot t = 1$ and its associated null basis (of null vectors) l, n, L, N where $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l.n = L.N = 1$ (and all other such inner products are zero) then, for example, $x \wedge y$ and $s \wedge t$ are spacelike, $x \wedge t$ $(= -l \wedge n)$ is a timelike, $l \wedge y$ and $l \wedge s$ are null and $l \wedge N$ and $n \wedge L$ are totally null 2-spaces at m. One may always extend an orthonormal basis at m to a smooth orthonormal basis on some open neighbourhood of m and hence, as above, extend a null basis at m to a smooth null basis in this neighbourhood. The manifold theory associated with the above bivector (and related) sets is discussed in [**12**] and one may identify PSB_m with the 4-dimensional, compact, connected Grassmann manifold G_m of all (real) 2-spaces at m and define its open submanifold $\overline{G}_m \equiv G_m \setminus N_m$ which is the disjoint union of the two open submanifolds S_m and T_m. In the usual manifold topology on G_m each member of N_m is a limit point of S_m and of T_m.

Now define the Lie algebras (under matrix commutation) $\overset{+}{S}_m \equiv \{A \in B_m : \overset{*}{A} = A\}$ and $\overset{-}{S}_m \equiv \{A \in B_m : \overset{*}{A} = -A\}$ which are Lie-isomorphic to each other and to $o(1,2)$. It is known that the Lie algebra B_m is the Lie algebra product $B_m \equiv \overset{+}{S}_m \oplus \overset{-}{S}_m$ and that the only simple members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are totally null [**22**] and conversely any totally null bivector lies in $\overset{+}{S}_m$ or $\overset{-}{S}_m$. Further, the blades of two totally null bivectors, one of which is in $\overset{+}{S}_m$ and the other in $\overset{-}{S}_m$, intersect in a unique null direction whereas if they are both in $\overset{+}{S}_m$ or both in $\overset{-}{S}_m$ their intersection is trivial. It is also useful to note that if $F \in B_m$ is *null* its unique null direction is orthogonal to all other members of its blade and that it may be written as $F = \overset{+}{F} + \overset{-}{F}$ for unique, nonzero, totally null members $\overset{+}{F} \in \overset{+}{S}_m$ and $\overset{-}{F} \in \overset{-}{S}_m$ and where the blades of $\overset{+}{F}$ and $\overset{-}{F}$ intersect in the unique null direction in the blade of F. If $P \in \overset{+}{S}_m$ and $Q \in \overset{-}{S}_m$, $P^{ab}Q_{ab} = 0$.

The Riemann tensor Riem with components $R^a{}_{bcd}$, its associated Ricci tensor Ricc with components $R_{ab} \equiv R^c{}_{acb}$, the Ricci scalar $R \equiv R_{ab}g^{ab}$ and the corresponding Weyl conformal tensor C with components $C^a{}_{bcd}$ are related at each $m \in M$ by

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{6}G_{abcd} \tag{1.1}$$

where G is the tensor with components $G_{abcd} = \frac{1}{2}[g_{ac}g_{bd} - g_{ad}g_{bc}]$,

$$E_{abcd} = \frac{1}{2}[\widetilde{R}_{ac}g_{bd} - \widetilde{R}_{ad}g_{bc} + \widetilde{R}_{bd}g_{ac} - \widetilde{R}_{bc}g_{ad}] \tag{1.2}$$

and where the tracefree Ricci tensor $\widetilde{\text{Ricc}}$ has been introduced with components $\widetilde{R}_{ab} = R_{ab} - \frac{R}{4}g_{ab}$. Then $E(m) = 0$ if and only if $\widetilde{\text{Ricc}}(m) = 0$, that is, Ricc $= \frac{R}{4}g$ at m, which is equivalent to the Einstein space condition at m.

The Weyl conformal tensor at m may be classified algebraically [**8**] in a manner similar to that given by Petrov in the Lorentz case [**16**] (the Petrov classification). The details of this are not needed except for the concept of a principal null direction of C at m. A null direction spanned by a null $l \in T_mM$ is called a *principal null direction for* $C(m) \neq 0$ if $C_{abcd}l^a l^c = l_b p_d + p_b l_d$ for some 1-form p at m. Such null directions are important in this classification and have the property (which is useful later) that if $k_0 \in T_mM$ spans a principal null direction of $C(m)$ then the null directions spanned by those null vectors in some open neighbourhood of k_0 in the topology of T_mM cannot all be principal null directions of $C(m)$ since they force the contradiction $C(m) = 0$. This result is trivially true in the Lorentz case since, for $C(m) \neq 0$, only finitely many such directions may exist, but less obviously true in the present case since it is possible for infinitely many such directions to exist in neutral signature even if $C(m) \neq 0$ [**8**].

2. The Sectional Curvature Function on (M,g)

The sectional curvature function at m is a real-valued function defined on the open submanifold $\overline{G}_m$ of G_m, $\sigma_m : \overline{G}_m \to \mathbb{R}$ and given, for a representative member F of SB_m, by

$$\sigma_m(F) \equiv \frac{R_{abcd}F^{ab}F^{cd}}{2G_{abcd}F^{ab}F^{cd}} = \frac{R_{abcd}F^{ab}F^{cd}}{2F^{ab}F_{ab}} \tag{2.1}$$

It is clear that this definition is independent of the representative member F used to fix the member of $\overline{G}_m$. (For the function σ_m the identification of PSB_m with G_m, through a representative member of SB_m, will always be made.) Now a *simple* bivector F is null *or* totally null if and only if $G_{abcd}F^{ab}F^{cd} (= F^{ab}F_{ab}) = 0$ and so such members of G_m must be excluded from the domain of σ_m. The interpretation of $\sigma_m(F)$ for $F \in \overline{G}_m$ is as follows; there exists a neighbourhood U of m such that the geodesics of ∇ starting from m with initial directions in the blade of F generate a 2-dimensional submanifold M' of U (and hence of M) such that M' has an induced metric h from g and $\sigma_m(F)$ is then the Gauss curvature of M' with respect to h at m. It seems that sectional curvature is essentially what Riemann had in mind in his famous address on curvature [**18**]. If σ_m is a constant function on

$\overline{G_m}$, (1.1) holds at m with $C = E = 0$ and σ_m is, in this case, trivially continuously extendible to a (constant) function on G_m.

If g is *positive definite* σ_m is defined on the whole of G_m for each m and provided it is not a constant function on G_m for any m in some open dense subset of M, the collection of functions σ_m at each $m \in M$ uniquely determines the metric on M from which it came (and this applies for all dimensions $\geqslant 4$) [**15**]. For *Lorentz signature*, all 2-spaces are either spacelike, timelike or null, as defined above, (with totally null ones impossible) and thus σ_m is only immediately defined on the set of non-null 2-spaces $\overline{G_m}$. If, however, σ_m is continuously extendible to a *single* null 2-space (in the usual manifold topology on G_m), *it is a constant function on* $\overline{G_m}$ and hence (trivially) continuously extendible to a constant function on the whole of G_m [**6, 7, 13**] and the components R_{abcd} are proportional to G_{abcd} at m. [In fact, to achieve the constancy of σ_m on $\overline{G_m}$ it is sufficient to achieve it on S_m or on T_m, as is easily checked (cf [**5**]).] Again in this case if σ_m is *not* a constant function on $\overline{G_m}$ for any $m \in M$ the collection of functions σ_m at each $m \in M$ uniquely determines each set $N_m = G_m \setminus \overline{G_m}$ and hence, as is easily shown, the null cone at each m. In fact it uniquely determines the metric on M from which it came except in certain very special circumstances [**6, 19, 7**]. Further, if g is Ricci-flat and *nonflat* (in the sense that Riem does not vanish over any nonempty open subset of M) the functions σ_m for each $m \in M$ uniquely determine the metric on M which gave rise to them in all cases [**6, 7**].

From now on, unless specified to the contrary, g has neutral signature.

LEMMA 1. (i) *If σ_m is a constant function on S_m or on T_m it is a constant function on $\overline{G_m}$ and hence continuously extendible to a constant function on G_m.*

(ii) *If $k_0 \in T_mM$ is null and $R_{ab}k^ak^b = 0$ for each* null *k in some neighbourhood of $k_0 \in T_mM$, (M, g) satisfies the Einstein space condition $R_{ab} = \frac{R}{4}g_{ab}$ at m.*

(iii) *Let $F_0 \in N_m$ and suppose $R_{abcd}F^{ab}F^{cd} = 0$ for each $F \in U \cap N_m$ for some open neighbourhood U of F_0 in G_m. Then the constant curvature condition $R_{abcd} = \frac{R}{6}G_{abcd}$ holds at m.*

PROOF. (i) The proof is straightforward. Suppose σ_m is a constant function on S_m with value d. In the usual basis let $Q = x \wedge t$ be timelike and construct the spacelike members $\overset{\alpha}{P} \equiv x \wedge (t + \alpha y)$ for $\alpha > 1$. Then define the tensor $H_{abcd} = R_{abcd} - 2dG_{abcd}$ so that $H_{abcd}\overset{\alpha}{P}{}^{ab}\overset{\alpha}{P}{}^{cd} = 0$ for each α. This is a polynomial in α and vanishes for all $\alpha > 1$. It follows that $H_{abcd}Q^{ab}Q^{cd} = 0$ and hence that $\sigma_m(Q) = d$. The proof when σ_m is a constant function on T_m is similar.

(ii) The set of null *directions* at m is a 2-dimensional, smooth submanifold of $P\mathbb{R}^3$. Using the null basis l, n, L, N for T_mM given in Section 1 and choosing projective coordinates y, z in this submanifold about the null direction spanned by $k_0(= l) = (1, 0, 0, 0)$, so that $k = (1, -yz, y, z)$ for suitable $y, z \in \mathbb{R}$, the condition $R_{ab}k^ak^b = 0$ becomes a polynomial relation $P(y, z) = 0$ on an open subset of $\mathbb{R}^2$ and hence all polynomial coefficients are zero. So $R_{01} = R_{23}$ with all other components of Ricc vanishing at m and the result follows from the completeness relation $g_{ab} = l_an_b + n_al_b + L_aN_b + N_aL_b$.

(iii) Suppose F_0 is null in the given condition for (iii). So in a null basis at m chosen so that, say, $F_0 = l \wedge y$, one has $R_{abcd}l^a y^b l^c y^d = 0$. Defining a symmetric tensor T at m by $T_{ab} = R_{cadb}l^c l^d$ and considering the members $l \wedge (y + \mu L)$ and $l \wedge (y + \nu N)$ with $\mu, \nu \in \mathbb{R}$ sufficiently small that these members are in $U \cap N_m$, one gets $T(y,y)(\equiv T_{ab}y^a y^b) = T(y,L) = T(L,L) = T(y,N) = T(N,N) = 0$ and, since $T_{ab}l^b = 0$, also $T(l,l) = T(l,n) = T(l,L) = T(l,N) = 0$, which imply $T(L,N) = 0$. It follows that $T_{ab} = al_a l_b + b(l_a L_b + L_a l_b) + c(l_a N_b + N_a l_b)$ and so $g^{ab}T_{ab} = 0$, that is, $R_{ab}l^a l^b = 0$ at m from the definition of T. Since this last equation then holds for each null vector in some neighbourhood of l in $T_m M$ it follows from part (ii) that the Einstein space condition holds at m. Then the remarks following (1.2) show that $E(m) = 0$ and so from (1.1) and the above form for T each null direction in some open neighbourhood of that spanned by l is a principal null direction for $C(m)$ as given at the end of Section 1 [**8**] with, allowing for an abuse of notation, the 1-form p a linear combination of l, L and N. It follows that $C(m) = 0$. If F_0 is totally null, say $F_0 = l \wedge N$ in the null basis, a similarly argument and result arise by considering the bivectors $l \wedge (N + \lambda L)$ for $\lambda \in \mathbb{R}$ sufficiently small. The result now follows and, for dimension 4, strengthens a result in [**2**]. □

Lemma 2. *For (M, g), if F and F' are bivector representatives of two distinct members of N_m, the bivector $F + \lambda F'$, for* some $0 \neq \lambda \in \mathbb{R}$, *(is simple and) also represents a member of N_m if and only if the 2-spaces represented by F and F' intersect in a null direction (and then the bivector $F + \lambda F'$ represents a member of N_m for* all $\lambda \in \mathbb{R}$*). Thus the subset $N_m \subset G_m$ may be used to fix the collection of null vectors of g at m.*

Proof. Let F, F' represent distinct members of N_m. Then F and F' are independent and simple and if their blades intersect in a *null* direction, say k, $F = k \wedge u$ and $G = k \wedge v$ for $u, v \in T_m M$. Then whether F and F' are null or totally null, $k.u = k.v = 0$ and it easily follows that $F + \lambda F'$ represents a member of N_m for *all* $\lambda \in \mathbb{R}$. Conversely, write $F = p \wedge q$, $F' = r \wedge s$ and $F + \lambda F' = e \wedge f$ for $p, q, r, s, e, f \in T_m M$, for some $0 \neq \lambda \in \mathbb{R}$ and with F, F' and $F + \lambda F'$ representing members of N_m. One may always choose $p.p = p.q = r.r = r.s = e.e = e.f = 0$ and then a contraction of $F + \lambda F' = e \wedge f$ with e gives $(e.q)p - (e.p)q + \lambda(s.e)r - \lambda(e.r)s = 0$. If p, q, r and s are independent in $T_m M$ one achieves the contradiction that e is orthogonal to each member of the basis p, q, r, s for $T_m M$ and so it follows that p, q, r and s are not independent and hence that the blades of F and F' intersect in a direction spanned, say, by $k \in T_m M$. Now write $F = k \wedge r'$ and $F' = k \wedge s'$ and suppose that k is not null. Then one may choose r' and s' null with each orthogonal to k. Then since F, F' and $H \equiv F + \lambda F'$ represent members of N_m, $F_{ab}F^{ab} = F'_{ab}F'^{ab} = H_{ab}H^{ab} = 0$ from which it follows that $F_{ab}F'^{ab} = 0$ and so $(k.k)(r'.s') = 0$. Since $k.k \neq 0$ one finds $r'.s' = 0$ and so $r' \wedge s'$ is totally null. But $k.r' = k.s' = 0$ and so $k \in r' \wedge s'$ which contradicts the fact that $k.k \neq 0$ (or, alternatively, that F and F' are independent). So k is null (and hence $F + \lambda F'$ is null or totally null for all $\lambda \in \mathbb{R}$). [It is remarked that a consideration of the null 2-spaces $l \wedge y$ and $n \wedge y$ in a null basis shows that it is insufficient for simple bivectors F, F' representing 2-spaces in N_m to have intersecting blades in order that $F + \lambda F'$ represent a 2-space in N_m for any $0 \neq \lambda \in \mathbb{R}$.] It is then clear that N_m, in particular those "pencils" of the form $F + \lambda F'$ which lie in N_m, reproduce the set of null vectors of g at m. □

It is remarked first that the 2-dimensional manifold of null directions which emerges in this lemma serves to distinguish the signature of g, being topologically $\mathbb{S}^1 \times \mathbb{S}^1$ for neutral signature and topologically $\mathbb{S}^2$ for Lorentz signature. Second, it is pointed out that *in the Lorentz case* the set Z of all members of all 2-spaces in N_m consists, apart from the zero vector, of all spacelike and all null vectors at m and, as an alternative to lemma 2, the null cone is then the boundary of this set (in the natural topology on T_mM) and is hence determined by it (cf [**17**]). This fails for neutral signature since Z is then equal to T_mM.

The next lemma considers the continuous extendibility of the sectional curvature function to N_m

Lemma 3. (i) *Consider (M,g) with $m \in M$, let $A \equiv S_m \cup T_m = \overline{G_m}$ and let F be a representative bivector in N_m. Suppose that $R_{abcd}F^{ab}F^{cd} \neq 0$. If U is any open neighbourhood of F in G_m, σ_m is unbounded on $A \cap U$.*

(ii) *If σ_m is continuously extendible to* a single member of N_m *it is a constant function on $\overline{G_m}$ and hence may be extended continuously to G_m. Thus the constant curvature condition holds at m.*

Proof. Since F is necessarily a limit point of A, the proof of part (i) is essentially the same as that given in [**13**] (see [**17**] and cf [**1**]). The idea is to consider the restriction f to S^5 of the natural projection $\mathbb{R}^6 \to P\mathbb{R}^5$ (that is, $B_m \to PB_m$) and to show that the map $h : S^5 \to \mathbb{R}$ given (using an identification mentioned earlier) by $h(Q) = R_{abcd}Q^{ab}Q^{cd}$ is bounded away from zero on some compact neighbourhood of $Q_0 \in f^{-1}(F)$ whilst noting that $G_{abcd}F^{ab}F^{cd} = 0$.

For part (ii), if σ_m is continuously extendible to $F_0 \in N_m$ then σ_m is continuous as a map $\{F_0\} \cup A \to \mathbb{R}$. Suppose $\sigma_m(F_0) = a$ and let $V = (a - \epsilon, a + \epsilon)$ for $0 < \epsilon \in \mathbb{R}$. Then if $U = \sigma_m^{-1}V$, $F_0 \in U$, U is open in $\{F_0\} \cup A$ with $U = (\{F_0\} \cup A) \cap W$ for W open in G_m and $U \cap A \neq \emptyset$ since F_0 is a limit point of A. Then σ_m is bounded on U and since $A \cap W \subset U$, σ_m is bounded on $A \cap W$. Also W is an open neighbourhood of F_0 in G_m and then it follows from part (i) that $R_{abcd}F_0^{ab}F_0^{cd} = 0$ and, in fact, this latter result is true for each member of $W \cap N_m$. Then Lemma 1(iii) shows that the constant curvature condition holds at m. Thus σ_m is a constant function on $\overline{G_m}$ and trivially continuously extendible to G_m. □

Theorem 1. *Suppose that g and g' are smooth metrics on a 4-dimensional, smooth, connected manifold M with g of neutral signature and that the (closed) subset $B \subset M$ of precisely those points where* Riem *(for g) vanishes has empty interior in the topology of M (so that (M,g) is* nonflat.*) Suppose also that g and g' determine the same sectional curvature function at each $m \in M$ and that for no $m \in M \setminus B$ is it a constant function. Then, using a prime to denote corresponding quantities with respect to g' one has, on M, for some real-valued, nowhere-zero function $\phi : M \to \mathbb{R}$*

(i) $g' = \phi g$, (iii) $R'^a{}_{bcd} = \phi R^a{}_{bcd}$, (v) $R' = R$,
(ii) $R'_{abcd} = \phi^2 R_{abcd}$, (iv) $R'_{ab} = \phi R_{ab}$, (vi) $C' = \phi C$.

PROOF. It is perhaps unrealistic to demand that Riem is nowhere zero on M and this is why the subset B is introduced (but with the reasonable nonflat assumption, $\operatorname{int} B = \emptyset$, where int denotes the interior operator in the manifold topology of M). The condition that the sectional curvature function σ_m for g is not constant at any $m \in M \setminus B$ and equals the sectional curvature function σ'_m for g' on this subset means (Lemma 3(ii)) that the subset of G_m on which σ_m and σ'_m are not defined is the same subset for both g and g' and, in fact, equals the set N_m as defined for g *and for* g'. Thus from lemma 2 the set of null vectors for these metrics agree everywhere on $M \setminus B$ and hence g' is conformally related to g on this subset. It follows that g' and g are conformally related on M (and that g' has neutral signature) and so part (i) of the theorem holds and for $\phi : M \to \mathbb{R}$ smooth since g and g' are. It is remarked that since Riem vanishes on B the sectional curvature functions σ'_m and σ_m are zero maps for each $m \in B$. It then follows that Riem$'$ vanishes on precisely the subset B. To see this note that (2.1) and part (i) of the theorem show that $R'_{abcd}F^{ab}F^{cd} = 0$ at each $m \in B$ and for each (simple) F representing a spacelike or timelike 2-space at m and hence, by continuity, for *each* simple F at each such m. Applying this for the simple bivectors $x \wedge y$, $x \wedge s$,...,$s \wedge t$ in $\overline{G_m}$ formed from the orthonormal basis x, y, s, t for T_m, together with some other judiciously chosen simple bivectors and the symmetries of R'_{abcd} one sees that Riem$'$ vanishes on B and, by reversing the argument, nowhere else. Now the equality of the sectional curvatures together with part (i) gives $(R'_{abcd} - \phi^2 R_{abcd})F^{ab}F^{cd} = 0$ for each simple F and then, by a similar argument to that given in the last sentence, one sees that (ii) holds on M. Then (iii), (iv) and (v) immediately follow. For (vi) use (1.1) and (1.2) to compute the Weyl conformal tensor directly using the results previously found (including $\tilde{R'_{ab}} = \phi \tilde{R_{ab}}$ and recalling that C has components $C^a{}_{bcd}$). [It is remarked that, although (i) above implies the equality of the Weyl tensors, they may each be zero. In fact, if one assumes that the Weyl tensor of g (and hence of g') is nowhere zero on M one immediately has $\phi = 1$ on M (from (vi)) and so $g' = g$ on M but this will not be assumed here. The result (vi) applies in all cases. $\square$

Continuing with the assumptions of theorem 1 the argument now follows that given in [**6, 7**]. Let $U \subset M$ be the open subset of M on which the Weyl tensor of g (and hence of g') is not zero (and so $U \subset M \setminus B$) and let $V \subset M$ be the open set on which $d\phi$ does not vanish. Then $\phi = 1$ on U, $U \cap V = \emptyset$ and ϕ is a (nonzero) constant on each component of $\operatorname{int}(M \setminus V)$. It follows that Riem$'$ = Riem on $\operatorname{int}(M \setminus V)$ since their metrics differ only by a constant conformal factor. Now Riem$'$ and Riem cannot vanish over any nonempty open subset of $\operatorname{int}(M \setminus V)$ (by the definition of B) and so part (iii) above shows that $\phi = 1$ on $\operatorname{int}(M \setminus V)$. Define the closed subset $W \subset M$ comprising precisely those points of M at which $\phi = 1$. Then $U \subset W$ and $\operatorname{int}(M \setminus V) \subset W$. Then *disjointly* decompose M as $M = V \cup \operatorname{int} W \cup K$ where the closed set K is defined by the disjointness of the decomposition and has empty interior in M since any nonempty open subset U' contained in K would necessarily satisfy $U' \cap V = \emptyset$ and so ϕ would be constant on each of its components. Thus, since $\operatorname{int} B = \emptyset$, U' is not contained in B (and so Riem is nonzero and $\phi = 1$ over the nonempty, open subset $U' \cap (M \setminus B)$ of U'). It follows that $U' \cap (\operatorname{int} W) \neq \emptyset$ contradicting the disjointness of the decomposition. Thus M has been disjointly decomposed into the open subset $\operatorname{int} W$ where $\phi = 1$ (and hence $g' = g$) together with the open subset V where $\mathrm{d}\phi$ does not vanish and which is conformally flat for g and g' (since $U \cap V = \emptyset$) and a closed subset K with empty interior. So leaving aside the subset K one is left with the open set $\operatorname{int} W$, where $g' = g$, together with (perhaps more interestingly) the conformally flat open subset V upon which $\mathrm{d}\phi$ is nowhere zero.

3. Analysis of the Subset V

On V one has a nowhere-zero closed 1-form $\mathrm{d}\phi$ with components ϕ_a and V is conformally flat for g and g', $C' = C = 0$. One could now argue as in [**7**] but the following is a little easier and gives the same results. From $C = 0$ and (1.1) and (1.2) one has at each $m \in M$ and for the metric g

$$R_{abcd}(\equiv g_{ae}R^{e}{}_{bcd}) = \frac{1}{2}[\widetilde{R}_{ac}g_{bd} - \widetilde{R}_{ad}g_{bc} + \widetilde{R}_{bd}g_{ac} - \widetilde{R}_{bc}g_{ad}] + \frac{R}{6}G_{abcd} \tag{3.1}$$

and also the conformally flat Bianchi identities, one for each of the connections ∇ of g and ∇' of g',

$$R_{ca;b} - R_{cb;a} = \frac{1}{6}[g_{ac}R_{,b} - g_{cb}R_{,a}] \qquad R'_{ca|b} - R'_{cb|a} = \frac{1}{6}[g'_{ac}R'_{,b} - g'_{cb}R'_{,a}] \tag{3.2}$$

where a comma denotes a partial derivative and a semi-colon and the symbol $|$ denote covariant derivatives with respect to ∇ and ∇', respectively. Now evaluate the second in (3.2), using Theorem 1 parts (iv) and (v), and subtract from it ϕ times the first in (3.2). The partial derivatives disappear and terms in $P^a_{bc} \equiv \Gamma'^a_{bc} - \Gamma^a_{bc}$ arise where Γ and Γ' are the Christoffel symbols from ∇ and ∇', respectively, and where, since $g' = \phi g$ (and $\phi^a \equiv g^{ab}\phi_b$)

$$P^a_{bc} = \frac{1}{2}\phi^{-1}[\phi_c\delta^a_b + \phi_b\delta^a_c - \phi^a g_{bc}] \tag{3.3}$$

One finds after a short calculation

$$\phi_b R_{ac} - \phi_a R_{bc} = \phi R_{ae}P^e_{bc} - \phi R_{be}P^e_{ac} = R_{be}\phi^e g_{ac} - R_{ae}\phi^e g_{bc} \tag{3.4}$$

A contraction with g^{ac} then shows that $\widetilde{R}_{ab}\phi^b = 0$ and a back substitution into (3.4) reveals that $\widetilde{R}_{ac}\phi_b - \widetilde{R}_{bc}\phi_a = 0$ and hence that $\widetilde{R}_{ab} = \psi\phi_a\phi_b$ for some function $\psi : V \to \mathbb{R}$. Thus $\psi(\phi^a\phi_a) = 0$ on V. If, at any $m \in V$, $\psi(m) = 0$ then $\widetilde{\mathrm{Ricc}}(m) = 0$ and so $E(m) = 0$ from (1.2). Then (1.1) shows that $R_{abcd} = \frac{R}{6}G_{abcd}$ at m. Now if $R(m) \neq 0$, σ_m is a constant function at $m \in M \setminus B$ and a contradiction is obtained and if $R(m) = 0$, $\mathrm{Riem}(m) = 0$. It follows, since $\operatorname{int} B = \emptyset$, that ψ cannot vanish over any nonempty open subset of V. So $\phi^a\phi_a = 0$ on V and hence ϕ^a is null on V (and, from Theorem 1(i), with respect to both metrics g and g'). [Thus in the positive definite case and from the definition of V, $V = \emptyset$ and the result in [**15**] is recovered.] A use of (3.3) now gives $\phi_{a;b} = \phi_{a|b} + \phi^{-1}\phi_a\phi_b$ and this enables further ∇ and ∇' covariant derivatives to be taken as follows.

$$\phi_{a;bc} = (\phi_{a|b})_{;c} - \phi^{-2}\phi_a\phi_b\phi_c + \phi^{-1}\phi_a\phi_{b;c} + \phi^{-1}\phi_{a;c}\phi_b \tag{3.5}$$

and so

$$(\phi_{a|b})_{;c} = \phi_{a|bc} = \phi_{a|e}P^e_{bc} + \phi_{e|b}P^e_{ac} = (2\phi)^{-1}[\phi_{a|c}\phi_b + \phi_{c|b}\phi_a + 2\phi_{a;b}\phi_c] \tag{3.6}$$

Putting (3.5) and (3.6) together gives

$$\phi_{a;bc} - \phi_{a;cb} = \phi_{a|bc} - \phi_{a|cb} + (2\phi)^{-1}[\phi_{a;c}\phi_b - \phi_{a;b}\phi_c] \tag{3.7}$$

(which corrects a minus sign error in equation (9.48) in [**7**] but which ultimately cancels out without problem in that reference). Using the Ricci identities for ∇ and ∇' in (3.7) followed by a contraction with ϕ^b (and noting that since $\phi_{a;b} = \phi_{b;a}$, $\phi_{a;b}\phi^b = 0$) one gets

$$\phi^b[\phi_d R^d{}_{abc}] = \phi^b[\phi_d R'^d{}_{abc}] = \phi[\phi^b\phi_d R^d{}_{abc}] \tag{3.8}$$

Then (3.1), the condition $\phi^a\phi_a = 0$ on V and the above expression for $\widetilde{\text{Ricc}}$ give $R_{abcd}\phi^b\phi^d = -\frac{R}{12}\phi_a\phi_c$. Thus if $R(m) \neq 0$ for some $m \in V$, R does not vanish in some open neighbourhood of m and hence $\phi = 1$ and dϕ vanishes on this neighbourhood, contradicting the definition of V. It follows that $R = 0$ (that is, $\widetilde{\text{Ricc}} = \text{Ricc}$) and $R_{ab} = \psi\phi_a\phi_b$ on V. Thus

$$R_{abcd}\phi^d = 0 \tag{3.9}$$

holds on V whilst (3.7) and the Ricci identities give $\phi_{a;c}\phi_b = \phi_{a;b}\phi_c$ on V. So $\phi_{a;b} = \alpha\phi_a\phi_b$ for some function α on V and another application of the Ricci identity together with (3.9) shows that $\alpha\phi_a$ is locally a gradient, $\alpha\phi_a = \rho_{,a}$ on some suitable open neighbourhood U'' of any $m \in V$ and for some function ρ on U''. Thus $\chi \equiv e^{-\rho}\mathrm{d}\phi$ is a parallel 1-form satisfying χ =du on U'' for $u : U'' \to \mathbb{R}$ and so $\phi = \phi(u)$ and $\rho = \rho(u)$. Finally it follows from (3.2) that $R_{ab} = \gamma\chi_a\chi_b$ for $\gamma : U'' \to \mathbb{R}$ with $\gamma = \gamma(u)$.

Since Riem cannot vanish over any nonempty open subset of M there exists $m \in V$ and a neighbourhood of m such that Riem does not vanish at any point of this neighbourhood. Construct a null basis l, n, L, N at m with l^a chosen equal to $g^{ab}\chi_b(m) \equiv \chi^a(m)$ and then a basis $l\wedge N$, $l\wedge n$, $n\wedge L$, $l\wedge L$, $L\wedge N$ and $n\wedge N$ for B_m in terms of symmetrized products of which Riem(m) may be written. Now since the Weyl tensor C and R vanish on V, (1.1) gives Riem $= E$ and so (see, e.g.[**4**]) * Riem $= -$ Riem* on V. Also, from (3.9) Riem(m) may be reduced to a sum of symmetrized products of $A \equiv l\wedge N$, $B \equiv l\wedge L$ and $C \equiv L\wedge N$ and the expression for Ricc shows that the terms in C vanish. Then the relation * Riem $= -$ Riem*, noting that $A \in \overset{+}{S}_m$ and $B \in \overset{-}{S}_m$ (see Section 1) shows that in some open neighbourhood V' of m (assumed to satisfy all the conditions placed on the above set U'') where Riem does not vanish and chosen to allow a smooth extension of the null basis to V' consistent with $l^a = \chi^a$ one has

$$R_{abcd} = \beta(A_{ab}B_{cd} + B_{ab}A_{cd}) \tag{3.10}$$

for a smooth nowhere-zero function $\beta : V' \to \mathbb{R}$ (and from which one now has $R_{ab} = 2\beta l_a l_b$). A consequence of this is that for any *totally null* 2-space represented by a totally null bivector F at $m \in V'$ (and which is necessarily in $\overset{+}{S}_m$ or $\overset{-}{S}_m$), $R_{abcd}F^{ab}F^{cd} = 0$, whereas if F is *null*, one may write $F = \overset{+}{F} + \overset{-}{F}$ with $\overset{+}{F} \in \overset{+}{S}_m$, $\overset{-}{F} \in \overset{-}{S}_m$ and with each totally null (see Section 1) and so $R_{abcd}F^{ab}F^{cd} = 2\beta(\overset{+}{F}{}^{ab}A_{ab})(\overset{-}{F}{}^{ab}B_{ab})$ which is not zero unless $\overset{+}{F}$ is a multiple of A *or* $\overset{-}{F}$ is a multiple of B (cf Lemma 3(i)). Also if X is a spacelike (respectively timelike) 2-space at m and Y its spacelike (respectively timelike) orthogonal complement and which are represented, respectively, by the (simple) bivectors F and $\overset{*}{F}$ whose blades are X and Y (and noting that $F + \overset{*}{F} \in \overset{+}{S}_m$), the conditions ${}^*\,\mathrm{Riem} = -\,\mathrm{Riem}^*$ and $F^{ab}F_{ab} = \overset{*}{F}{}^{ab}\overset{*}{F}_{ab}$ show that the sectional curvatures of X and Y differ only in sign.

Returning to (3.10) one has $A_{ab}l^b = 0$ and hence $(A_{ab}l^b)_{;c} = 0$ on V'. Thus, since l is parallel on V', $(A_{ab;c}k^c)l^b = 0$ for any $k \in T_mM$ ($m \in V'$) and so, since $A_{ab;c}k^c$ is in $\overset{+}{S}_m$ for each k, A is a *recurrent* 2-form on V', $A_{ab;c} = A_{ab}p_c$ and similarly for B, $B_{ab;c} = B_{ab}q_c$ for 1-forms p and q on V'. One can then compute from the Ricci identity and (3.10)

$$A_{ab;cd} - A_{ab;dc}[= A_{ab}(p_{c;d} - p_{d;c})] = A_{eb}R^e{}_{acd} + A_{ae}R^e{}_{bcd} = 0 \tag{3.11}$$

to see that p, (and similarly q) are locally gradients of, say, the functions a and b, respectively and which are defined on the, possibly reduced, subset V'. Thus A and B may be locally scaled to parallel bivectors $A' \equiv e^{-a}A$ and $B' = e^{-b}B$, respectively. Thus Riem is *recurrent* on V', $R_{abcd;e} = R_{abcd}r_e$, on V' where $r \equiv p+q$ is a gradient (as it must be by Walker's theorem [**21**]). Clearly Ricc ($R_{ab} = 2\beta l_a l_b$) is also recurrent on V', $R_{ab;c} = R_{ab}r_c$ with $r_a = \beta^{-1}\beta_a$ and (3.2) with $R = 0$ reveals that $\beta_{,a}$ is proportional to l_a and is hence null. Thus the infinitesimal holonomy group of (M, g) has constant dimension ($= 2$) on the open subset V' of V and so equals the holonomy group on any component of V'. This holonomy group is represented by the (2-dimensional) holonomy algebra labelled $2g$ in [**22**]. [It is remarked that the above calculation shows that, on V', Ricc is of Segre type $\{(211)\}$ and recurrent on V' (but not properly recurrent since it is (locally) proportional to a parallel, symmetric second order tensor on V') and again this holonomy result follows from [**10**].] One thus has

Theorem 2. *Let M be a 4-dimensional manifold with nonflat metric g of neutral signature. Let g' be any other metric on M whose sectional curvature function σ'_m at $m \in M$ equals that of g at each $m \in M$. Suppose that for no $m \in M \setminus B$ is σ_m a constant function. Then g and g' are conformally related on M (and hence g' has neutral signature). Also one may disjointly decompose $M = V \cup \operatorname{int} W \cup K$ such that $g' = g$ on $\operatorname{int} W$, K is a closed subset of M with $\operatorname{int} K = \emptyset$ and V is an open subset of M on which g and g' are conformally flat and the above general analysis of this section holds.*

One can add more here. From [**21**] one sees that on the open dense subset of V where Riem does not vanish the conditions of Walker's nonsimple K_n^* spaces are satisfied and so in some connected, open neighbourhood of any point of this neighbourhood one may choose coordinates u, v, x, y with u as above such that the metric takes the form

$$ds^2 = H(u,x,y)du^2 + 2du\,dv + dx^2 - dy^2 \tag{3.12}$$

where $l_a = u_{,a}$ and, from the conformally flat condition, $H(u,x,y) = \delta(u)(x^2 - y^2)$ for some smooth function δ. This is the analogue, for this signature, of the conformally flat plane waves of general relativity [**20**], the latter appearing in a similar way in the study of sectional curvature in Lorentz signature [**6, 19**]. To see that non-trivially related pairs g and g' of solutions exist with identical sectional curvature functions suppose that $M = \mathbb{R}^4$ with metric g as in (3.12). Then since the function ϕ above is a function only of u and may be chosen positive, let $g' = e^{2\rho(u)}g$ for some smooth function ρ on M. A computation of Riem$'$ and use of Theorem 1(ii) shows that the sectional curvatures of g and g' are equal if and only if

$$\ddot{\rho} - \dot{\rho}^2 = \delta(u)(e^{2\rho} - 1)$$

(where a dot denotes d/du) solutions of which are known (see [**19**]).

4. Sectional Curvature Preserving Vector Fields

Let f be a smooth map on some open neighbourhood W of M into M, let u, v be independent members of T_mM for $m \in W$ and let $m' \in M$ with $f(m) = m'$. With f_* the usual differential of f and assumed to be an isomorphism at each $m \in W$, suppose the sectional curvatures of $u \wedge v$ at m and $f_*u \wedge f_*v$ at m' are equal for each such choice of m, u and v (in the sense that neither is defined or each is defined and equality holds). Then f is called *sectional curvature preserving on* W. Similarly a smooth vector field X on M is called *sectional curvature preserving* on M if each of its local flows is sectional curvature preserving in the above sense [**6**]. It is now easily checked from theorem 1 that the collection of all sectional preserving vector fields SCP(M) on M is a subalgebra of the (finite-dimensional) Lie algebra of conformal vector fields on M (each member satisfying, from theorem 1, $\mathcal{L}_X g = wg$ and $\mathcal{L}_X \operatorname{Ricc} = w \operatorname{Ricc}$ where $\mathcal{L}$ denotes the Lie derivative and w is some function on M). Clearly SCP(M) contains the Killing algebra of M but contains no proper homothetic vector fields. Now consider a (connected) coordinate neighbourhood W' in the open subset V (as described in the last section) of the type in which (3.12) is written. Then $R_{ab} = -2\delta(u)l_a l_b$ and $X \in \mathrm{SCP}(W')$ if and only if *either* (i) $\mathcal{L}_X g = wg$ *and* $\mathcal{L}_X \operatorname{Ricc} = w \operatorname{Ricc}$ for some function $w(u)$ (and necessarily $\ddot{w} = 2w\delta$), *or* (ii) $\mathcal{L}_X g = wg$ where $w(u)$ satisfies $\ddot{w} = 2w\delta$ (and necessarily $\mathcal{L}_X \operatorname{Ricc} = w \operatorname{Ricc}$). The details of the Lie algebra SCP(W') may then be explored in a similar way to that in the Lorentz case [**9, 14**]. This should be compared with, for example, a study of such vector fields on any component of the open subset $\operatorname{int} W$ of Section 2 where SCP($\operatorname{int} W$) coincides with the Killing algebra on $\operatorname{int} W$.

5. Final Remarks Including the Ricci-Flat Case

The restriction regarding the nonconstancy of σ_m in Theorem 2 can, in a special case, be removed in the following way. Suppose, with the above notation, that (M, g) is *nonflat* and *Ricci flat*. Then $C = \text{Riem}$ on M and so C and Riem are each nowhere zero on an open dense subset $U \subset M$ and each vanish on $M \smallsetminus U$ where the latter subset, because of the nonflat condition, (is closed and) has empty interior in M. Then σ_m is necessarily nowhere a constant function on U (otherwise the Ricci-flat condition would force the contradiction $\text{Riem} = 0$ at such points). Also $\phi = 1$ on U and hence on M. Thus one has

THEOREM 3. *Let M be a 4-dimensional manifold with a nonflat, Ricci flat metric g of neutral signature. Let g' be any other metric on M whose sectional function σ_m equals that of g' at each $m \in M$. Then $g' = g$ on M.*

It is noted that, quite generally for this dimension and signature, ${}^*G^* = G$, ${}^*C^* = C$ and ${}^*E^* = -E$ (see e.g. [**4**]) and hence, from (1.1), $\text{Riem} - {}^*\text{Riem}^* = 2E$. So, for any spacelike or timelike 2-space represented by the simple bivector F and with orthogonal complement represented by $\overset{*}{F}$, $F^{ab}F_{ab} = \overset{*}{F}{}^{ab}\overset{*}{F}_{ab}$ and the following holds

$$\sigma_m(F) - \sigma_m(\overset{*}{F}) = \frac{R_{abcd}F^{ab}F^{cd}}{2G_{abcd}F^{ab}F^{cd}} - \frac{{}^*R^*_{abcd}F^{ab}F^{cd}}{2G_{abcd}F^{ab}F^{cd}} = \frac{E_{abcd}F^{ab}F^{cd}}{G_{abcd}F^{ab}F^{cd}}$$

It follows that for each such 2-space the difference $\sigma_m(F) - \sigma_m(\overset{*}{F})$ is controlled by the tensor E (that is, by $\widetilde{\text{Ricc}}(m)$) (see e.g. [**11**]) and that, just as in the Lorentz case [**3**], $\sigma_m(F) = \sigma_m(\overset{*}{F})$ for each such 2-space if and only if $E(m) = 0$, that is, if and only if the Einstein space condition holds at m.

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