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FLAT DOUBLE ROTATIONAL SURFACES IN EUCLIDEAN AND LORENTZ–MINKOWSKI 4-SPACE

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Abstract. A new type of surfaces in 4-dimensional Euclidean and Lorentz–Minkowski space is constructed by performing two simultaneous rotations on a planar curve. In analogy with rotational surfaces, the resulting surfaces are called double rotational surfaces. Classification theorems of flat double rotational surfaces are proved. These classifications contain amongst other cones over 4-dimensional Clelia curves. As a side product these new kinds of curves in 4-space are defined.

1. Introduction

Rotational surfaces or surfaces of revolution in Euclidean 3-space are well-known study subjects in classical differential geometry (see e.g. [**5**]). Their intuitive construction, being the trace of a rotated planar curve, is appealing to geometers and prompting alteration. For instance, the helicoidal surfaces or generalized helicoids arise when a planar curve is rotated about an axis in its supporting plane and simultaneously translated in the direction of that rotation axis, see e.g. [**5**].

Together with Van de Woestyne the author studied another generalization of rotational surfaces, namely, twisted surfaces, see [**4**] and the references therein. Later, a second kind of twisted surfaces was defined in [**6**]. Twisted surfaces were first defined in [**5**] and are constructed by rotating a planar curve about an axis in its supporting plane while simultaneously rotating it in its supporting plane. Here this construction is carried over to 4-space since there is 'more space to twist' there, or, as prof. B. Rouxel wrote in a private communication: '... il y a plus de place!'.

Already in the previous century, Moore [**7**] and Vranceanu [**8**] defined and studied surfaces of revolution in Euclidean 4-space. Later several properties of these rotational surfaces in Euclidean 4-space were treated in e.g. [**1**]. Also, equivalent notions are defined and examined in pseudo-Euclidean 4-spaces, see e.g. [**2**].

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In this contribution a generalization of these rotational surfaces is considered, namely, double rotational surfaces in 4-space are constructed by performing two simultaneous rotations on a planar curve.

This article is organized as follows. First some preliminaries on Euclidean and Lorentz–Minkowski 4-space are repeated. Then double rotational surfaces in these spaces are defined. Following, a classification of flat double rotational surfaces is proved. Related to this last part, a 4-dimensional version of Clelia curves is defined.

2. Preliminaries

Euclidean 4-space $\mathbb{E}^4$ is $\mathbb{R}^4 = \{(w,x,y,z) \mid w,x,y,z \in \mathbb{R}\}$ endowed with the Euclidean scalar product $\langle u,v\rangle = \sum_{i=1}^4 u_i v_i$, while Lorentz–Minkowski 4-space $\mathbb{E}^4_1$ is obtained by endorsing $\mathbb{R}^4$ with the Lorentz–Minkowski scalar product $\langle u,v\rangle_1 = -u_1v_1 + \sum_{i=2}^4 u_iv_i$, both for vectors $u = (u_1,u_2,u_3,u_4), v = (v_1,v_2,v_3,v_4) \in \mathbb{R}^4$.

In $\mathbb{E}^4$ there exist two types of rotations: on the one hand rotations about a point, which leave invariant two perpendicular planes, and on the other hand rotations about a plane. Since the latter type is a special case of the former type, only rotations about a point are considered in this work. For instance, see e.g. [**7**],

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{2.1}$$

where $\theta, \varphi \in \mathbb{R}$ are the rotation angles, represents a rotation about the origin that leaves invariant the wx-plane and the yz-plane. As is clear, if either $\theta = 0$ or $\varphi = 0$, this reduces to a rotation about the wx-plane or about the yz-plane, respectively.

In $\mathbb{E}^4_1$ a vector v is called spacelike if $\langle v,v\rangle_1 > 0$ or $v = 0$, timelike if $\langle v,v\rangle_1 < 0$ and lightlike or null if $\langle v,v\rangle_1 = 0$ with $v \neq 0$. Hence, the w-axis is timelike while the x-, y- and z-axis are spacelike. Furthermore, the wx-, wy- and wz-plane are timelike, while the xy-, xz- and yz-plane are spacelike. A rotation in $\mathbb{E}^4_1$ about the origin that leaves the wx-plane and the yz-plane invariant is given by a matrix similar to (2.1) where in the upper left 2×2 block the trigonometric functions are replaced by the hyperbolic functions and the minus sign is removed, see e.g., [**2**].

For a surface in $\mathbb{E}^4$ or $\mathbb{E}^4_1$ parameterized by $\psi(s,t) : U \subset \mathbb{R}^2 \to \mathbb{E}^4_{(1)}$, the components of the first fundamental form are $E = \langle\psi_s,\psi_s\rangle_{(1)}$, $F = \langle\psi_s,\psi_t\rangle_{(1)}$ and $G = \langle\psi_t,\psi_t\rangle_{(1)}$. The Gaussian curvature K of a non-degenerate (i.e., $EG - F^2 \neq 0$) surface is given by Brioschi's formula, see e.g., [**5**], $K = k(EG - F^2)^{-2}$ where

$$k = \begin{vmatrix} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s & -\frac{1}{2}E_t + F_s \\ F_t - \frac{1}{2}G_s & E & F \\ \frac{1}{2}G_t & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_t & \frac{1}{2}G_s \\ \frac{1}{2}E_t & E & F \\ \frac{1}{2}G_s & F & G \end{vmatrix}. \tag{2.2}$$

A surface which has identically vanishing Gaussian curvature is called a *flat surface*.

Here and in what follows, the parameters s and t are often dropped for notational purposes. The subscript (1) means no subscript for the Euclidean case and subscript 1 for the Lorentz–Minkowski case. A subscript s or t at a function $Z(s,t)$ denotes the partial derivative of Z with respect to s or t, respectively.

3. Double rotational surfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_1$

Double rotational surfaces are on the one hand studied as a continuation of the research on twisted surfaces in 3-space, see [**4**] and the references therein, and on the other hand they are a generalization of the rotational surfaces in 4-space, see e.g. [**1, 2, 7, 8**]. Combining these two viewpoints leads to the following definition.

Definition 3.1. A *double rotational surface* is the trace of a planar curve, the *profile curve*, that is subjected to two different simultaneous rotations, possibly at different rotation speeds. One of these rotations must leave invariant the supporting plane of the profile curve, hence rotates the profile curve in its supporting plane.

Now an explicit parameterization of these double rotational surfaces is stated.

3.1. Double rotational surfaces in $\mathbb{E}^4$. Without losing generality assume that the profile curve α lies in the wx-plane, $\alpha(t) = (f(t) + a_1, g(t) + a_2, 0, 0)$ with f and g real-valued functions. This profile curve is subjected to a rotation about the point $(a_1, a_2, 0, 0)$ which leaves invariant the wx-plane and the yz-plane,

$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(bs) & -\sin(bs) & 0 & 0 \\ \sin(bs) & \cos(bs) & 0 & 0 \\ 0 & 0 & \cos(\tilde{b}s) & -\sin(\tilde{b}s) \\ 0 & 0 & \sin(\tilde{b}s) & \cos(\tilde{b}s) \end{pmatrix} \begin{pmatrix} f(t) \\ g(t) \\ 0 \\ 0 \end{pmatrix}$$

where $a_1, a_2, b, \tilde{b} \in \mathbb{R}$. Simultaneously, α is rotated about another point which we can, up to a translation, choose to be the origin, using a rotation with rotation angles $c, d \in \mathbb{R}$ that leaves invariant the xy-plane and the wz-plane, that is,

$$\begin{pmatrix} \cos(cs) & 0 & 0 & -\sin(cs) \\ 0 & \cos(ds) & -\sin(ds) & 0 \\ 0 & \sin(ds) & \cos(ds) & 0 \\ \sin(cs) & 0 & 0 & \cos(cs) \end{pmatrix} \begin{pmatrix} a_1 + f(t)\cos(bs) - g(t)\sin(bs) \\ a_2 + f(t)\sin(bs) + g(t)\cos(bs) \\ 0 \\ 0 \end{pmatrix}.$$

Thus, up to transformations, a double rotational surface in $\mathbb{E}^4$ is parameterized by

$$\psi(s,t) = \begin{pmatrix} \cos(cs)\,(a_1 + f(t)\cos(bs) - g(t)\sin(bs)) \\ \cos(ds)\,(a_2 + f(t)\sin(bs) + g(t)\cos(bs)) \\ \sin(ds)\,(a_2 + f(t)\sin(bs) + g(t)\cos(bs)) \\ \sin(cs)\,(a_1 + f(t)\cos(bs) - g(t)\sin(bs)) \end{pmatrix} \tag{3.1}$$

with $a_1, a_2, b, c, d \in \mathbb{R}$. Here it is always assumed that c and d are non-zero since otherwise the surface is a twisted surface that lies in a 3-space.

If $b = 0$, that is, the profile curve is not rotated in its supporting plane, this construction and hence parameterization (3.1) reduces to that of a rotational surface in $\mathbb{E}^4$, which is studied in e.g. [**1**]. See also the special examples there and in [**8**].

3.2. Double rotational surfaces in $\mathbb{E}^4_1$. In $\mathbb{E}^4_1$ a distinction between the different possibilities for the causal character (spacelike, timelike or lightlike/null) of the supporting plane of the profile curve must be made. The existence of these different causal characters of vectors and planes in $\mathbb{E}^4_1$ leads to deviating and unexpected results when studying surfaces and curvature properties of them in $\mathbb{E}^4_1$.

Indeed, as is clear from e.g. [**4, 6**], including profile curves in lightlike planes and performing rotations about lightlike axes lead to necessary adjustments of the construction of twisted surfaces and to substantially different results. Hence, in this work only profile curves in non-null planes and rotations keeping invariant non-null planes are taken into account. That way, two distinct parameterizations of double rotational surfaces in $\mathbb{E}^4_1$ can be given.

3.2.1. *A profile curve in a timelike plane.* Without losing generality, take a profile curve $\alpha(t) = (f(t) + a_1, g(t) + a_2, 0, 0)$, with f and g real-valued functions, in the timelike wx-plane. Rotate this profile curve in its supporting plane about the point $(a_1, a_2, 0, 0)$ while leaving invariant the wx-plane and the yz-plane. Simultaneously the profile curve is rotated about the origin using a rotation that leaves invariant the wz-plane and the xy-plane. Then, up to transformations, a first possible double rotational surface in $\mathbb{E}^4_1$ is parameterized by

$$\psi(s,t) = \begin{pmatrix} \cosh(cs)\,(a_1 + f(t)\cosh(bs) + g(t)\sinh(bs)) \\ \cos(ds)\,(a_2 + f(t)\sinh(bs) + g(t)\cosh(bs)) \\ \sin(ds)\,(a_2 + f(t)\sinh(bs) + g(t)\cosh(bs)) \\ \sinh(cs)\,(a_1 + f(t)\cosh(bs) + g(t)\sinh(bs)) \end{pmatrix} \tag{3.2}$$

where $a_1, a_2, b, c, d \in \mathbb{R}$ and c and d are both non-zero.

3.2.2. *A profile curve in a spacelike plane.* One can, again without losing generality, start with a profile curve $\alpha(t) = (0, f(t) + a_1, 0, g(t) + a_2)$ in the spacelike xz-plane. This profile curve is then rotated about the point $(0, a_1, 0, a_2)$ using a rotation that leaves invariant the wy-plane and the xz-plane. Simultaneously, rotate the profile curve using a rotation about the origin leaving the wz-plane and the xy-plane invariant. Therefore, up to transformations, a second possible double rotational surface in $\mathbb{E}^4_1$ is parameterized by

$$\psi(s,t) = \begin{pmatrix} \sinh(cs)\,(a_2 + f(t)\sin(bs) + g(t)\cos(bs)) \\ \cos(ds)\,(a_1 + f(t)\cos(bs) - g(t)\sin(bs)) \\ \sin(ds)\,(a_1 + f(t)\cos(bs) - g(t)\sin(bs)) \\ \cosh(cs)\,(a_2 + f(t)\sin(bs) + g(t)\cos(bs)) \end{pmatrix} \tag{3.3}$$

where $a_1, a_2, b, c, d \in \mathbb{R}$ and c and d are both non-zero.

In the case $b = 0$, parameterizations (3.2) and (3.3) reduce to that of a rotational surface in $\mathbb{E}^4_1$. These are examined in e.g. [**2**] where the name 'double rotational surface' is used differently to denote a rotational surface in $\mathbb{E}^4_1$.

4. Flat double rotational surfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_1$

Classification theorems of flat double rotational surfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_1$ are proved by carrying out the manipulations of the equations with the computer algebra system Maple. In order to reduce the work in the calculations it is assumed that $f(t) = t$. Using Maple it is easy to see that this can be done without losing generality since the alternative choice $g(t) = t$ leads to the same equations.

Theorem 4.1 (Flat double rotational surfaces in $\mathbb{E}^4$). *Excluding the twisted surfaces in $\mathbb{E}^3$ and the rotational surfaces in $\mathbb{E}^4$, a double rotational surface in $\mathbb{E}^4$ parameterized by* (3.1) *is flat if and only if it is either, up to transformation,*

- *a cone over the curve (here is $p \in \mathbb{R}$)*

$$\big(\cos(cs)\left(\cos(bs) - p\sin(bs)\right), \cos(ds)\left(\sin(bs) + p\cos(bs)\right), \sin(ds)\left(\sin(bs) + p\cos(bs)\right), \sin(cs)\left(\cos(bs) - p\sin(bs)\right)\big), \tag{4.1}$$

- *parameterized by* (3.1) *with* $a_1 = a_2 = 0$, $d = \pm c$, $f(t) = t$ *and* g *implicitly defined by, where* $p, q \in \mathbb{R}$ *and* p *non-zero,*

$$p\log\Big(1 + \frac{g^2(t)}{t^2}\Big) - 2\arctan\Big(\frac{g(t)}{t}\Big) = q - 2p\log t. \tag{4.2}$$

Proof. For a surface parameterized by (3.1) with $f(t) = t$ we calculate

$$E = (b^2 + c^2)g^2(t) + (b^2 + d^2)t^2 + a_1^2c^2 + a_2^2d^2 - 2(a_1c^2g(t) - a_2d^2t)\sin(bs) + (2(d^2 - c^2)tg(t)\sin(bs) + 2(a_1c^2t + a_2d^2g(t)))\cos(bs) + (c^2 - d^2)(t^2 - g^2(t))\cos^2(bs),$$

$$F = b(tg'(t) - g(t)) \quad \text{and} \quad G = 1 + {g'}^2(t).$$

Inserting these in equation (2.2) for the numerator of the Gaussian curvature, one obtains a quiet large condition for $K \equiv 0$ which can be summarized by

$$\sum_{i=0}^{4} A_i(t)\cos^i(bs) + \sum_{i=0}^{3} B_i(t)\cos^i(bs)\sin(bs) = 0. \tag{4.3}$$

Here the coefficients A_i and B_j are expressions in a_1, a_2, b, c, d, t, $g(t)$ and up to second order derivatives of g. Because of the linear independency of the cosine and sine functions all the coefficients $A_i(t)$ and $B_j(t)$ for $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{0, 1, 2, 3\}$ must vanish. Therefore,

$$A_4 = (c - d)^2(c + d)^2g''(t)\left((t^3 - 3tg^2(t))g'(t) + 3t^2g(t) - g^3(t)\right) = 0,$$
$$B_3 = (c - d)^2(c + d)^2g''(t)\left((-3t^2g(t) + g^3(t))g'(t) + t^3 - 3tg^2(t)\right) = 0.$$

Hence four different cases must be examined.

Case 1 or 2 $d = c$ **or** $d = -c$**.** Using this, one obtains that

$$A_2 = 2c^4\left((Pg'(t) + Q)g''(t) + R_1\right) = 0 = B_1 = 2c^4\left((-Qg'(t) + P)g''(t) + R_2\right)$$

where P, Q, R_1 and R_2 are terms in a_1, a_2, t, $g(t)$ and $g'(t)$. Now $QA_2 + PB_1$ and $PA_2 - QB_1$ lead to two new equations, which on their turn lead to two subcases,

$$(a_1^2 + a_2^2)^2\bigg((t^2 + g^2(t))g''(t) + \frac{1 + {g'}^2(t)}{2}(g(t)(1 - {g'}^2(t)) - 2tg'(t))\bigg) = 0, \tag{4.4}$$

$$(a_1^2 + a_2^2)^2\bigg((t^2 + g^2(t))g'(t)g''(t) + \frac{1 + {g'}^2(t)}{2}(t(1 - {g'}^2(t)) + 2g(t)g'(t))\bigg) = 0. \tag{4.5}$$

Case 1 *or* 2 *A* $a_1 = a_2 = 0$. Using this, one obtains that

$$A_0 = c^2(b^2 + c^2)(tg'(t) - g(t))\Big((t^2 + g^2(t))g''(t) - (tg'(t) - g(t))(1 + {g'}^2(t))\Big) = 0.$$

If $tg'(t) = g(t)$ then $g(t) = pt$ with $p \in \mathbb{R}$, hence one obtains a cone over the curve (4.1). If $(t^2 + g^2(t))g''(t) = (tg'(t) - g(t))(1 + {g'}^2(t))$, then rewrite it and integrate

$$\frac{g''(t)}{1 + {g'}^2(t)} = \frac{(g(t)/t)'}{1 + (g(t)/t)^2} \quad \text{so} \arctan g'(t) = \arctan\frac{g(t)}{t} + q \quad \text{with } q \in \mathbb{R}.$$

If $q = 0$, then $g(t) = pt$ for $p \in \mathbb{R}$ leads again to a cone over the curve (4.1). Else

$$g'(t) = \tan\Big(\arctan\frac{g(t)}{t} + q\Big) = \frac{\tan\big(\arctan(g(t)/t)\big) + \tan q}{1 - \tan\big(\arctan(g(t)/t)\big)\tan q} = \frac{g(t)/t + p}{1 - pg(t)/t}$$

with $p = \tan q$. Using the substitution $g(t) = tu(t)$ and integrating this leads to the implicit expression (4.2) for g.

Case 1 *or* 2 *B* $a_1 \neq 0 \neq a_2$. Combine the last terms of (4.4) and (4.5) to obtain $g(t)g'(t) = -t$ and derive a contradiction when used in the last term of (4.4).

Case 3 $g''(t) = 0$**.** Use $g(t) = pt + q$ with $p, q \in \mathbb{R}$ and consider two subcases,

$$A_1 = 2(p^2+1)q\left(((a_1p - a_2)d^2 + a_1pb^2)c^2 - a_2b^2d^2\right) = 0, \tag{4.6}$$

$$B_0 = 2(p^2+1)q\left(((a_2p + a_1)d^2 + a_1b^2)c^2 + a_2pb^2d^2\right) = 0. \tag{4.7}$$

Case 3*A* $q = 0$. If this is used, then $B_1 = -2c^2d^2(1+p^2)(a_1p-a_2)(a_2p+a_1) = 0$. If $a_2 = a_1p$, then $A_2 = a_1^2c^2d^2(1+p^2)^3 = 0$, hence $a_1 = a_2 = 0$ and we obtain a cone over curve (4.1). Similarly if $a_1 = -a_2p$, then $A_2 = -a_2^2c^2d^2(1+p^2)^3 = 0$ implies $a_1 = a_2 = 0$ so again a cone over curve (4.1) is found.

Case 3*B* $q \neq 0$. Solve from the last term of (4.7) an expression for a_1. Inserting that in the last term of (4.6) one finds $-(p^2+1)(b^2+c^2)a_2d^2 = 0$. Hence $a_1 = a_2 = 0$, which leads to $B_1 = -2b^2pq^2(c^2 - d^2) = 0$. The only case that is not treated yet is the one where $p = 0$. But with that $A_0 = -(b^2+d^2)c^2q^2 = 0$ which only leads to either a contradiction or a previous case.

Case 4 $Pg'(t) + Q = 0$ **and** $-Qg'(t) + P = 0$ **where** $P = t^3 - 3tg^2(t)$ **and** $Q = 3t^2g(t) - g^3(t)$**.** Multiplying the first condition by Q and adding the second condition times P leads to the contradiction $t^2 + f^2(t) = 0$.

Therefore, a flat double rotational surface is one of the surfaces in the statement of the theorem. Vice versa it is calculated straightforwardly that the surfaces in the statement of the theorem are flat. □

Remark that this classification result is not equivalent to the classification result of flat twisted surfaces in $\mathbb{E}^3$, see [**3**] and the references therein. Both classifications contain similar cones, but those cones are the only flat twisted surfaces in $\mathbb{E}^3$ when excluding the surfaces of revolution, while there exist flat double rotational surfaces parameterized by (3.1) with $f(t) = t$ and g implicitly defined by (4.2).

Theorem 4.2 (Flat double rotational surfaces in $\mathbb{E}^4_1$). *Excluding the twisted surfaces in* $\mathbb{E}^3_1$ *and the rotational surfaces in* $\mathbb{E}^4_1$*, a double rotational surface in* $\mathbb{E}^4_1$ *parameterized by* (3.2) *is flat if and only if it is either, up to transformation,*

- *a cone over the curve (here is* $p \in \mathbb{R}$*)*

$$\begin{aligned}(\cosh(cs)\left(\cosh(bs) + p\sinh(bs)\right), \cos(ds)\left(\sinh(bs) + p\cosh(bs)\right),\\ \sin(ds)\left(\sinh(bs) + p\cosh(bs)\right), \sinh(cs)\left(\cosh(bs) + p\sinh(bs)\right)\big),\end{aligned} \tag{4.8}$$

- *parameterized by* (3.2) *with* $f(t) = t$ *and* $g(t) = \pm t$,
- *parameterized by* (3.2) *with* $a_2 = 0$, $d = \pm b$, $f(t) = t$ *and* $g(t) = pt + q$ *with* $p^2 = 1 + \frac{b^2+c^2}{a_1^2c^2}q^2$ *where* $q \in \mathbb{R}$,

and one parameterized by (3.3) *is flat if and only if it is either, up to transformation,*

- *a cone over the curve (here is* $p \in \mathbb{R}$*)*

$$\begin{aligned}(4.9)\qquad &\big(\sinh(cs)\,(\sin(bs) + p\cos(bs))\,, \cos(ds)\,(\cos(bs) - p\sin(bs))\,,\\ &\sin(ds)\,(\cos(bs) - p\sin(bs))\,, \cosh(cs)\,(\sin(bs) + p\cos(bs))\big),\end{aligned}$$

- *parameterized by* (3.3) *with* $a_2 = 0$, $c = \pm b$, $f(t) = t$, $g(t) = pt + q$ *and* $a_1 = \pm\frac{q}{d}\sqrt{\frac{b^2+d^2}{1+p^2}}$ *where* $p, q \in \mathbb{R}$.

Proof. For a surface parameterized by (3.2) the vanishing of the Gaussian curvature leads to an equation of the form (4.3) but with the trigonometric functions replaced by their hyperbolic counterparts. Since the hyperbolic sine and the hyperbolic cosine functions are linearly independent, all coefficients A_i and B_j for $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{0, 1, 2, 3\}$ must be zero, leading in this case to

$$A_4 = (c^2 + d^2)^2 g''(t)((t^3 + 3tg(t)^2)g'(t) + 3t^2 g(t) + g(t)^3) = 0,$$
$$B_3 = (c^2 + d^2)^2 g''(t)((3t^2 g(t) + g(t)^3)g'(t) + t^3 + 3tg(t)^2) = 0.$$

Hence only two cases must be considered. The rest of the proof as well as that for a flat surface parameterized by (3.3) is carried out similarly to the Euclidean case. □

Remark that also this classification differs from the result on flat twisted surfaces in $\mathbb{E}^3_1$, see [**3**] and the references therein. Moreover, it also deviates from the result in $\mathbb{E}^4$ and there is a difference between the results for the two possible parameterizations of double rotational surfaces in $\mathbb{E}^4_1$. These last differences are due to the fact that the roles of c and d are not interchangeable in $\mathbb{E}^4_1$ but are in $\mathbb{E}^4$.

4.1. Clelia curves. In $\mathbb{E}^3$, a Clelia curve is a spherical curve which is characterized by the linear dependency of its coordinates when it is parameterized using spherical coordinates. Although Clelia curves were studied already in the 18th century, they have more or less been forgotten about, except for the special cases of Viviani's curve and Pappus' spiral. Surprisingly, Clelia curves turned out to be the curves determining cones which are flat twisted surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$, see [**3, 4**].

Analogously to a Clelia curve in $\mathbb{E}^3$, a Clelia curve in $\mathbb{E}^4$ can be defined.

Definition 4.1. A *Clelia curve* in $\mathbb{E}^4$ is a curve on the hypersphere with linearly dependent coordinates when parameterized using hyperspherical coordinates.

The hypersphere $\mathbb{S}^3(r) = \{x \in \mathbb{E}^4 \mid \langle x, x\rangle = r^2\}$ of radius r is parameterized using hyperspherical coordinates, which are called Hopf coordinates, as

$$x(\theta, \nu, \varphi) = r\,(\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi\cos\nu, \sin\varphi\sin\nu)\,.$$

In order to infer that it is a Clelia curve, rewrite parameterization (4.1) as shown in [**3**]. Remark that if $c = \pm d$ in parameterization (4.1), then the Clelia curve lies on a flat torus, which lies itself on a hypersphere.

The previous definition can be extended to one of the Clelia curves in $\mathbb{E}^4_1$.

Definition 4.2. A *Clelia curve* in $\mathbb{E}^4_1$ is a curve that lies either on the pseudosphere $\mathbb{S}^3_1(r) = \{p \in \mathbb{E}^4_1 \mid \langle p, p\rangle_1 = r^2\}$ or on the hyperbolic space $\mathbb{H}^3(r) = \{p \in \mathbb{E}^4_1 \mid$

$\langle p,p\rangle_1 = -r^2\}$ and has linearly dependent coordinates when parameterized using coordinates adapted to the surrounding hypersurface.

Use Hopf-like coordinates to parameterize $\mathbb{S}^3_1(r)$ and for $\mathbb{H}^3(r)$. Parameterizations (4.8) and (4.9) can be rewritten, as shown in [**3**], to see that it are Clelia curves.

5. Conclusion and acknowledgments

In this work double rotational surfaces are introduced as a new kind of surfaces in the 4-space. Classifications of flat double rotational surfaces are proved and turn out to differ from equivalent results on flat twisted surfaces in 3-space. This advertises double rotational surfaces as interesting subjects for further research.

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