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ON NONLOCAL GRAVITY WITH CONSTANT SCALAR CURVATURE

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Abstract. A class of nonlocal gravity models, where nonlocal term contains an analytic function of the d'Alembert operator $\Box$, is considered. For simplicity, these models are considered without matter sector. Related equations of motion for gravitational field $g_{\mu\nu}(x)$ are presented and analyzed for a constant scalar curvature R. The corresponding solutions for the cosmological scale factor $a(t)$ of the FLRW universe are found and discussed.

1. Introduction

General Relativity (GR), which is Einstein theory of gravity (ETG), is a dominant theory of gravitational phenomena in the last hundred years. It is successfully confirmed in the Solar system, and predicted black holes, gravitational lensing and gravitational waves, which have been observed. It also predicts dark matter and dark energy, which are very interesting proposals but not yet experimentally confirmed.

Despite its nice theoretical properties and great phenomenological achievements, Einstein GR is not a complete theory of gravity and it should be modified in its geometrical sector. Motivations for modification of ETG come from quantum gravity, string theory, astrophysics and cosmology. Modifications started already at the early days of GR and have been intensified after discovery of accelerated expansion of the universe in 1998 (for a review, see [**2, 14**]). There are no theoretical principle that could tell us which of huge number of possibilities is an appropriate modification of ETG. Therefore, modification consists in replacement of scalar curvature R in the Einstein–Hilbert action $S = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x$ by a covariant

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scalar construction in the framework of the pseudo-Riemannian geometry. A modified gravity theory to be valuable it has in a limit to contain Einstein GR and give an acceptable answer to problems of the ETG. One of actual and promising modifications is nonlocal modified gravity (e.g. see review [**9**] and [**13, 11**]).

Here we present cosmological solutions for a nonlocal gravity model (2.1) with constant scalar curvature ($R = R_0$) in equations of motion (2.2). $R = R_0$ is a restriction on possible solutions of equations (2.3) and (2.4) for the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. It is not so restrictive as the condition that the Hubble parameter (H) is a constant, but it is a stronger condition than ansätze already used in nonlocal gravity models [**9, 1, 12**], e.g., $\Box R = aR+b$.

2. Nonlocal gravity

In this paper, we consider a class of nonlocal gravity models without matter, given by the modified Einstein–Hilbert action in the form

$$S = \frac{1}{16\pi G}\int \big(R - 2\Lambda + \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\big)\sqrt{-g}\; d^4x, \tag{2.1}$$

where $\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)$ is a nonlocal term with $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ as an analytic function of the d'Alembert operator $\Box$. $\mathcal{H}(R)$ and $\mathcal{G}(R)$ are some differentiable functions of the scalar curvature R and Λ is the cosmological constant. Inspiration to take nonlocal $\mathcal{F}(\Box)$ in the analytic form comes from string theory, particularly from p-adic string theory (e.g. see recent review [**10**])

By variation of action (2.1) with respect to the metric $g^{\mu\nu}$, we obtain the equations of motion (EOM) for $g_{\mu\nu}$ (e.g., see [**6**] for details), i.e.,

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + R_{\mu\nu}\Phi - K_{\mu\nu}\Phi \\ + \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1}\big(g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Box^\ell\mathcal{H}(R)\partial_\beta\Box^{n-1-\ell}\mathcal{G}(R) \\ - 2\partial_\mu\Box^\ell\mathcal{H}(R)\partial_\nu\Box^{n-1-\ell}\mathcal{G}(R) + g_{\mu\nu}\Box^\ell\mathcal{H}(R)\Box^{n-\ell}\mathcal{G}(R)\big) = 0, \end{aligned} \tag{2.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, and

$$K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box, \quad \Phi = \mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R) + \mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R).$$

Here, the sign $'$ denotes derivative with respect to R.

In the case of a homogeneous and isotropic metric of the universe, i.e., the FLRW metric, equation (2.2) is equivalent to the following system of equations (trace and 00 component):

$$\begin{aligned} 4\Lambda - R - 2\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + R\Phi + 3\Box\Phi \\ + \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1}\big(\partial_\mu\Box^\ell\mathcal{H}(R)\partial^\mu\Box^{n-1-\ell}\mathcal{G}(R) + 2\Box^\ell\mathcal{H}(R)\Box^{n-\ell}\mathcal{G}(R)\big) = 0, \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
(2.4) \qquad & G_{00} + \Lambda g_{00} - \frac{1}{2} g_{00} \mathcal{H}(R) \mathcal{F}(\Box) \mathcal{G}(R) + R_{00} \Phi - K_{00} \Phi \\
& + \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1} \big(g_{00} g^{\alpha\beta} \partial_\alpha \Box^\ell \mathcal{H}(R) \partial_\beta \Box^{n-1-\ell} \mathcal{G}(R) \\
& - 2 \partial_0 \Box^\ell \mathcal{H}(R) \partial_0 \Box^{n-1-\ell} \mathcal{G}(R) + g_{00} \Box^\ell \mathcal{H}(R) \Box^{n-\ell} \mathcal{G}(R) \big) = 0.
\end{aligned}
$$

The search for a general solution of the scale factor $a(t)$ of equations (2.3) and (2.4) is a hard task. We find here particular cosmological solutions for the case of constant scalar curvature R_0.

3. Constant scalar curvature

Since we use the FLRW metric $ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\big)$, where $k = \pm 1,\ 0$ is curvature constant and the speed of light $c = 1$, then scalar curvature is $R = 6\Big(\frac{\ddot{a}}{a} + \big(\frac{\dot{a}}{a}\big)^2 + \frac{k}{a^2}\Big)$ and $\Box = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t}$, where $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

We want to find the solution of equations of motion (2.3) and (2.4) for cosmological scale factor $a(t)$ when $R = R_0 =$ constant. It is useful to start from the differential equation

$$6\Big(\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big) = R_0.$$

The change of variable $b(t) = a^2(t)$ yields a second order linear differential equation with constant coefficients

$$3\ddot{b} - R_0 b + 6k = 0. \tag{3.1}$$

Depending on the sign of R_0, we have the following general solutions for $b(t)$:

$$
\begin{aligned}
& R_0 > 0, \quad b(t) = \frac{6k}{R_0} + \sigma \cosh \sqrt{R_0/3}\, t + \tau \sinh \sqrt{R_0/3}\, t, \\
& R_0 = 0, \quad b(t) = -kt^2 + \sigma t + \tau, \\
& R_0 < 0, \quad b(t) = \frac{6k}{R_0} + \sigma \cos \sqrt{-R_0/3}\, t + \tau \sin \sqrt{-R_0/3}\, t,
\end{aligned} \tag{3.2}
$$

where σ and τ are some constants.

Assumption that scalar curvature is constant simplifies the equations of motion (2.3) and (2.4) considerably and they take the form

$$
\begin{aligned}
& -2U + R_0 U' = R_0 - 4\Lambda, \\
& \frac{1}{2} U + R_{00} U' = \Lambda - G_{00},
\end{aligned} \tag{3.3}
$$

where $U = f_0 \mathcal{G}(R)\mathcal{H}(R)|_{R=R_0}$ and $U' = f_0 \frac{\partial}{\partial R}(\mathcal{G}(R)\mathcal{H}(R))|_{R=R_0}$.

System of equations (3.3) has two solutions:

$$U = 2\Lambda - R_0, \quad U' = -1, \tag{3.4}$$

$$U = \frac{1}{2}(R_0 U' + 4\Lambda - R_0), \quad R_0 + 4R_{00} = 0. \tag{3.5}$$

The above solution (3.4) for U and U' presents constraint on the form of nonlocal term in (2.1), while solution (3.5) is also a constraint on parameters σ, τ, k and R_0 in the expressions for $b(t)$ in (3.2).

3.1. Solution (3.4)**.** Solution (3.4) implies constraint on functions $\mathcal{G}(R)$ and $\mathcal{H}(R)$ in action (2.1) in the form $f_0\mathcal{G}(R)\mathcal{H}(R)|_{R=R_0} = 2\Lambda - R_0$. One of possibilities is $f_0 = 1$ and $\mathcal{G}(R) = \mathcal{H}(R) = \sqrt{2\Lambda - R + Y(R)}$, where $Y(R_0) = Y'(R)|_{R=R_0} = 0$ and all higher derivatives of $Y(R)$ at the point $R = R_0$ are arbitrary. Then the Einstein term in action (2.1) disappears and it can be rewritten as

$$S = \frac{1}{16\pi G}\int \big(Y(R) + \sqrt{2\Lambda - R + Y(R)}\, F(\Box)\, \sqrt{2\Lambda - R + Y(R)}\,\big)\sqrt{-g}\, d^4x, \tag{3.6}$$

where now $F(\Box) = \sum_{n=1}^{\infty} f_n \Box^n$. With respect to solution (3.4), parameters in expressions (3.2) are arbitrary.

3.2. Solution (3.5)**: condition** $R_0 + 4R_{00} = 0$**.** Consider now constraints which equation $R_0 + 4R_{00} = 0$ implies on the parameters σ, τ, k and R_0 in (3.2). Since

$$R_{00} = -3\frac{\ddot{a}}{a} = \frac{3}{4}\frac{(\dot{b})^2 - 2b\ddot{b}}{b^2},$$

one has the following connections between the parameters:

$$\begin{aligned} &R_0 > 0, \quad 36k^2 = R_0^2(\sigma^2 - \tau^2),\\ &R_0 = 0, \quad \sigma^2 + 4k\tau = 0,\\ &R_0 < 0, \quad 36k^2 = R_0^2(\sigma^2 + \tau^2). \end{aligned} \tag{3.7}$$

3.2.1. *Case* $R_0 > 0$. In this case, it is convenient to take $R_0 = 4\Lambda$, $\Lambda > 0$. Hence, scale factor $a(t)$ (3.2) is

$$a(t) = \sqrt{\frac{3k}{2\Lambda} + \sigma\cosh 2\sqrt{\Lambda/3}\, t + \tau\sinh 2\sqrt{\Lambda/3}\, t}\,, \quad k = \pm 1.$$

According to (3.7), in this case

$$\sigma^2 - \tau^2 = \frac{9k^2}{4\Lambda^2} = \frac{9}{4\Lambda^2}$$

and we can introduce φ such that

$$\cosh 2\varphi = \frac{\sigma}{\sqrt{\sigma^2 - \tau^2}} = \frac{2\sigma\Lambda}{3}, \quad \sinh 2\varphi = \frac{\tau}{\sqrt{\sigma^2 - \tau^2}} = \frac{2\tau\Lambda}{3}.$$

Now we can write $a(t)$ as

$$a(t) = \sqrt{\frac{3}{2\Lambda}\big(k + \cosh 2\big(\sqrt{\Lambda/3}\, t + \varphi\big)\big)}, \quad k = \pm 1. \tag{3.8}$$

After further transformations of (3.8) we obtain

$$\begin{aligned} a(t) &= \sqrt{3/\Lambda}\cosh\Big(\sqrt{\Lambda/3}\, t + \varphi\Big), \quad k = +1,\\ a(t) &= \sqrt{3/\Lambda}\,\big|\sinh\big(\sqrt{\Lambda/3}\, t + \varphi\big)\big|, \quad k = -1 \end{aligned}$$

which are the same solutions as in the Einstein gravity with cosmological constant Λ.

Now, let $\sigma^2 - \tau^2 = 0$; then the scale factor takes the form

$$a(t) = \sqrt{\sigma}\, e^{\pm\sqrt{\Lambda/3}\, t}, \quad k = 0,\ \sigma > 0.$$

3.2.2. *Case* $R_0 = 0$. Taking into account condition $\sigma^2 + 4k\tau = 0$, there are two solutions for the cosmological scale factor $a(t)$ for all values of time t:

(1) $k = 0$: $a(t) = \sqrt{\tau}$, $\tau > 0$.
(2) $k = -1$: $a(t) = t$, $\sigma = \tau = 0$, $(c = 1)$.

Solution (1) corresponds to the Minkowski space. Solution (2) is related to the Milne model of the universe.

3.2.3. *Case* $R_0 < 0$. Taking $R_0 = 4\Lambda,\ \Lambda < 0$, the corresponding scale factor $a(t)$ in (3.2) is

$$a(t) = \sqrt{\frac{3k}{2\Lambda} + \sigma \cos 2\sqrt{-\Lambda/3}\, t + \tau \sin 2\sqrt{-\Lambda/3}\, t}\,, \quad \Lambda < 0. \tag{3.9}$$

According to (3.7), in this case

$$\sigma^2 + \tau^2 = \frac{9k^2}{4\Lambda^2} = \frac{9}{4\Lambda^2}$$

and we can introduce φ such that

$$\cos 2\varphi = \frac{\sigma}{\sqrt{\sigma^2+\tau^2}} = -\frac{2\sigma\Lambda}{3}, \quad \sin 2\varphi = \frac{\tau}{\sqrt{\sigma^2+\tau^2}} = -\frac{2\tau\Lambda}{3}.$$

We can now rewrite (3.9) as

$$a(t) = \sqrt{-\frac{3}{2\Lambda}\Big(- k + \cos 2\big(\sqrt{-\Lambda/3}\, t - \varphi\big)\Big)}\,, \quad \Lambda < 0\,, \quad k = -1. \tag{3.10}$$

By standard trigonometric transformation, solution (3.10) can be rewritten in the usual form

$$a(t) = \sqrt{-3/\Lambda}\cos\big(\sqrt{-\Lambda/3}\, t - \varphi\big)\,, \quad \Lambda < 0, \quad (k = -1).$$

3.3. Solution (3.6): condition on $\mathcal{G}(R)$ and $\mathcal{H}(R)$. Here we consider some functions $\mathcal{G}(R)$ and $\mathcal{H}(R)$ which satisfy the relation

$$U = \frac{1}{2}\left(R_0 U' + 4\Lambda - R_0\right), \tag{3.11}$$

where

$$U = f_0 \mathcal{G}(R)\mathcal{H}(R)|_{R=R_0}, \quad U' = f_0 \frac{\partial}{\partial R}(\mathcal{G}(R)\mathcal{H}(R))|_{R=R_0}.$$

Let us start with $\mathcal{H}(R) = R^p$ and $\mathcal{G}(R) = R^q$, where p and q are some integers. Then $U = f_0 R_0^{p+q}$, $U' = f_0(p+q)R_0^{p+q-1}$ and equality (3.11) becomes

$$f_0 R_0^{p+q}(2 - p - q) = 4\Lambda - R_0.$$

3.3.1. *Case.* 1: $p = q = 1$. In this case, action (2.1) becomes

$$S = \frac{1}{16\pi G} \int \big(R - 2\Lambda + R\mathcal{F}(\Box)R\big)\sqrt{-g}\, d^4x$$

and R_0 is related to the cosmological constant Λ by equality $R_0 = 4\Lambda$. Some cosmological solutions of nonlocal model (3.3.1) are considered in [**1, 12, 7**].

3.3.2. *Case.* 2: $p + q = 0$. Now action (2.1) takes the form

$$S = \frac{1}{16\pi G} \int \Big(R - 2\Lambda + R^{-p}\mathcal{F}(\Box)R^{p}\Big)\sqrt{-g}\, d^4x$$

with condition $f_0 = 2\Lambda - \frac{R_0}{2}$. Studies of this model can be found in [**4, 3**]. The special case when $p = 1$ is introduced and investigated in [**5**].

3.3.3. *Case.* 3: $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\Lambda}$. In this case we have

$$U = f_0(R_0 - 2\Lambda), \quad U' = f_0,$$

and equality (3.11) becomes

$$f_0(R_0 - 4\Lambda) = 4\Lambda - R_0.$$

If $R_0 = 4\Lambda$, then f_0 is an arbitrary parameter. When $f_0 = -1$ then the corresponding action coincides with (3.6). However, if $f_0 = 0$, then we can write the action as

$$S = \frac{1}{16\pi G} \int (R - 2\Lambda + \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda}\,)\, \sqrt{-g}\, d^4x, \tag{3.12}$$

where $F(\Box) = \sum_{n=1}^{\infty} f_n \Box^n$. Note that (3.12) can be rewritten in the form

$$S = \frac{1}{16\pi G} \int \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box)\, \sqrt{R - 2\Lambda}\, \sqrt{-g}\, d^4x, \tag{3.13}$$

where $\mathcal{F}(\Box) = 1 + F(\Box) = 1 + \sum_{n=1}^{\infty} f_n \Box^n$.

4. Concluding remarks

When scalar curvature R is a constant, we obtained cosmological solutions for scale factor $a(t)$ of nonlocal gravity model (2.1). These solutions are interesting as backgrounds for cosmological perturbations, e.g. see [**8**]. Presented $a(t)$ solutions in Subsection 3.2 are the same as in the local Einstein gravity.

The relations $U = f_0 \mathcal{G}(R)\mathcal{H}(R)|_{R=R_0}$ and $U' = f_0 \frac{\partial}{\partial R}(\mathcal{G}(R)\mathcal{H}(R))|_{R=R_0}$ related to action (2.1) have to satisfy two conditions. When $U = 2\Lambda - R_0$ and $U' = -1$ then model (2.1) does not contain Einstein gravity and there is no restriction on parameters σ, τ and k in solutions (3.2). Relation $U = \frac{1}{2}(R_0 U' + 4\Lambda - R_0)$ is another possibility, which is consistent with obtained standard solutions in Subsection 3.2.

A consideration similar to the above one was given in [**4**], but the present analysis is more general and complete. For further investigation, we find very promising nonlocal gravity model presented by action (3.13).

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