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KENMOTSU MANIFOLDS WITH GENERALIZED TANAKA–WEBSTER CONNECTION

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Abstract. We study the g-Tanaka–Webster connection associated to a Kenmotsu structure. With the help of g-Tanaka–Webster connection we characterize Kenmotsu manifolds and find certain curvature properties of this connection on Kenmotsu manifolds. Finally an illustrative example is given to verify some results.

1. Introduction

The Tanaka–Webster connection has been introduced by Tanno [**20**] as a generalization of the well-known connection defined at the end of the 1970's by Tanaka in [**21**] and independently by Webster in [**26**]. This connection coincides with the Tanaka–Webster connection if the associated CR-structure is integrable. The Tanaka–Webster connection is defined as the canonical affine connection on a non-degenarate, pseudo-Harmitian CR-manifold. For a real hypersurface in a Kähler manifold with almost contact structure (ϕ, ξ, η, g), Cho [**6, 7**] adapted Tanno's g-Tanaka–Webster connection for a non-zero real number k. Using the g-Tanaka–Webster connection, some geometers have studied some characterizations of real hypersurfaces in complex space forms [**22**]. Recently in [**1**] Bilal et al. study g-Tanaka–Webster connection in Kenmotsu manifolds.

A Riemannian manifold is called semisymmetric if the curvature tensor satisfies

$$R(X,Y)\cdot R = 0, \tag{1.1}$$

where $R(X,Y)$ is considered as a field of linear operators, acting on R. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowicz, R. S. Couty and N. S. Sinjukov. A Riemannian manifold is said to be Ricci semisymmetric if the curvature

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tensor satisfies

$$R(X,Y)\cdot S=0, \tag{1.2}$$

where $R(X,Y)$ is considered as a field of linear operators, acting on R and S is the Ricci tensor of type $(0,2)$. The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S=0$) as a proper subset. Ricci semisymmetric manifolds were investigated by several authors. Every semisymmetric manifold is Ricci semisymmetric. The converse statement is not true. However, under some additional assumptions (1.1) and (1.2) are equivalent. Semisymmetric manifolds were classified by Szabó, locally in [**19**]. A fundamental study on Riemannian semisymmetric manifolds was made by Szabó [**19**], Boeckx et al. [**5**] and Kowalski [**15**].

An example of a curvature condition of semisymmetry type is $Q\cdot R=0$, where Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$. A natural extension of such curvature conditions form curvature conditions of pseudosymmetry type. The curvature condition $Q\cdot R=0$ has been studied by Verstraelen et al. in [**25**]. Motivated by the above studies in the present paper we characterize Kenmotsu manifolds admitting the g-Tanaka–Webster connection.

This paper is organized in the following way: In Section 2, we recall some basic formulas and results. In Section 3, we mention the expressions of the curvature tensor and Ricci tensor $\bar{R}$ and $\bar{S}$ with respect to the generalized Tanaka–Webster connection and then prove some interesting results. Section 4, deals with the study of Ricci semisymmetric Kenmotsu manifolds and prove that Ricci semisymmetry with respect to $\bar{\nabla}$ and ∇ are equivalent if and only if the manifold is an Einstein manifold. Next it is shown that the curvature conditions $\bar{Q}\cdot\bar{R}=0$ and $Q\cdot R=0$ are equivalent if and only if the manifold is an Einstein manifold, where Q and $\bar{Q}$ are respectively Ricci operator defined by $S(X,Y)=g(QX,Y)$ and $\bar{S}(X,Y)=g(\bar{Q}X,Y)$. Next in Section 6, we prove that the concirular curvature tensor with respect to the g-Tanaka–Webster connection and Levi-Civita connection are equal. In this section we also prove that $\bar{\mathcal{Z}}\cdot\bar{S}=0$ if and only if the manifold is an Einstein manifold. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting the g-Tanaka–Webster connection in order to verify some results.

2. Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying [**3**]

$$\phi^2=-I+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0, \tag{2.1}$$

$$\eta\circ\phi=0,\quad g(X,Y)=g(\phi X,\phi Y)+\eta(X)\eta(Y), \tag{2.2}$$

$$g(X,\phi Y)=-g(\phi X,Y),\quad g(X,\xi)=\eta(X), \tag{2.3}$$

for all vectors field X,Y. An almost contact metric manifold M is called a Kenmotsu manifold if it satisfies [**12**]

$$(\nabla_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X, \tag{2.4}$$

for all vector fields X, Y, where ∇ is the Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{2.6}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy [**2**, **12**, **13**]

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.8}$$

$$Q\xi = -2n\xi. \tag{2.9}$$

A Kenmotsu manifold is normal (that is, the Nijienhuis tensor of ϕ equals to $-2d\eta \otimes \xi$, but not Sasakian. Moreover, it is also not compact, since from (2.8) we get $\operatorname{div} \xi = 2n$. Kenmotsu [**12**] showed:

(a) that locally a Kenmotsu manifold is a Warped product $I \times_f N$ of an interval I and a Kaehler manifold N with wrapping function $f(t) = se^t$, where s is a non-zero constant;
(b) that a Kenmotsu manifold of constant ϕ sectional curvature is a space of constant curvature -1 and so it is locally hyperbolic space.

Kenmotsu manifolds have been studied by several authors such as Pitis [**17**], De [**8**], Özgür and De [**16**], De and Majhi [**10**], De, Yildiz and Yaliniz [**9**], Hong et al. [**11**], Umnova [**24**] and many others.

3. Curvature tensor and Ricci tensor with respect to the generalised Tanaka–Webster connection

The g-Tanaka–Webster connection $\bar{\nabla}$ defined by Tanno for contact metric manifolds is given by [**20**],

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y,$$

for any X, Y tangent to M.

With the help of (2.5) and (2.6) the above equation takes the form,

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X - \eta(X)\phi Y. \tag{3.1}$$

Putting $Y = \xi$ in (3.1) and using (2.1) we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \eta(X)\xi - X, \tag{3.2}$$

Using (2.5) in (3.2) we get $\bar{\nabla}_X \xi = 0$. Now

$$(\bar{\nabla}_X \eta)Y = \bar{\nabla}_X \eta(Y) - \eta(\bar{\nabla}_X Y), \tag{3.3}$$

From (3.1) and (3.3) we get

$$(\bar{\nabla}_X \eta)Y = \nabla_X \eta(Y) - g(X,Y) + \eta(Y)\eta(X). \tag{3.4}$$

With the help of (2.6), from the above equation, it follows that $(\bar{\nabla}_X \eta)Y = 0$. Again

$$(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z). \tag{3.5}$$

Finally using (3.1) in (3.5) yields

$$(\bar{\nabla}_X g)(Y,Z) = 0. \tag{3.6}$$

Now in this position we can state the following:

Proposition 3.1. *In a Kenmotsu manifold ξ, η, g are parallel with respect to the g-Tanaka–Webster connection.*

Proposition 3.2. *In a Kenmotsu manifold the g-Tanaka–Webster connection is a metric connection.*

Proposition 3.3. *In a Kenmotsu manifold the integral curves of the vector field ξ are geodesic with respect to the generalized Tanaka–Webster connection.*

Here we also obtain an interesting result stated below.

Theorem 3.1. *The g-Tanaka-webster connection $\bar{\nabla}$ associated to the Levi-Civita connection is just the only one affine connection, which is metric and its torsion is of the form*

$$\bar{T}(X,Y) = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X.$$

Proof. We see in (3.6) that the Tanaka–Webster coonection is a metric connection. Now the torsion tensor $\bar{T}$ of $\bar{\nabla}$ is given by $\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X$. Using (3.1) in the previous relation we get

$$\bar{T}(X,Y) = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X. \tag{3.7}$$

Now we recall the famous result stating with:

Any metric connection can be expressed with the help of its torsion $\bar{T}$ in the following way:

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \tfrac{1}{2}[g(\bar{T}(X,Y),Z) - g(\bar{T}(X,Z),Y) - g(\bar{T}(Y,Z),X)].$$

Applying (3.7) in the above relation yields

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \eta(Z)g(X,Y) - \eta(Y)g(X,Z) - \eta(X)g(\phi Y, Z).$$

Contracting Z in the above equation, we get

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X - \eta(X)\phi Y. \qquad \square$$

Let R and $\bar{R}$ denote the curvature tensors ∇ and $\bar{\nabla}$ respectively. Then

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z \tag{3.8}$$

Using (3.1) in (3.8) yields [**1**]

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(Z,X)Y. \tag{3.9}$$

Using (2.7) and putting $Z = \xi$ in (3.9) we get $\bar{R}(X,Y)\xi = 0$. Also by the help of (2.7) and (3.9) we can easily obtain $\bar{R}(\xi,Y)Z = 0$ and $\bar{R}(X,\xi)Y = 0$, for all vector fields X, Y, Z.

Taking the inner product with W in (3.9),

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(Y,Z)g(X,W) - g(Z,X)g(Y,W). \tag{3.10}$$

Let $\{e_1, e_2, e_3, \dots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = W = e_i$ in (3.10) and taking summation over i, $1 \leqslant i \leqslant (2n+1)$, we obtain

$$\bar{S}(Y,Z) = S(Y,Z) + 2ng(Y,Z), \tag{3.11}$$

where $\bar{S}$ and S are the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

Let $\bar{r}$ and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively. Let $\{e_1, e_2, e_3, \dots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $Y = Z = e_i$ and taking summation over i, $1 \leqslant i \leqslant (2n+1)$ we have

$$\bar{r} = r + 2n(2n+1). \tag{3.12}$$

Therefore we can state the following:

Proposition 3.4. *For a Kenmotsu manifold M admitting generalised Tanaka–Webster connection $\bar{\nabla}$*

(i) *The curvature tensor $\bar{R}$ of $\bar{\nabla}$ is given by* (3.9)*,*
(ii) *The Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is given by* (3.11)*,*
(iii) *The scalar curvature $\bar{r}$ of $\bar{\nabla}$ is given by* (3.12)*,*
(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,
(v) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,
(vi) *The Ricci tensor $\bar{S}$ is symmetric,*

Suppose $\bar{\nabla}$ is flat. Then from (3.9), we get

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y],$$

which implies that the manifold is of constant sectional curvature -1.

This leads to the following:

Proposition 3.5. *The manifold M^{2n+1} is flat with respect to the generalized Tanaka–Webster connection if and only if M^{2n+1} is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.*

It is known [**12**] that if M be a conformally flat Kenmotsu manifold of dimension $\geqslant 5$, then M has constant sectional curvature equal to -1. The converse is also true. Hence from the above proposition we have the following theorem:

Theorem 3.2. *Let M be a Kenmotsu manifold admitting the g-Tanaka–Webster connection. Then the following assertions are equivalent:*

(i) *$\bar{\nabla}$ is flat.*
(ii) *M is of constant sectional curvature -1.*
(iii) *M is conformally flat of dimension $\geqslant 5$.*

4. Ricci semisymmetry with respect to $\bar{\nabla}$ and ∇

In this section we characterize the Ricci semisymmetry in Kenmotsu manifolds with respect to the generalized Tanaka–Webster connection. Now

$$(\bar{R}(X,Y) \cdot \bar{S})(U,V) = -\bar{S}(\bar{R}(X,Y)U,V) - \bar{S}(U,\bar{R}(X,Y)V). \tag{4.1}$$

Therefore using (3.9) in the above equation, we have

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V) + g(X,U)S(Y,V) + g(X,V)S(U,Y) - g(Y,U)S(X,V) - g(Y,V)S(U,X). \tag{4.2}$$

Suppose $(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V)$, then from the above equation, it follows that

$$g(X,U)S(Y,V) + g(X,V)S(U,Y) - g(Y,U)S(X,V) - g(Y,V)S(U,X) = 0.$$

Putting $X = U = e_i$, $1 \leqslant i \leqslant (2n+1)$ in the above equation, we have

$$S(Y,V) = \frac{r}{(2n+1)}g(Y,V).$$

Again if $S(Y,V) = \frac{r}{(2n+1)}g(Y,V)$, then from (4.2), it follows that

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V).$$

This leads to the following:

THEOREM 4.1. *Ricci semisymmetries with respect to $\bar{\nabla}$ and ∇ are equivalent if and only if the manifold M is an Einstein manifold with respect to the Levi-Civita connection.*

5. Kenmotsu manifolds satisfying $\bar{Q}\cdot\bar{R} = 0$ with respect to the g-Tanaka–Webster connection

In this section we characterize $\bar{Q}\cdot\bar{R} = 0$ and $Q\cdot R = 0$ in a Kenmotsu manifold with respect to the g-Tanaka–Webster connection and Levi-Civita connection. Now

$$(\bar{Q}\cdot\bar{R})(X,Y)Z = \bar{Q}(\bar{R}(X,Y)Z) - \bar{R}(\bar{Q}X,Y)Z - \bar{R}(X,\bar{Q}Y)Z - \bar{R}(X,Y)\bar{Q}Z. \tag{5.1}$$

From (3.11), it follows that

$$\bar{Q}X = QX + 2nX. \tag{5.2}$$

Using (3.9) and (5.2) in (5.1) yields

$$(\bar{Q}\cdot\bar{R})(X,Y)Z = (Q\cdot R)(X,Y)Z + 2S(X,Z)Y - 2S(Y,Z)X - 2n[g(Y,Z)X - g(X,Z)Y]. \tag{5.3}$$

Suppose $\bar{Q}\cdot\bar{R} = 0$ and $Q\cdot R = 0$ are equivalent in a Kenmotsu manifold M. Then from (5.3) it follows that

$$S(X,Z)Y - S(Y,Z)X - n[g(Y,Z)X - g(X,Z)Y] = 0.$$

Contracting Y from the above equation, we get $S(X,Z) = ng(X,Z)$, which implies that the manifold M^{2n+1} is an Einstein manifold.

Conversely, let the manifold M^{2n+1} be an Einstein manifold. Then from (5.3), it follows that $\bar{Q}\cdot\bar{R} = Q\cdot R$.

This leads to the following:

THEOREM 5.1. *The curvature properties $\bar{Q}\cdot\bar{R} = 0$ and $Q\cdot R = 0$ are equivalent in a Kenmotsu manifold M if and only if M is an Einstein manifold with respect to the Levi-Civita connection.*

6. Concircular Curvature tensor with respect to ∇ and $\bar{\nabla}$

A transformation in an $(2n+1)$ dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle of M, is said to be a concircular transformation [**14**, **23**, **27**]. A concircular transformation is always a conformal transformation [**14**]. Here, we mean a geodesic circle by a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformation is a generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism [**4**]. An important invariant of concircular transformation is the concircular curvature tensor $\mathcal{Z}$, defined by [**27**]

$$\mathcal{Z}(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y],$$

for all $X, Y, W \in \chi(M)$, where R is the Riemannian curvature tensor and r is the scalar curvature with respect to the Levi-Civita connection.

Now the concircular curvature tensor with respect to the g-Tanaka–Webster connection is given by

$$\bar{\mathcal{Z}}(X,Y)W = \bar{R}(X,Y)W - \frac{\bar{r}}{2n(2n+1)}[g(Y,W)X - g(X,W)Y], \tag{6.1}$$

for all $X, Y, W \in \chi(M)$, where $\bar{R}$ is the Riemannian curvature tensor and $\bar{r}$ is the scalar curvature with respect to the g-Tanaka-webster connection.

Using (3.9) and (3.12) in the above equation, we get $\bar{\mathcal{Z}}(X,Y)W = \mathcal{Z}(X,Y)W$.

Therefore we can state the following:

THEOREM 6.1. *The concirular curvature tensors with respect to the g-Tanaka–Webster connection and Levi-Civita connection are equal.*

Now suppose $\bar{\mathcal{Z}}(X,Y) \cdot \bar{S} = 0$. Then we have

$$\bar{S}(\bar{\mathcal{Z}}(X,Y)U, V) + \bar{S}(U, \bar{\mathcal{Z}}(X,Y)V) = 0,$$

for all $X, Y, U, V \in \chi(M)$. Substituting X by ξ in the above equation yields

$$\bar{S}(\bar{\mathcal{Z}}(\xi,Y)U, V) + \bar{S}(U, \bar{\mathcal{Z}}(\xi,Y)V) = 0, \tag{6.2}$$

for all $X, Y, U, V \in \chi(M)$. Using (3.9), (3.12) and (6.1) in (6.2) yields

$$\eta(U)\bar{S}(Y,V) - \eta(V)\bar{S}(U,Y) = 0. \tag{6.3}$$

Putting $U = \xi$ in (6.3) implies $\bar{S}(Y,V) = 0$. Hence from (3.11) it follows that

$$S(Y,V) = -2ng(Y,V).$$

This leads to the following:

THEOREM 6.2. *A Kenmotsu manifold satisfies the condition $\bar{\mathcal{Z}}(X,Y) \cdot \bar{S} = 0$ with respect to the g-Tanaka–Webster connection if and only if the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

7. Example of a 5-dimensional Kenmotsu manifold admitting g-Tanaka–Webster connection

Consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields $e_1 = e^{-v}\frac{\partial}{\partial x}$, $e_2 = e^{-v}\frac{\partial}{\partial y}$, $e_3 = e^{-v}\frac{\partial}{\partial z}$, $e_4 = e^{-v}\frac{\partial}{\partial u}$, $e_5 = e^{-v}\frac{\partial}{\partial v}$, which are linearly independent at each point of M. Let g be the Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i, j = 1, 2, 3, 4, 5$ and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_5)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M. Let ϕ be the $(1, 1)$-tensor field defined by $\phi e_1 = e_3$, $\phi e_2 = e_4$, $\phi e_3 = -e_1$, $\phi e_4 = -e_2$, $\phi e_5 = 0$. Using the linearity of ϕ and g, we have $\eta(e_5) = 1$, $\phi^2 Z = -Z + \eta(Z)e_5$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed.

We have $\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}) = -e^{2v}$. Hence we obtain $\Omega = -e^{2v} dx \wedge dz$. Thus, $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold.

Now we have $[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0$, $[e_1, e_5] = e_1$, $[e_4, e_5] = e_4$, $[e_2, e_4] = [e_3, e_4] = 0$, $[e_2, e_5] = e_2$, $[e_3, e_5] = e_3$.

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\ \multicolumn{5}{l}{\nabla_{e_5} e_1 = \nabla_{e_5} e_2 = \nabla_{e_5} e_3 = \nabla_{e_5} e_4 = \nabla_{e_5} e_5 = 0.} \end{array}$$

Using the above relations in (3.1), we obtain

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{e_1} e_2 = \bar{\nabla}_{e_1} e_3 = \bar{\nabla}_{e_1} e_4 = \bar{\nabla}_{e_1} e_5 = 0, \\ \bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{e_2} e_2 = \bar{\nabla}_{e_2} e_3 = \bar{\nabla}_{e_2} e_4 = \bar{\nabla}_{e_2} e_5 = 0, \\ \bar{\nabla}_{e_3} e_1 = \bar{\nabla}_{e_3} e_2 = \bar{\nabla}_{e_3} e_3 = \bar{\nabla}_{e_3} e_4 = \bar{\nabla}_{e_3} e_5 = 0, \\ \bar{\nabla}_{e_4} e_1 = \bar{\nabla}_{e_4} e_2 = \bar{\nabla}_{e_4} e_3 = \bar{\nabla}_{e_4} e_4 = \bar{\nabla}_{e_4} e_5 = 0, \\ \bar{\nabla}_{e_5} e_1 = \bar{\nabla}_{e_5} e_2 = \bar{\nabla}_{e_5} e_3 = \bar{\nabla}_{e_5} e_4 = \bar{\nabla}_{e_5} e_5 = 0. \end{aligned}$$

By the above results, we can easily obtain that the non-vanishing components of the curvature tensor with respect to the Levi-Civita connection are

$$\begin{aligned}
&R(e_1,e_2)e_2 = R(e_1,e_3)e_3 = R(e_1,e_4)e_4 = R(e_1,e_5)e_5 = -e_1,\\
&R(e_1,e_2)e_1 = e_2, R(e_1,e_3)e_1 = R(e_5,e_3)e_5 = R(e_2,e_3)e_2 = e_3,\\
&R(e_2,e_3)e_3 = R(e_2,e_4)e_4 = R(e_2,e_5)e_5 = -e_2, R(e_3,e_4)e_4 = -e_3,\\
&R(e_2,e_5)e_2 = R(e_1,e_5)e_1 = R(e_4,e_5)e_4 = R(e_3,e_5)e_3 = e_5,\\
&R(e_1,e_4)e_1 = R(e_2,e_4)e_2 = R(e_3,e_4)e_3 = R(e_5,e_4)e_5 = e_4.
\end{aligned}$$

Therefore the manifold M has a constant sectional curvature -1.

Now the components of the curvature tensor with respect to the g-Tanaka–Webster connection are

$$\begin{aligned}
&\bar{R}(e_1,e_2)e_2 = \bar{R}(e_1,e_3)e_3 = \bar{R}(e_1,e_4)e_4 = 0,\\
&\bar{R}(e_1,e_2)e_1 = \bar{R}(e_1,e_3)e_1 = \bar{R}(e_2,e_3)e_2 = 0,\\
&\bar{R}(e_2,e_3)e_3 = \bar{R}(e_2,e_4)e_4 = \bar{R}(e_2,e_5)e_5 = 0,\\
&\bar{R}(e_3,e_4)e_4 = \bar{R}(e_2,e_5)e_2 = \bar{R}(e_1,e_5)e_1 = 0,\\
&\bar{R}(e_3,e_5)e_3 = \bar{R}(e_1,e_4)e_1 = \bar{R}(e_2,e_4)e_2 = 0,\\
&\bar{R}(e_1,e_5)e_5 = \bar{R}(e_3,e_5)e_5 = \bar{R}(e_4,e_5)e_5 = 0.
\end{aligned}$$

Hence Theorem 3.2 is verified.

With the help of the above results we get the components of the Ricci tensor as follows:

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4, \tag{7.1}$$

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_4,e_4) = \bar{S}(e_5,e_5) = 0. \tag{7.2}$$

Therefore $r = \sum_{i=1}^{5} S(e_i,e_i) = -20$ and $\bar{r} = \sum_{i=1}^{5} \bar{S}(e_i,e_i) = 0$.

Again from the expressions of the curvature tensor and Ricci tensor we can easily verify Proposition 3.4.

Also from (7.1) we see that the manifold is an Einstein manifold with respect to the Levi-Civita connection. Hence Theorem 4.1 is verified.

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