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TOPOLOGICALLY BOOLEAN AND $g(x)$-CLEAN RINGS

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Abstract. Let R be a ring with identity and let $g(x)$ be a polynomial in $Z(R)[x]$ where $Z(R)$ denotes the center of R. An element $r \in R$ is called $g(x)$-clean if $r = u + s$ for some $u, s \in R$ such that u is a unit and $g(s) = 0$. The ring R is $g(x)$-clean if every element of R is $g(x)$-clean. We consider $g(x) = x(x - c)$ where c is a unit in R such that every root of $g(x)$ is central in R. We show, via set-theoretic topology, that among conditions equivalent to R being $g(x)$-clean, is that R is right (left) c-topologically boolean.

1. Introduction

Let R be a ring with identity and let $g(x)$ be a polynomial in $Z(R)[x]$ where $Z(R)$ denotes the center of R. Let $\mathrm{Id}(R)$ and $U(R)$ denote the set of idempotents and the set of units in R, respectively. The notion of $g(x)$-clean rings first appeared in a 2002 paper of Camillo and Simón [**1**], where an element $r \in R$ is called $g(x)$-clean if $r = u + s$ for some $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. The ring R is $g(x)$-clean if every element of R is $g(x)$-clean. Note that if $r \in R$ is $g(x)$-clean and $g(x)$ is a factor of a polynomial $h(x) \in Z(R)[x]$, then r is also $h(x)$-clean.

Clearly, if $g(x) = x^2 - x$, then $g(x)$-clean rings are clean. However, in general, $g(x)$-clean rings are not necessarily clean. A well-known example is the group ring $\mathbb{Z}_{(7)}C_3$ where $\mathbb{Z}_{(7)} = \{m/n \mid m, n \in \mathbb{Z},\ \gcd(7, n) = 1\}$ and C_3 is the cyclic group of order 3. By [**7**, Example 2.7], $\mathbb{Z}_{(7)}C_3$ is (x^4-1)-clean. However, Han and Nicholson [**4**] have shown that $\mathbb{Z}_{(7)}C_3$ is not clean.

Conversely, for a clean ring R, there may exist a $g(x) \in Z(R)[x]$ such that R is not $g(x)$-clean (see [**3**, Example 2.3]). Indeed, let R be a Boolean ring containing more than two elements. Let $c \in R$ where $0 \neq c \neq 1$ and let $g(x) = x^2+(1+c)x+c = (x + 1)(x + c)$. Since R is Boolean, so it is clean. Suppose that R is $g(x)$-clean. Then $c = u + s$ for some $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. Note that $u = 1$ since R is Boolean. Therefore, $s = c + 1$. However, $g(c + 1) = c \neq 0$ which contradicts the assumption that $g(s) = 0$. Hence, it follows that R is clean but not $g(x)$-clean.

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In [**2**], a ring R (not necessarily commutative) is said to be right (left) topologically boolean, or a right (left) tb-ring for short, if for every pair of distinct maximal right (left) ideals of R, there is a nontrivial idempotent in exactly one of them. The case where R is commutative has been considered earlier in [**5**]. Now let $g_c(x) = x(x-c) \in Z(R)[x]$. Here, we define a ring R to be right (left) c-topologically boolean, or a right (left) c-tb ring for short, if for every pair of distinct maximal right (left) ideals of R, there is a root of $g_c(x)$ in exactly one of them. We say that R is a c-tb ring if it is both right and left c-tb. Clearly, when $c = 1$, a right (left) c-tb ring is just a right (left) tb-ring.

In this paper we consider $g(x) = x(x-c) \in Z(R)[x]$ where c is a unit in R such that every root of $g(x)$ is central in R and show via set-theoretic topology that among conditions equivalent to R being $g(x)$-clean is that R is right (left) c-tb. Throughout this paper, all rings are assumed to be associative with identity.

2. Some preliminaries

Let n be a positive integer. For a ring R and polynomial $g(x) \in Z(R)[x]$, an element $r \in R$ is said to be $(n, g(x))$-clean if $r = u_1 + \cdots + u_n + s$ for some $u_1, \ldots, u_n \in U(R)$ and $s \in R$ such that $g(s) = 0$. The ring R is $(n, g(x))$-clean if all of its elements are $(n, g(x))$-clean. Clearly, a $(1, g(x))$-clean ring is $g(x)$-clean. In [**8**], an element $r \in R$ is said to be n-clean if $r = e + u_1 + \cdots + u_n$ for some $e \in \mathrm{Id}(R)$ and $u_1, \ldots, u_n \in U(R)$. The ring R is n-clean if all of its elements are n-clean.

In [**7**, Theorem 2.1], Wang and Chen showed that if $g(x) = (x-a)(x-b) \in Z(R)[x]$ with $b - a \in U(R)$, then R is $g(x)$-clean if and only if R is clean. In [**3**, Theorem 3.2], Fan and Yang gave another proof of the same result. In the following, we give an extension to n-clean rings as follows:

THEOREM 2.1. *Let R be a ring and let $g(x) = (x-a)(x-b)h(x) \in Z(R)[x]$ such that $b - a \in U(R)$. If R is n-clean, then R is $(n, g(x))$-clean ($n \in \mathbb{N}$).*

PROOF. Let $r \in R$. Since R is n-clean, then $(r-a)(b-a)^{-1} = e + u_1 + \cdots + u_n$ for some $e \in \mathrm{Id}(R)$ and $u_i \in U(R)$ $(i = 1, \ldots, n)$. Thus, $r = (e(b-a) + a) + u_1(b-a) + \cdots + u_n(b-a)$, where $u_i(b-a) \in U(R)$ $(i = 1, \ldots, n)$. Note that $g((e(b-a)+a) = e(b-a)(e(b-a)-(b-a))h(e(b-a)+a) = 0 \cdot h(e(b-a)+a) = 0$. Hence, $e(b-a)+a$ is a root of $g(x)$. It follows that R is $(n, g(x))$-clean. $\square$

By Theorem 2.1 and the fact that clean rings are n-clean for any integer $n \geqslant 1$ (by [**9**, Lemma 2.1]), we obtain the following:

COROLLARY 2.1. *Let R be a ring and let $g(x) = (x-a)(x-b) \in Z(R)[x]$ such that $b - a \in U(R)$. Then R is $g(x)$-clean if and only if R is n-clean for all positive integers n.*

Let R be a ring and let $g(x) \in Z(R)[x]$. An element $r \in R$ is called weakly $g(x)$-clean if $r = u + s$ or $r = u - s$ for some $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. We say that R is weakly $g(x)$-clean if every element in R is weakly $g(x)$-clean. Clearly, a $g(x)$-clean ring is weakly $g(x)$-clean. It is also clear that if

R is a weakly $g(x)$-clean ring and $g(x)$ is a factor of a polynomial $h(x) \in Z(R)[x]$, then R is also a weakly $h(x)$-clean ring.

In the following we obtain some results which generalise parts of Theorem 3.5 in [**3**].

Proposition 2.1. *Let R be a ring which is weakly $x(x-c)$-clean where $c \in Z(R)$. Then $c \in U(R)$.*

Proof. Let $g(x) = x(x-c) \in Z(R)[x]$. Since R is weakly $g(x)$-clean, $c = u+s$ or $c = u-s$ for some $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. For the case $c = u+s$, we have that $s = -u + c$ and hence, $s^2 = (-u+c)^2 = u^2 + cr$ for some $r \in R$. Since $0 = g(s) = s(s-c)$, we also have $s^2 = cs$. Thus, $c(s-r) = u^2 \in U(R)$. This implies that $c \in U(R)$. For the case $c = u - s$, we have that $s = u - c$ and hence, $s^2 = (u-c)^2 = u^2 - cr$ for some $r \in R$. Since $0 = g(s) = s(s-c)$, we also have $s^2 = cs$. Thus, $c(s+r) = u^2 \in U(R)$ which implies that $c \in U(R)$. □

Lemma 2.1. *Let R be a ring and let $g(x) = ax^m - bx^n$, $h(x) = ax^m + bx^n \in Z(R)[x]$ where m, n are positive integers of different parity. Then R is $g(x)$-clean if and only if R is $h(x)$-clean.*

Proof. ($\Rightarrow$): Assume that R is a $g(x)$-clean ring. Then for any $r \in R$, $-r = u+s$ where $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. It follows that $r = (-u)+(-s)$. Note that

$$\begin{aligned} h(-s) &= a(-s)^m + b(-s)^n = (-1)^m as^m + (-1)^n bs^n \\ &= \begin{cases} as^m - bs^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(as^m - bs^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases} \\ &= 0. \end{aligned}$$

It follows that r is $h(x)$-clean.

($\Leftarrow$): Suppose that R is $h(x)$-clean. Then for any $r \in R$, $-r = u + s$ where $u \in U(R)$ and $s \in R$ such that $h(s) = 0$. It follows that $r = (-u) + (-s)$. Then since

$$\begin{aligned} g(-s) &= a(-s)^m - b(-s)^n = (-1)^m as^m - (-1)^n bs^n \\ &= \begin{cases} as^m + bs^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(as^m + bs^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases} \\ &= 0, \end{aligned}$$

we have that r is $g(x)$-clean. □

Theorem 2.2. *Let R be a ring and let $c \in Z(R)$. Then the following are equivalent:*

(a) *R is $x(x-c)$-clean;*
(b) *R is $x(x+c)$-clean;*
(c) *R is n-clean for all positive integers n and $c \in U(R)$.*

PROOF. (a)$\Leftrightarrow$ (b): This follows readily by Lemma 2.1.
(a)$\Rightarrow$ (c): Assume (a). By Proposition 2.1, we have $c \in U(R)$. It follows by Corollary 2.1 that R is n-clean for all positive integers n.
(c)$\Rightarrow$ (a): This follows readily by Theorem 2.1 (take $n = 1$). $\square$

LEMMA 2.2. *Let R be a ring, let $c \in U(R)$ and let all roots of $g(x) = x(x - c)$ in R be central. For any $a, b \in R$, if $ab = c$, then $ba = c$.*

PROOF. Let $a, b \in R$ such that $ab = c$. Since c is a root of $g(x)$, we have that c is central and therefore, $ba(ba-c) = baba-c(ba) = b(ab)a-c(ba) = c(ba)-c(ba) = 0$. Thus, ba is a root of $g(x)$ and hence, ba is also central. Then $ca = (ab)a = a(ba) = baa$ and it follows that $c^2 = c(ab) = (ca)b = (baa)b = bac$. Since $c \in U(R)$ (by the hypothesis), it follows that $c = ba$. $\square$

3. Some equivalent conditions for $x(x - c)$-clean rings

Let R be a ring. A proper right (left) ideal P of R is said to be prime if $aRb \subseteq P$ with $a, b \in R$ implies that $a \in P$ or $b \in P$. Given a ring R, let $\mathrm{Spec}_r(R)$ be the set of all proper right ideals of R which are prime. It has been shown in [**10**, Corollary 2.8] that if R is not a right quasi-duo ring, then $\mathrm{Spec}_r(R)$ is a topological space with the weak Zariski topology but not with the Zariski topology. For a right ideal I of R, let $\mathcal{U}_r(I) = \{P \in \mathrm{Spec}_r(R) \mid P \not\supseteq I\}$ and $\mathcal{V}_r(I) = \mathrm{Spec}_r(R) \setminus \mathcal{U}_r(I)$. Let $\tau = \{\mathcal{U}_r(I) \mid I \text{ is a right ideal of } R\}$. Then τ contains the empty set and $\mathrm{Spec}_r(R)$. In general, τ is just a subbase of the weak Zariski topology on $\mathrm{Spec}_r(R)$. For any element $a \in R$, let $\mathcal{U}_r(a) = \mathcal{U}_r(aR)$ and $\mathcal{V}_r(a) = \mathcal{V}_r(aR)$. Then $\mathcal{U}_r(a) = \{P \in \mathrm{Spec}_r(R) \mid a \notin P\}$ and $\mathcal{V}_r(a) = \{P \in \mathrm{Spec}_r(R) \mid a \in P\}$. The left prime spectrum $\mathrm{Spec}_l(R)$ and the weak Zariski topology associated with it are defined analogously. Let $\mathrm{Max}_r(R)$ ($\mathrm{Max}_l(R)$) be the set of all maximal right (left) ideals of R. Since maximal right (left) ideals are prime right (left) ideals (see [**6**]), $\mathrm{Max}_r(R)$ ($\mathrm{Max}_l(R)$) inherits the weak Zariski topology on $\mathrm{Spec}_r(R)$ ($\mathrm{Spec}_l(R)$). Let $U_r(I) = \mathrm{Max}_r(R) \cap \mathcal{U}_r(I)$ and $V_r(I) = \mathrm{Max}_r(R) \cap \mathcal{V}_r(I)$ for any right ideal I of R. Then, in particular, $U_r(a) = \mathrm{Max}_r(R) \cap \mathcal{U}_r(a)$ and $V_r(a) = \mathrm{Max}_r(R) \cap \mathcal{V}_r(a)$ for any $a \in R$.

Recall that a clopen set in a topological space is a set which is both open and closed. A topological space is said to be zero-dimensional if it has a base consisting of clopen sets.

We begin with the following lemmas.

LEMMA 3.1. *Let R be a ring, let $g(x) = x(x - c) \in Z(R)[x]$ where $c \in U(R)$ and let $s \in R$ be a central root of $g(x)$. Let N be a maximal right ideal of R. If $s \notin N$, then $c - s \in N$.*

PROOF. Since $g(s) = 0$, we have that $s(s - c) = 0 \in P$ for any prime right ideal P of R. Then since s is central, it follows that every prime right ideal of R contains either s or $s - c$. Now since $c = s + (c - s)$ and $c \in U(R)$, we have that $1 = sc^{-1} + (c-s)c^{-1}$. Hence, every prime right ideal of R contains either s or $c-s$ but not both. Since maximal right ideals are prime right ideals (by [**6**]), it follows that if $s \notin N$, then $c - s \in N$. $\square$

Lemma 3.2. *Let R be a ring and let $g(x) = x(x-c) \in Z(R)[x]$ where $c \in U(R)$. Let $s, t \in R$ be central roots of $g(x)$. Then $c^{-1}st$, $s + t - c^{-1}st$ and $c - s$ are also roots of $g(x)$.*

Proof. We first note that since $s(s - c) = 0$ and $t(t - c) = 0$, we thus have $s = c^{-1}s^2$ and $t = c^{-1}t^2$. Then

$$\begin{aligned} g(c^{-1}st) &= c^{-1}st(c^{-1}st - c) = c^{-2}(st)^2 - st \\ &= c^{-2}(st)^2 - c^{-1}s^2t = c^{-2}(s^2t)(t - c) = 0. \end{aligned}$$

We also have that

$$\begin{aligned} g(s + t - c^{-1}st) &= (s + t - c^{-1}st)(s + t - c^{-1}st - c) \\ &= s(s - c) + s(t - c^{-1}st) + t(s - c^{-1}st) \\ &\quad + t(t - c) - c^{-1}st(s - c^{-1}st) - c^{-1}st(t - c) = 0. \end{aligned}$$

Finally, we note that $g(c - s) = (c - s)((c - s) - c) = (s - c)s = g(s) = 0$. $\square$

Let R be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ where $c \in U(R)$. Let $\xi = \{U_r(s) \mid s \in R \text{ is a central root of } g(x) = x(x - c)\}$. By Lemma 3.2 and the following lemma, we may deduce that ξ is closed under intersection and union.

Lemma 3.3. *Let R be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ is central in R. If s, $t \in R$ are roots of $g(x)$, then the following hold.*

(a) *$U_r(s) \cap U_r(t) = U_r(c^{-1}st)$;*
(b) *$U_r(s) \cup U_r(t) = U_r(s + t - c^{-1}st)$;*
(c) *$U_r(s) = V_r(c - s)$. In particular, every set in ξ is clopen.*

Proof. (a) Let $P \in \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$. Then $P \in \mathrm{Spec}_r(R)$ with $s, t \notin P$. Note that $c \notin P$. Since c, s, t are central in R and P is a prime right ideal of R, it follows that $c^{-1}st \notin P$. Hence, $P \in \mathcal{U}_r(c^{-1}st)$ and therefore, $\mathcal{U}_r(s) \cap \mathcal{U}_r(t) \subseteq \mathcal{U}_r(c^{-1}st)$. Conversely, suppose that $P \in \mathcal{U}_r(c^{-1}st)$. If s or t belongs to P, then since s, t are central in R and P is a right ideal of R, it follows that $c^{-1}st \in P$; a contradiction. Thus s and t do not belong to P, that is, $P \in \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$. Hence, $\mathcal{U}_r(c^{-1}st) \subseteq \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$. The equality $\mathcal{U}_r(s) \cap \mathcal{U}_r(t) = \mathcal{U}_r(c^{-1}st)$ thus follows. Then $U_r(s) \cap U_r(t) = \mathcal{U}_r(s) \cap \mathcal{U}_r(t) \cap \mathrm{Max}_r(R) = \mathcal{U}_r(c^{-1}st) \cap \mathrm{Max}_r(R) = U_r(c^{-1}st)$.

(b) Let $P \in \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$. Then $s \notin P$ or $t \notin P$. Without loss of generality, suppose that $s \notin P$. Since $s(s - c) = 0 \in P$ and $s \notin P$ with s central in R, it follows that $s - c \in P$. Then $(1 - c^{-1}s)t = -c^{-1}(s - c)t \in P$. If $s + (1 - c^{-1}s)t \in P$, then it will follow that $s \in P$; a contradiction. Thus, $s + (1 - c^{-1}s)t \notin P$ and hence, $P \in \mathcal{U}_r(s + (1 - c^{-1}s)t)$. The inclusion $\mathcal{U}_r(s) \cup \mathcal{U}_r(t) \subseteq \mathcal{U}_r(s + (1 - c^{-1}s)t)$ therefore holds. For the reverse inclusion, suppose that $P \in \mathcal{U}_r(s + (1 - c^{-1}s)t)$. Then $s + (1 - c^{-1}s)t \notin P$. If s and t both belong to P, then $s + (1 - c^{-1}s)t \in P$; a contradiction. Hence, either $s \notin P$ or $t \notin P$, that is, $P \in \mathcal{U}_r(s)$ or $P \in \mathcal{U}_r(t)$. Therefore, $P \in \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$ and the inclusion $\mathcal{U}_r(s + (1 - c^{-1}s)t) \subseteq \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$

follows. Hence, $\mathcal{U}_r(s) \cup \mathcal{U}_r(t) = \mathcal{U}_r(s + (1 - c^{-1}s)t)$. It follows that

$$\begin{aligned} U_r(s) \cup U_r(t) &= (\mathcal{U}_r(s) \cap \mathrm{Max}_r(R)) \cup (\mathcal{U}_r(t) \cap \mathrm{Max}_r(R)) \\ &= (\mathcal{U}_r(s) \cup \mathcal{U}_r(t)) \cap \mathrm{Max}_r(R) \\ &= \mathcal{U}_r(s + (1 - c^{-1}s)t) \cap \mathrm{Max}_r(R) = U_r(s + (1 - c^{-1}s)t). \end{aligned}$$

(c) By using Lemma 3.1, we have $U_r(s) = \mathrm{Max}_r(R) \setminus U_r(c - s) = V_r(c - s)$. It follows that every set in ξ is clopen. □

Next, we extend Proposition 2.4 in [**2**] as follows:

Proposition 3.1. *Let R be an $x(x - c)$-clean ring with $c \in Z(R)$ such that every root of $x(x - c)$ is central in R. Then R is a right c-tb ring.*

Proof. By Proposition 2.1, $c \in U(R)$. Let M and N be distinct maximal right ideals of R. Then there exists $a \in M \setminus N$ and $N + aR = R$. Hence, $1 - ar \in N$ for some $r \in R$. Since N is a right ideal of R, $c - arc = (1 - ar)c \in N$. Let $y = arc$. Then $c - y \in N$ and $y \in M \setminus N$. Since R is $x(x - c)$-clean, there exist a unit $u \in R$ and a root $s \in R$ of $x(x - c)$ such that $y = u + s$. If $s \in M$, then $u = y - s \in M$ from which it follows that $M = R$; a contradiction since M is a maximal right ideal of R. Thus, $s \notin M$. If $s \notin N$, then $c - s \in N$ (by Lemma 3.1) and hence, $u = y - s = (y - c) + (c - s) \in N$. It follows that $N = R$ which is also not possible since N is a maximal right ideal of R. We thus have that s is a root of $x(x - c)$ belonging to N only. Hence, R is a right c-tb ring. □

Proposition 3.2. *Let R be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ in R is central. If R is a right c-tb ring, then ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$. In particular, $\mathrm{Max}_r(R)$ is a compact, zero-dimensional Hausdorff space.*

Proof. Note that if M_1 and M_2 are two distinct maximal right ideals of R, then since R is a right c-tb ring, there exists a root $s \in R$ of $g(x)$ such that $s \notin M_1$, $s \in M_2$ (that is, $M_1 \in U_r(s)$, $M_2 \notin U_r(s)$). The points in $\mathrm{Max}_r(R)$ can therefore be separated by disjoint clopen sets belonging to ξ. Hence, $\mathrm{Max}_r(R)$ is Hausdorff. By [**2**, Lemma 2.1] we have that $\mathrm{Max}_r(R)$ is compact.

To show that ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$, let $K \subseteq \mathrm{Max}_r(R)$ be a closed subset and take $M \notin K$. For each $N \in K$, since $N \neq M$, there exists a clopen set $U_r(s_N) \in \xi$ separating M and N, say $N \in U_r(s_N)$. The collection $\{U_r(s_N) \mid N \in K\}$ is therefore an open cover of the set K. Since K is compact, it has a finite subcover, that is, K is contained in a finite cover of sets of the form $U_r(s_N)$ with $N \in K$. By Lemma 3.3, there exists a clopen set $C \in \xi$ separating M from K. Hence, ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$. Since every set in ξ is clopen (by Lemma 3.3), it follows that $\mathrm{Max}_r(R)$ is zero-dimensional. □

Proposition 3.3. *Let R be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ in R is central. If ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$, then for any $a \in R$, there exists a root s of $g(x)$ such that $s \notin M$ for every $M \in V_r(a)$ and $s \in N$ for every $N \in V_r(a - c)$.*

PROOF. Consider the disjoint closed sets $V_r(a)$ and $V_r(a-c)$. Since ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$ and $\mathrm{Max}_r(R)$ is compact, there is a clopen set $U_r(s) \in \xi$ separating the sets $V_r(a)$ and $V_r(a-c)$. Without loss of generality, assume that $V_r(a) \subseteq U_r(s)$ and $V_r(a-c) \subseteq V_r(s)$. Then it follows that $s \notin M$ for every $M \in V_r(a)$ and $s \in N$ for every $N \in V_r(a-c)$. □

PROPOSITION 3.4. *Let R be a ring and let $g(x) = x(x-c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ in R is central. If for every $a \in R$ there exists a root $s \in Z(R)$ of $g(x)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a-c) \subseteq V_r(s)$, then R is $g(x)$-clean.*

PROOF. Let $a \in R$. By the hypothesis, there exists a root $s \in Z(R)$ of $g(x)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a-c) \subseteq V_r(s)$. We claim that $a-s$ is a unit. Let M be a maximal right ideal of R. Note that if $a \in M$, then $a-s \notin M$, since $s \notin M$. Next, suppose that $a \notin M$. If $a-s \in M$, then $s \notin M$, and hence, $c-s \in M$ (by Lemma 3.1). Then since $(a-c)+(c-s) = a-s \in M$, it follows that $a-c \in M$ and hence, $s \in M$ (because $V_r(a-c) \subseteq V_r(s)$); a contradiction. Thus, $a-s \notin M$. We have therefore shown that $a-s \notin M$ for any maximal right ideal M of R. Hence, $a-s$ has a right inverse, that is, $(a-s)v = 1$ for some $v \in R$. Then $(a-s)(vc) = c$ and by Lemma 2.2, we have that $(vc)(a-s) = c$. Since $c \in U(R) \cap Z(R)$, we can conclude that $a-s$ is a unit in R. Hence, a is the sum of a unit and a root of $g(x)$ in R. Since a is arbitrary in R, it follows that R is $g(x)$-clean. □

We are now ready for the main result.

THEOREM 3.1. *Let R be a ring and let $x(x-c) \in Z(R)[x]$ with $c \in U(R)$. If every root of $x(x-c)$ is central in R, then the following conditions are equivalent.*

(a) *R is $x(x-c)$-clean;*
(b) *R is $x(x+c)$-clean;*
(c) *R is n-clean for all positive integers n;*
(d) *R is a right c-tb ring;*
(e) *The collection $\xi = \{U_r(s) \mid s \in R$ is a root of $x(x-c)\}$ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$;*
(f) *For every $a \in R$, there exists a root $s \in Z(R)$ of $x(x-c)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a-c) \subseteq V_r(s)$;*
(g) *R is a left c-tb ring;*
(h) *The collection $\xi = \{U_l(s) \mid s \in R$ is a root of $x(x-c)\}$ forms a base for the weak Zariski topology on $\mathrm{Max}_l(R)$.*

PROOF. By Theorem 2.2, it follows readily that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). By Proposition 3.1, we readily have (a) $\Rightarrow$ (d). The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (f) follow by Propositions 3.2 and 3.3, respectively. The implication (f) $\Rightarrow$ (a) is straightforward by using Proposition 3.4. By using the left analogue of the arguments in the proofs of (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a), we obtain the equivalence (a) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h). □

A ring R is said to be strongly clean if every element of R is the sum of an idempotent and a unit which commute with one another. A strongly clean ring is therefore clean and hence, $x(x-1)$-clean. On the other hand, an abelian $x(x-1)$-clean ring is clearly strongly clean. We thus have the following as a consequence of Theorem 3.1:

COROLLARY 3.1. *Let R be an abelian ring. The following conditions are equivalent:*

(a) *R is clean;*
(b) *R is strongly clean;*
(c) *R is $x(x+1)$-clean;*
(d) *R is n-clean for all positive integers n;*
(e) *R is a right tb-ring;*
(f) *The collection $\xi = \{U_r(s) \mid s \in \mathrm{Id}(R)\}$ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$;*
(g) *For every $a \in R$, there exists $s \in \mathrm{Id}(R)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a-1) \subseteq V_r(s)$;*
(h) *R is a left tb-ring;*
(i) *The collection $\xi = \{U_l(s) \mid s \in \mathrm{Id}(R)\}$ forms a base for the weak Zariski topology on $\mathrm{Max}_l(R)$.*

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