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ON A PROPERTY OF STIRLING POLYNOMIALS

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ABSTRACT. We positively answer a question posed in 1960 by D. S. Mitrinović and R. S. Mitrinović (Publ. Fac. Electrotech. Univ. Belgrade, Ser. Math. Phys. **34** (1960), 1–23) about the Stirling numbers of the first kind.

1. Introduction

We bring a positive answer to a question posed by D. S. Mitrinović and R. S. Mitrinović in 1960 [**4**] about the Stirling numbers of the first kind $s(n,k)$ defined by $x(x-1)\cdots(x-n+1)=\sum_{k\geqslant 0} s(n,k)x^k$. In what follows, we denote by $\lfloor x\rfloor$ the integer part of a real number x and by $\operatorname{mod}(n,2)$ the remainder of the division of n by 2. We recall that a primitive polynomial is a polynomial over an integral domain R such that no non-invertible element of R divides all its coefficients at once.

THEOREM 1.1. *Let $(m_n)_{n\geqslant 0}$ be the sequence defined by*

$$m_n := \frac{1}{(n+1)!}\prod_{p\ prime\ and\ p\leqslant n+1} p^{\left\lfloor \frac{n}{p-1}\right\rfloor+\left\lfloor \frac{n}{p(p-1)}\right\rfloor+\left\lfloor \frac{n}{p^2(p-1)}\right\rfloor+\cdots}. \tag{1.1}$$

Then for any $n\geqslant 0$, m_n is a non-negative integer and for any positive integer k, we have

$$s(n,n-2k)=\frac{1}{m_{2k}}\binom{n}{2k+1}P_{2k}(n),\quad (n\geqslant 2k) \tag{1.2}$$

$$s(n,n-2k-1)=\frac{1}{m_{2k+1}}\binom{n}{2k+2}n(n-1)P_{2k+1}(n)\quad (n\geqslant 2k+1), \tag{1.3}$$

where $P_{2k}(x)$ and $P_{2k+1}(x)$ are two primitive polynomials over $\mathbb{Z}$ satisfying

$$P_{2k}(0)=P_{2k+1}(0). \tag{1.4}$$

The sequence $(m_n)_{n\geqslant 0}=(1,1,4,2,48,16,576,\ldots)$ is the sequence $A163176$ in the OEIS [**6**]. The first few expressions of the polynomials $P_n(x)$ for $2\leqslant n\leqslant 9$ are

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$$
\begin{aligned}
P_2(x) &= 3x - 1,\\
P_3(x) &= -1,\\
P_4(x) &= 15x^3 - 30x^2 + 5x + 2,\\
P_5(x) &= -3x^2 + 7x + 2,\\
P_6(x) &= 63x^5 - 315x^4 + 315x^3 + 91x^2 - 42x - 16,\\
P_7(x) &= -9x^4 + 54x^3 - 51x^2 - 58x - 16,\\
P_8(x) &= 135x^7 - 1260x^6 + 3150x^5 - 840x^4 - 2345x^3 - 540x^2 + 404x + 144,\\
P_9(x) &= -15x^6 + 165x^5 - 465x^4 - 17x^3 + 648x^2 + 548x + 144.
\end{aligned}
$$

In [**4**], Mitrinović and Mitrinović state the relations (1.2), (1.3) and (1.4) for $k \in \{1, 2, 3, 4, 5, 6\}$, and then raise the question of whether these equalities are true in the general case. Theorem 1.1, main result of our paper, brings a positive answer to this question.

2. Proof of Theorem 1.1

To prove the theorem, it is useful to note the following three lemmas. The proof is mainly based on the properties of the Nörlund polynomials and the sequence $(m_n)_{n\geqslant 0}$. The Nörlund polynomials $B_n^{(x)}$ are defined by [**5**]

$$\left(\frac{z}{e^z - 1}\right)^x = \sum_{n=0}^{\infty} B_n^{(x)} \frac{z^n}{n!}.$$

For any positive integer n, $B_n^{(x)}$ is a polynomial over the rational number field of degree n and it is divisible by x.

The Bernoulli numbers B_n are defined for $n \geqslant 0$ by $B_n = B_n^{(1)}$. It is well known that

$$B_{2n+1} = 0 \quad (n \geqslant 1). \tag{2.1}$$

Lemma 2.1. *For any positive integer n, we have*

$$
\begin{aligned}
\big(B_n^{(x)}\big) &= (-1)^{n-1}\frac{B_n}{n},\\
[x^2]\big(B_{2n+1}^{(x)}\big) &= \frac{2n+1}{4n} B_{2n}
\end{aligned}
\tag{2.2}
$$

Proof. Let $n \geqslant 1$. In [**3**, Theorems 1 and 2], Liu and Srivastava have determined explicitly the coefficients of the polynomial $B_n^{(x)}$. They proved that for $1 \leqslant k \leqslant n$, the coefficient of the x^k term in $B_n^{(x)}$ is given by

$$[x^k]B_n^{(x)} = (-1)^{n-k}\frac{n!}{k!}\sum \frac{B_{\nu_1}\cdots B_{\nu_k}}{(\nu_1\cdots\nu_k)\nu_1!\cdots\nu_k!},$$

where the sum is taken over all positive integers $\nu_1, \dots, \nu_k$ that satisfy $\nu_1 + \dots + \nu_k = n$. Applying this for $k = 1$, we obtain easily (2.2), and for $k = 2$ we get

$$\begin{aligned}[x^2]\big(B_{2n+1}^{(x)}\big) &= -\frac{1}{2}\sum_{j=1}^{2n}\binom{2n+1}{j}\frac{B_j B_{2n+1-j}}{j(2n+1-j)}\\ &= -\frac{1}{2}\binom{2n+1}{1}\frac{B_1 B_{2n}}{2n} - \frac{1}{2}\binom{2n+1}{2n}\frac{B_{2n}B_1}{2n}\\ &\quad - \frac{1}{2}\sum_{j=2}^{2n-1}\binom{2n+1}{j}\frac{B_j B_{2n+1-j}}{j(2n+1-j)}\\ &= \frac{2n+1}{4n}B_{2n} - \frac{1}{2}\sum_{j=2}^{2n-1}\binom{2n+1}{j}\frac{B_j B_{2n+1-j}}{j(2n+1-j)}.\end{aligned}$$

The proof will be complete if we can show that

$$\sum_{j=2}^{2n-1}\binom{2n+1}{j}\frac{B_j B_{2n+1-j}}{j(2n+1-j)} = 0.$$

As the case $n = 1$ is obvious, let us consider the case $n \geqslant 2$. Firstly we remind that after B_1, all the Bernoulli numbers with odd index vanish. Note that j and $(2n+1-j)$ have a different parity, so we deduce that at least one of B_j and B_{2n+1-j} has an odd index greater than 1 and therefore $B_j B_{2n+1-j} = 0$ for any $2 \leqslant j \leqslant 2n-1$. □

In the proof of Theorem 1.1, the following lemma is essential.

Lemma 2.2. *For any integer $n \geqslant 2$, we have*

$$\binom{x-1}{n}B_n^{(x)} = \frac{1}{m_n}\binom{x}{n+1}(x(x-1))^{\bmod (n,2)}P_n(x) \tag{2.3}$$

where $P_n(x)$ is a primitive polynomial over $\mathbb{Z}$.

Proof. Let n be a positive integer. For any prime number p, let $r_p(n)$ be the highest power of p that divides $n!$. Adelberg [**1**, Corollary 3] shows that if we set $n_p = p\lfloor\frac{n}{p-1}\rfloor$ and

$$d_n = \frac{1}{n!}\prod_{p \text{ prime and } p\leqslant n+1} p^{r_p(n_p)}, \tag{2.4}$$

then $d_n B_n^{(x)}$ is a primitive polynomial over $\mathbb{Z}$.

By the Legendre's formula [**7**, p. 31], we obtain that for any prime number p less than or equal to $n+1$

$$r_p(n_p) = \sum_{k\geqslant 0}\Big\lfloor\frac{1}{p^k}\Big\lfloor\frac{n}{p-1}\Big\rfloor\Big\rfloor = \sum_{k\geqslant 0}\Big\lfloor\frac{n}{p^k(p-1)}\Big\rfloor. \tag{2.5}$$

From (2.4), (2.5) and (1.1), we obtain $d_n = (n+1)m_n$. On the other hand, from the definition of the Nörlund polynomials, it is easy to see that $B_n^{(x)}$ is divisible by x. Moreover, from (2.2) and (2.1) we have that for any odd integer $n \geqslant 3$, $[x](B_n^{(x)}) = (-1)^{n-1}\frac{B_n}{n} = 0$ and $B_n^{(1)} = B_n = 0$. It follows that in $\mathbb{Z}[x]$, the primitive polynomial $(n+1)m_nB_n^{(x)}$ is divisible by $x(x(x-1))^{\operatorname{mod}(n,2)}$ for $n \geqslant 2$. The quotient $P_n(x)$ of these two polynomials is also a primitive polynomial over $\mathbb{Z}$ and then we have

$$(n+1)m_nB_n^{(x)} = x(x(x-1))^{\operatorname{mod}(n,2)}P_n(x), \quad (n \geqslant 2). \tag{2.6}$$

Multiplying both sides of (2.6) by $\frac{1}{(n+1)m_n}\binom{x-1}{n}$, we deduce (2.3). $\square$

The following lemma states some properties of the sequence $(m_n)_{n\geqslant 0}$ defined by (1.1).

Lemma 2.3. *For any non-negative integer n, we have*

(1) *m_n is an integer*
(2) $m_{2n} = (n+1)m_{2n+1}$

Proof. Let n be a non-negative integer. For any prime number $p \leqslant n+1$ and for any integer $k \geqslant 0$, we have $\frac{n}{p^k(p-1)} - \frac{n+1}{p^{k+1}} = \frac{n+1-p}{p^{k+1}(p-1)} \geqslant 0$ and then $\lfloor \frac{n}{p^k(p-1)} \rfloor - \lfloor \frac{n+1}{p^{k+1}} \rfloor \geqslant 0$. By the Legendre's formula, we have

$$v_p(m_n) = \sum_{k\geqslant 0}\Big(\Big\lfloor \frac{n}{p^k(p-1)}\Big\rfloor - \Big\lfloor \frac{n+1}{p^{k+1}}\Big\rfloor\Big) \geqslant 0.$$

It follows that m_n is an integer.

Let $p \leqslant n+1$ be a prime number. It is clear that for all positive integers x and y we have

$$\Big\lfloor \frac{x+1}{y}\Big\rfloor - \Big\lfloor \frac{x}{y}\Big\rfloor = \begin{cases} 1 \text{ if } y \text{ divides } x+1, \\ 0 \text{ otherwise.}\end{cases}$$

Thus

$$v_p\Big(\frac{(n+1)m_{2n+1}}{m_{2n}}\Big) = v_p(2^{-1}) + \sum_{k\geqslant 0}\Big\lfloor \frac{2n+1}{p^k(p-1)}\Big\rfloor - \Big\lfloor \frac{2n}{p^k(p-1)}\Big\rfloor.$$

Since, $p^k(p-1)$ divide $2n+1$ if and only if $p=2$ and $k=0$, then for any prime p, $v_p\big(\frac{(n+1)m_{2n+1}}{m_{2n}}\big) = 0$, which is equivalent to state that $\frac{(n+1)m_{2n+1}}{m_{2n}} = 1$. $\square$

Now we can prove the theorem by using these lemmas.

Let k be a positive integer. It is well known that (cf. [**2**, p. 329])

$$s(n, n-j) = \binom{n-1}{j}B_j^{(n)}, \quad \text{for } n \geqslant j \geqslant 0. \tag{2.7}$$

By Lemma 2.2, (2.7) can be written as

$$s(n,n-j) = \frac{1}{m_j}\binom{n}{j+1}(n(n-1))^{\operatorname{mod}(j,2)}P_j(n) \quad (n \geqslant j \geqslant 2),$$

Since $P_j(x)$ is a primitive polynomial over $\mathbb{Z}$, applying this result for $j = 2k$ (resp. $j = 2k+1$) gives (1.2), (resp. (1.3)). Furthermore, by taking at first $n = 2k$ in (2.6) and then $n = 2k+1$ we get

$$(2k+1)m_{2k}B_{2k}^{(x)} = xP_{2k}(x),$$
$$(2k+2)m_{2k+1}B_{2k+1}^{(x)} = x^3P_{2k+1}(x) - x^2P_{2k+1}(x).$$

It follows that

$$P_{2k}(0) = [x]((2k+1)m_{2k}B_{2k}^{(x)}),$$
$$P_{2k+1}(0) = [x^2](-(2k+2)m_{2k+1}B_{2k+1}^{(x)}).$$

By Lemma 2.1, these last two equalities can be written as

$$P_{2k}(0) = -m_{2k}(2k+1)\frac{B_{2k}}{2k},$$
$$P_{2k+1}(0) = -(k+1)m_{2k+1}(2k+1)\frac{B_{2k}}{2k}.$$

Given this and the second relation of Lemma 2.3, we can deduce that $P_{2k+1}(0) = P_{2k}(0)$. This establishes (1.4). The proof of the theorem is now complete.

Remark 2.1. As we know that for any non-negative integer n, $B_n^{(x)}$ is a polynomial of degree n, by using relation (2.6) we deduce that for any $k \geqslant 1$, we have $\deg(P_{2k}) = 2k-1$ and $\deg(P_{2k+1}) = 2k-2$.

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