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FRACTIONAL BLACK–SCHOLES MODEL WITH REGULARIZED PRABHAKAR DERIVATIVE

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Abstract. We introduce a fractional type Black–Scholes model in European options including the regularized Prabhakar derivative. We apply the reconstruction of variational iteration method to get the approximate analytical solutions for some models of generalized fractional Black–Scholes equations in terms of the generalized Mittag-Leffler functions.

1. Introduction

The Black–Scholes model is the most well known mathematical model for pricing financial derivatives. It was introduced by Black and Scholes in the year 1973 as a partial differential equation and became so popular and almost universally accepted by the option traders for the estimating and valuing European or American options over time. This equation is widely used in global financial markets by traders and investors and is used to calculate values of both call and put options. It is also applied to determine a fair price for a call or put option based on factors such as underlying stock volatility, days to expiration and others. The Black–Scholes model for the value of an option is described by the following equation

$$\frac{\partial v}{\partial t}+\frac{\sigma^2x^2}{2}\frac{\partial^2 v}{\partial x^2}+(r-\tau)x\frac{\partial v}{\partial x}-rv=0,\quad (x,t)\in\mathbb{R}^+\times(0,T),$$

where $v(x,t)$ is the put option at asset price x and at time t, T is the maturity, r is the risk free interest rate, τ is the dividend yield and σ represents the volatility function of underlying asset. Also, we denote $v_c(x,t)$ and $v_p(x,t)$ as the value of European call and put options, respectively. Moreover, the payoff functions are

$$v_c(x,t)=\max\{x-K,0\},\qquad v_p(x,t)=\max\{K-x,0\},$$

where K denotes the expiration price for the option.

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From another general point of view, with the developments of theory of fractional calculus (integral and differential operations of non integer order) in various fields of science and engineering [**5**–**7**, **11**, **16**, **19**, **21**, **30**–**32**], the problem of the fractional Black–Scholes equation has been treated by some researchers [**2**, **3**, **8**–**10**, **17**, **22**, **28**, **29**, **37**, **39**, **42**]. In these studies, numerical and analytical tools were employed to obtain solutions of the corresponding equations stated by fractional Riemann–Liouville and Caputo derivatives.

In this paper, we intend to study the fractional Black–Scholes equation with a generalized fractional derivative (regularized Prabhakar derivative) [**12**, **15**, **18**].

For this purpose, we use the reconstruction of variational iteration method [**1**] and develop the analytical solution of generalized fractional Black–Scholes equation for European option pricing problems. This equation is described by the following equation

$$\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+}v}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}}+\frac{\sigma^2x^2}{2}\frac{\partial^2 v}{\partial x^2}+(r-\tau)x\frac{\partial v}{\partial x}-rv=0,\quad (x,t)\in\mathbb{R}^+\times(0,T),$$

where the operator $\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+}}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}}$ indicates the regularized Prabhakar fractional derivative (generalization of the Caputo derivative) and $\rho,\mu,\omega,\gamma\in\mathbb{C}$, $0<\mu\leqslant 1$.

2. Preliminaries

In this section, we introduce some basic definitions and properties of generalized fractional calculus and generalized Mittag-Leffler function which will be used in this work.

2.1. The generalized Mittag-Leffler function. In 1971, Prabhakar introduced the generalized Mittag-Leffler function (Mittag-Leffler function with three parameters) on his study on singular integral equations as follows [**36**]

$$E^{\gamma}_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\rho k+\mu)}\frac{z^k}{k!},\quad \gamma,\rho,\mu\in\mathbb{C},\mathrm{Re}(\rho)>0, \tag{2.1}$$

where $(\gamma)_k$ is the Pochhammer symbol [**13**]

$$(\gamma)_0=1,\qquad (\gamma)_k=\gamma(\gamma+1)\dots(\gamma+k-1),\quad k=1,2,\dots.$$

For $\gamma=1$, we get the two-parameter Mittag-Leffler function $E_{\rho,\mu}(z)$ defined by

$$E_{\rho,\mu}(z):=E^1_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)},\quad \rho,\mu\in\mathbb{C},\mathrm{Re}(\rho)>0,$$

and for $\gamma=\mu=1$, this function coincides with the classical Mittag-Leffler function $E_\rho(z)$ [**33**, **34**]

$$E_{\rho}(z):=E^1_{\rho,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+1)},\quad \rho\in\mathbb{C},\mathrm{Re}(\rho)>0.$$

Many researchers have studied the generalized Mittag-Leffler function especially the theory of fractional calculus and detected some applications in physics, engineering and applied sciences. For example, new definitions of generalized fractional derivatives for the fractional differential and integral equations were introduced and solutions of the Cauchy-type initial and boundary value problems were presented in terms of the generalized Mittag-Leffler functions [**4**, **18**–**20**, **24**–**27**, **38**, **40**, **41**]. Eshaghi and Ansari used this function for presenting the solutions of autoconvolution equations and established a Lyapunov inequality for fractional differential equations [**14**, **15**]. D'Ovidio and Polito presented the stochastic solution to a generalized fractional partial differential equation involving a regularized operator related to the Prabhakar operator [**12**]. Polito and Tomovski also obtained some Opial and Hardy-type inequalities for the integrals and derivatives containing Mittag-Leffler function [**35**].

Lemma 2.1. *The Laplace transforms of generalized Mittag-Leffler function* (2.1) *has the following form* [**36**]

$$\mathcal{L}\big[x^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega x^{\rho})\big](s)=s^{-\mu}(1-\omega s^{-\rho})^{-\gamma},\quad |\omega s^{-\rho}|<1, \tag{2.2}$$

where $\gamma,\rho,\mu,\omega,s\in\mathbb{C}$, $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(s)>0$.

Theorem 2.1. *Let* $\gamma,\rho,\mu,\nu,\sigma,\omega\in\mathbb{C}$ $(\mathrm{Re}(\rho),\mathrm{Re}(\mu),\mathrm{Re}(\nu)>0)$*, then* [**26**]

$$\int_0^t (t-\eta)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\eta)^{\rho})\eta^{\nu-1}E^{\sigma}_{\rho,\nu}(\omega\eta^{\rho})\,d\eta=t^{\mu+\nu-1}E^{\gamma+\sigma}_{\rho,\mu+\nu}(\omega t^{\rho}),$$

and in the special case $\sigma=0$*, we have*

$$\int_0^t (t-\eta)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\eta)^{\rho})\eta^{\nu-1}d\eta=\Gamma(\nu)t^{\mu+\nu-1}E^{\gamma}_{\rho,\mu+\nu}(\omega t^{\rho}). \tag{2.3}$$

2.2. Prabhakar integral and derivative. After studying of some properties of the generalized Mittag-Leffler function, Prabhakar introduced an integral operator with generalized Mittag-Leffler function in kernel as follows.

Definition 2.1 (Prabhakar integral). Let $f\in L^1[0,b]$, $0<x<b\leqslant\infty$. The Prabhakar integral operator with generalized Mittag-Leffler function in its kernel is defined as follows [**18**]

$$E^{\gamma}_{\rho,\mu,\omega,0+}f(x)=\int_0^x (x-u)^{\mu-1}E^{\gamma}_{\rho,\mu}\big(\omega(x-u)^{\rho}\big)f(u)\,du,\quad x>0, \tag{2.4}$$

where $\rho,\mu,\omega,\gamma\in\mathbb{C}$, $\mathrm{Re}(\rho),\mathrm{Re}(\mu)>0$.

Remark 2.1. We note that for $\gamma=0$, Prabhakar integral operator (2.4) coincides with the Riemann–Liouville fractional integral of order μ

$$E^{0}_{\rho,\mu,\omega,0+}f=I^{\mu}_{0+}f.$$

Definition 2.2 (Prabhakar derivative). Let $f\in L^1[0,b]$, $0<x<b\leqslant\infty$. The Prabhakar derivative is defined by [**18**]

$$D^{\gamma}_{\rho,\mu,\omega,0+}f(x)=\frac{d^m}{dx^m}E^{-\gamma}_{\rho,m-\mu,\omega,0+}f(x), \tag{2.5}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$, $\mathrm{Re}(\rho), \mathrm{Re}(\mu) > 0$. Further, its regularized Caputo counterpart for function $f \in AC^m[0, b]$, $0 < x < b < \infty$, is given by

$$\begin{aligned} {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+} f(x) &= E^{-\gamma}_{\rho,m-\mu,\omega,0+} \frac{d^m}{dx^m} f(x) \\ &= D^{\gamma}_{\rho,\mu,\omega,0+} f(x) - \sum_{k=0}^{m-1} x^{k-\mu} E^{-\gamma}_{\rho,k-\mu+1}(\omega x^{\rho}) f^{(k)}(0+). \end{aligned} \tag{2.6}$$

REMARK 2.2. It is evident that the Prabhakar derivative (2.5) and regularized Prabhakar derivative (2.6) generalize the Riemann–Liouville and the Caputo fractional derivatives of order μ, respectively.

LEMMA 2.2. *For $m - 1 < \mu \leqslant m$, the Laplace transform of regularized Prabhakar derivative* (2.6) *has the form*

$$\mathcal{L}\{{}^{C}D^{\gamma}_{\rho,\mu,\omega,0+} f(x); s\} = s^{\mu}(1-\omega s^{-\rho})^{\gamma} F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma} f^{(k)}(0), \tag{2.7}$$

where $F(s)$ is the Laplace transform of $f(x)$.

PROOF. By applying the Laplace transform operator on the regularized Prabhakar derivative (2.6), we have

$$\begin{aligned} \mathcal{L}\{{}^{C}D^{\gamma}_{\rho,\mu,\omega,0+} f(x); s\} &= \mathcal{L}\Big\{E^{-\gamma}_{\rho,m-\mu,\omega,0+} \frac{d^m}{dx^m} f(x); s\Big\} \\ &= \mathcal{L}\big\{x^{m-\mu-1} E^{-\gamma}_{\rho,m-\mu}(\omega x^{\rho})\big\} \mathcal{L}\Big\{\frac{d^m}{dx^m} f(x); s\Big\} \\ &= s^{\mu-m}(1-\omega s^{-\rho})^{\gamma} \Big[s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0)\Big] \\ &= s^{\mu}(1-\omega s^{-\rho})^{\gamma} F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma} f^{(k)}(0). \ \square \end{aligned}$$

3. Reconstruction of variational iteration method with Laplace transform

Recently, Hesameddini and Latifizadeh [**23**] proposed the reconstruction of variational iteration algorithms by using the Laplace transform for solving the differential equations of integer order. In this section, we generalize this method for solving the fractional Black–Scholes equation including the regularized Prabhakar derivative. This method provides the analytical solution of the fractional Black–Scholes equation for a European option pricing problem with initial condition. Therefore, in order to present the procedure of the reconstruction of variational iteration method, we consider the general form of Black–Scholes equation as follows

$$\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+} f(x,t)}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}} = g\Big(t, x, f, \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x^2}\Big), \quad \rho, \mu, \omega, \gamma \in \mathbb{C},\ \mathrm{Re}(\rho) > 0,\ 0 < \mu < 1, \tag{3.1}$$

with the initial conditions $f(x,0) = 0$, in which the operator $\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+}}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}}$ indicates the regularized Prabhakar fractional derivative. We apply the Laplace transform to both sides of the above equation with respect to the independent variable t and then use relation (2.7) to get

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\mathcal{L}\{f(x,t)\} = \mathcal{L}\Big\{g\Big(t,x,f,\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}\Big)\Big\}.$$

We then set $\mathcal{L}\{g(t,x,f,\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2})\} = G(s,x)$ to obtain

$$\mathcal{L}\{f(x,t)\} = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}G(s,x). \tag{3.2}$$

Now, by applying the inverse Laplace transform on both sides of equation (3.2) and using relation (2.2) and the convolution theorem, we obtain

$$\begin{aligned} f(x,t) &= \mathcal{L}\{s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}G(s,x)\} = t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho}) * g\Big(t,x,f,\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}\Big) \\ &= \int_0^t (t-\eta)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\eta)^{\rho})g\Big(\eta,x,f,\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}\Big)d\eta. \end{aligned}$$

From the reconstruction of the variational iteration method introduced in [**23**] and using the initial conditions, an iteration formula for (3.1) can be constructed as

$$f_{n+1}(x,t) = f_0(x,t) + \int_0^t (t-\eta)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\eta)^{\rho})g\Big(\eta,x,f_n,\frac{\partial f_n}{\partial x}\frac{\partial^2 f_n}{\partial x^2}\Big)d\eta,$$

where $f_0(x,t)$ is an initial solution. Therefore, by using the above iteration formula, we can immediately evaluate several approximations exactly. The exact solution accordingly is given as follows

$$f(x,t) = \lim_{n\to\infty} f_n(x,t).$$

4. Numerical Examples

We now discuss the implementation of the proposed method by presenting some examples. To this end, by applying the reconstruction of the variational iteration method, we solve some examples of the generalized fractional Black–Scholes differential equations including the regularized Prabhakar derivative and show the solutions of these examples presented as a summation of the generalized Mittag-Leffler functions. In the particular case, the solution of the generalized fractional Black–Scholes equation including the regularized Prabhakar derivative gives the solution of the fractional Black–Scholes differential equation in the sense of the Caputo derivative. We first present two examples on which for anytime period, the value of dividend yield τ is considered to be zero.

Remark 4.1. For a generalization of the fractional Black–Scholes model with the Caputo derivative to the fractional Black–Scholes model with the regularized Prabhakar derivative, the initial condition in the following examples is similar to that of the initial condition given in [**1**].

EXAMPLE 4.1. The solution of the fractional Black–Scholes option pricing equation [**22**]

$$\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+}v}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \quad \rho,\mu,\omega,\gamma \in \mathbb{C}, \ \ \mathrm{Re}(\rho) > 0, \ \ 0 < \mu \leqslant 1, \tag{4.1}$$

with the initial condition $v(x,0) = \max\{e^x - 1, 0\}$, is given by

$$\begin{aligned} v(x,t) = \max\{e^x - 1, 0\} &\sum_{i=0}^{\infty} (-kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i \\ &+ \max\{e^x, 0\} \sum_{i=1}^{\infty} (-1)^{i-1}(kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i. \end{aligned}$$

PROOF. By using the presented method, we get the following recursive formula

$$v_{n+1}(x,t) = v_0(x,t) + \int_0^t (t-\eta)^{\mu-1} E^{\gamma}_{\rho,\mu}\big(\omega(t-\eta)^{\rho}\big) g(x,\eta,v_n)\, d\eta,$$

where

$$v_0(x,t) = \max\{e^x - 1, 0\}, \qquad g(x,\eta,v_n) = \frac{\partial^2 v_n}{\partial x^2} + (k-1)x\frac{\partial v_n}{\partial x} - kv_n.$$

Now, by using relation (2.3) for $\nu = 1$, we obtain the following successive approximations

$$\begin{aligned} v_0(x,t) &= \max\{e^x - 1, 0\} \\ v_1(x,t) &= \max\{e^x - 1, 0\}\big(1 - kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big) + \max\{e^x, 0\}\big(kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big), \\ v_2(x,t) &= \max\{e^x - 1, 0\}\big(1 - kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}) + k^2t^{2\mu}(E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^2\big) \\ &\quad + \max\{e^x, 0\}\big(kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}) - k^2t^{2\mu}(E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^2\big), \\ &\ \ \vdots \\ v_n(x,t) &= \max\{e^x - 1, 0\}\sum_{i=0}^{n}\big(-kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big)^i \\ &\quad + \max\{e^x, 0\}\sum_{i=1}^{n}(-1)^{i-1}\big(kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big)^i. \end{aligned}$$

Therefore, the solution is given by

$$\begin{aligned} v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\{e^x - 1, 0\} &\sum_{i=0}^{\infty}\big(-kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big)^i \\ &+ \max\{e^x, 0\}\sum_{i=1}^{\infty}(-1)^{i-1}\big(kt^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})\big)^i. \qquad \square \end{aligned}$$

Equation (4.1) is solved numerically for values $\gamma = 0.2$, $\rho = 0.9$, $\mu = 0.95$, $\omega = 0.3$ and $k = 1$. The results of $v(x,t)$ are presented in Figure 1.

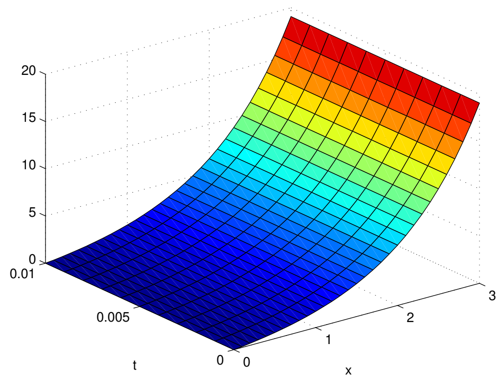

FIGURE 1.

REMARK 4.2. In the special case $\gamma = 0$, by using the definition of the Mittag-Leffler function in one parameter, the above solution becomes

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\{e^x - 1, 0\}E_\mu(-kt^\mu) + \max\{e^x, 0\}(1 - E_\mu(kt^\mu)).$$

which confirms the solution given in [**1**]. Furthermore, in the subcase $\mu = 1$, the exact solution of classical Black–Scholes equation (4.1) gives rise to

$$v(x,t) = \max\{e^x - 1, 0\}e^{-kt} + \max\{e^x, 0\}(1 - e^{-kt}).$$

EXAMPLE 4.2. Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $0 < \mu \leqslant 1$. Then the following fractional Black–Scholes option pricing equation

$$\frac{\partial^{\gamma}_{\rho,\mu,\omega,0+} v}{\partial t^{\gamma}_{\rho,\mu,\omega,0+}} + 0.08(2 + \sin x)^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \tag{4.2}$$

with the initial condition $v(x,0) = \max\{x - 25e^{-0.06}, 0\}$, has the solution

$$v(x,t) = \max\{x - 25e^{-0.06}, 0\}\sum_{i=0}^{\infty}\big(-0.06t^\mu E^{\gamma}_{\rho,\mu+1}(\omega t^\rho)\big)^i$$
$$- x\sum_{i=1}^{\infty}\big(-0.06t^\mu E^{\gamma}_{\rho,\mu+1}(\omega t^\rho)\big)^i.$$

PROOF. Similar to the previous example, by applying the proposed method, we get the following recursive formula

$$v_{n+1}(x,t) = v_0(x,t) - \int_0^t (t-\eta)^{\mu-1} E^{\gamma}_{\rho,\mu}\big(\omega(t-\eta)^\rho\big) g(x,\eta,v_n)\, d\eta,$$

where

$$v_0(x,t) = \max\{x - 25e^{-0.06}, 0\},$$

$$g(x, \eta, v_n) = 0.08(2 + \sin x)^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x \frac{\partial v}{\partial x} - 0.06v.$$

By straight computations, we obtain the following successive approximations

$$\begin{aligned}
v_0(x,t) &= \max\{x - 25e^{-0.06}, 0\} \\
v_1(x,t) &= \max\{x - 25e^{-0.06}, 0\}(1 - 0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})) + 0.06xt^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}), \\
v_2(x,t) &= \max\{x - 25e^{-0.06}, 0\}(1 - 0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}) + (0.06)^2 t^{2\mu} (E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^2) \\
&\quad + x(0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}) - (0.06)^2 t^{2\mu} (E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^2), \\
&\ \ \vdots \\
v_n(x,t) &= \max\{x - 25e^{-0.06}, 0\} \sum_{i=0}^{n} (-0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i \\
&\quad - x \sum_{i=1}^{n} (-0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i.
\end{aligned}$$

Therefore, the solution of (4.2) is given by

$$\begin{aligned}
v(x,t) = \lim_{n\to\infty} v_n(x,t) &= \max\{x - 25e^{-0.06}, 0\} \sum_{i=0}^{\infty} (-0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i \\
&\quad - x \sum_{i=1}^{\infty} (-0.06t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}))^i. \qquad \square
\end{aligned}$$

Equation (4.2) is solved numerically for values $\gamma = 0.2$, $\rho = 0.9$, $\mu = 0.95$ and $\omega = 0.3$. The result of $v(x,t)$ is presented in Figure 2.

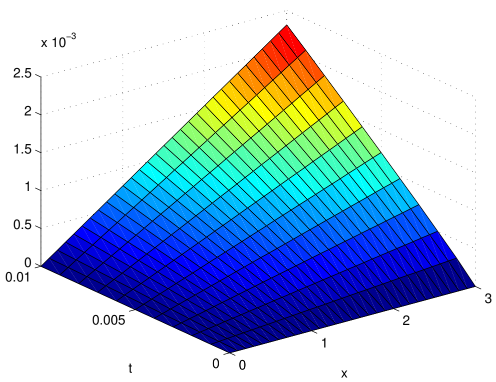

FIGURE 2.

Remark 4.3. When $\gamma = 0$, the solution of (4.2) takes the form

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\{x - 25e^{-0.06}, 0\} E_\mu(-0.06t^\mu) + x(1 - E_\mu(-0.06t^\mu)).$$

For the special case $\mu = 1$, the exact solution of (4.2) becomes

$$v(x,t) = \max\{x - 25e^{-0.06}, 0\} e^{-0.06t} + x(1 - e^{-0.06t}).$$

Example 4.3. Let $\rho, \mu, \omega, \gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $0 < \mu \leqslant 1$. Then the solution to the fractional Black–Scholes option pricing equation

$$\frac{\partial^\gamma_{\rho,\mu,\omega,0+} v}{\partial t^\gamma_{\rho,\mu,\omega,0+}} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} + (r-\tau)x\frac{\partial v}{\partial x} - rv = 0, \tag{4.3}$$

with the initial condition $v(x,0) = \max\{\alpha x - \beta, 0\}$, $\quad \alpha, \beta \in \mathbb{R}^+$, is given by

$$v(x,t) = \max\{\alpha x - \beta, 0\} \sum_{i=0}^{\infty} (rt^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i - \alpha x \sum_{i=1}^{\infty} (r^i - \tau^i)(t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i.$$

Proof. By applying the proposed method, a recursive formula is obtained as follows

$$v_{n+1}(x,t) = v_0(x,t) + \int_0^t (t-\eta)^{\mu-1} E^\gamma_{\rho,\mu}(\omega(t-\eta)^\rho) g(x,\eta,v_n)\, d\eta,$$

where

$$v_0(x,t) = \max\{\alpha x - \beta, 0\},$$
$$g(x,\eta,v_n) = -\frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} - (r-\tau)x\frac{\partial v}{\partial x} + rv.$$

The approximations are obtained as

$$\begin{aligned}
v_0(x,t) &= \max\{\alpha x - \beta, 0\}\\
v_1(x,t) &= \max\{\alpha x - \beta, 0\}(1 + rt^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho)) - \alpha x(r-\tau)t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho),\\
v_2(x,t) &= \max\{\alpha x - \beta, 0\}(1 + rt^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho) + r^2 t^{2\mu}(E^\gamma_{\rho,\mu+1}(\omega t^\rho))^2)\\
&\quad - \alpha x((r-\tau)t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho) + (r^2-\tau^2)t^{2\mu}(E^\gamma_{\rho,\mu+1}(\omega t^\rho))^2),\\
&\ \vdots\\
v_n(x,t) &= \max\{\alpha x - \beta, 0\} \sum_{i=0}^{n} (rt^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i - \alpha x \sum_{i=1}^{n} (r^i - \tau^i)(t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i.
\end{aligned}$$

Therefore, the solution of (4.3) is given by

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\{\alpha x - \beta, 0\} \sum_{i=0}^{\infty} (rt^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i - \alpha x \sum_{i=1}^{\infty} (r^i - \tau^i)(t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho))^i. \qquad \square$$

Equation (4.1) is solved numerically for values $\gamma = 0.2$, $\rho = 0.9$, $\mu = 0.95$, $\omega = 0.3$, $\alpha = 1$, $\beta = 10$, $\rho = 0.25$, $\tau = 0.2$. The result of $v(x,t)$ is presented in Figure 3.

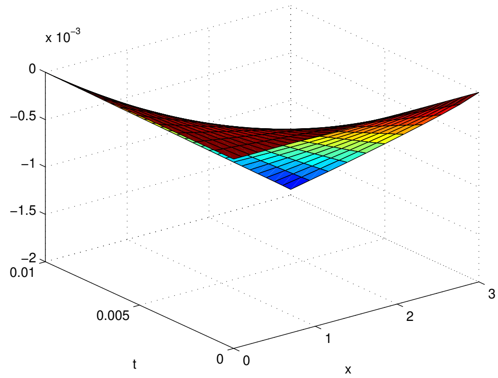

FIGURE 3. The solution of (4.3)

REMARK 4.4. When $\gamma = 0$, then the solution of (4.3) takes the form

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\{\alpha x - \beta, 0\} E_\mu(rt^\mu) - \alpha x(E_\mu(rt^\mu) - E_\mu(\tau t^\mu)).$$

For $\mu = 1$, the exact solution of (4.3) becomes

$$v(x,t) = \max\{\alpha x - \beta, 0\} e^{rt} - \alpha x(e^{rt} - e^{\tau t}).$$

REMARK 4.5. We observe that the convergence speed of the solution of the fractional Black–Scholes equation equations with the regularized Prabhakar derivative depends on the three-parameter generalized Mittag-Leffler functions. For example, for the case $-\beta t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho)$ we see that

$$\frac{d}{dt}(e^{-\alpha t}) \mid_{t=0} = -\alpha e^{-\alpha t} \mid_{t=0} = -\alpha,$$

$$\frac{d}{dt}(-\beta t^\mu E^\gamma_{\rho,\mu+1}(\omega t^\rho)) \mid_{t=0} = -\beta t^{\mu-1} E^\gamma_{\rho,\mu}(\omega t^\rho) \mid_{t=0} = \pm\infty,$$

which decreases much faster than $e^{-\alpha t}$ near the origin for $\rho, \mu, \omega, \gamma \in \mathbb{R}$, $0 < \mu \leqslant 1$ and $\alpha, \beta \in \mathbb{R}$.

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