PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 102(116) (2017), 61–71 DOI: https://doi.org/10.2298/PIM1716061M

CONTACT BLOW UP AND CYLINDRICAL CONTACT HOMOLOGY OF TORIC CONTACT MANIFOLDS OF REEB TYPE

Aleksandra Marinković

Abstract. Let (V, ξ) be a toric contact manifold of Reeb type that is a prequantization of a toric symplectic manifold (M, ω). A contact blow up of (V, ξ) is the prequantization of a symplectic blow up of (M, ω). Thus, a contact blow up of (V, ξ) is a new toric contact manifold of Reeb type. In some special cases we are able to compute the cylindrical contact homology for the contact blow-up using only the cylindrical contact homology of the contact manifold we started with.

1. Introduction

A toric contact manifold of *Reeb type* is a compact toric contact manifold with an invariant contact form whose Reeb vector field corresponds to an element of the Lie algebra of the torus. This large family of toric contact manifolds was introduced by Boyer and Galicki in [**6**]. They showed that toric contact manifolds of Reeb type are analogue of toric symplectic manifolds. More precisely, in a similar way as Delzant [**8**] showed that any compact toric symplectic manifold is a symplectic reduction of $(\mathbb{C}^d, \frac{i}{2}\sum_{j=1}^{d} dz_j \wedge d\bar{z}_j)$, for some $d \in \mathbb{N}$, Boyer and Galicki showed that any toric contact manifold of Reeb type is a contact reduction of the standard contact sphere $(S^{2d-1}, \frac{i}{4}\sum_{j=1}^{d}(z_j d\bar{z}_j - \bar{z}_j dz_j))$, for some $d \in \mathbb{N}$ [**6**, Theorem 5.1]. Moreover, while compact toric symplectic manifolds are classified by their moment polytopes, toric contact manifolds of Reeb type are classified by their moment cones [**13**].

A prequantization of an integral toric symplectic manifold is a toric contact manifold of the Reeb type. We do a symplectic blow up of appropriate size of the underlying toric symplectic manifold to obtain another integral toric symplectic manifold. Taking a prequantization of the latter we construct a new toric contact

2010 *Mathematics Subject Classification:* 53D10; 57R17.

Key words and phrases: toric contact manifold, contact blow up, cylindrical contact homology.

Communicated by Vladimir Dragović.

manifold of the Reeb type. This procedure is called *a contact blow up* of toric contact manifolds of the Reeb type. In term of cones, a contact blow up is given by choping an edge of the corresponding moment cone, see Section 2. We remark that a blow up of contact manifolds was introduced by Gromov in [**11**] and elaborated by Casals, Pancholi and Presas in [**7**].

In Section 3 we recall a cylindrical contact homology of toric contact manifolds of Reeb type that was computed by Abreu and Macarini in [**1**]. As they show the Conley–Zehnder index of the Reeb orbits that generate the chain groups is even number. Thus the boundary map in the chain is trivial and contact homology groups are isomorphic to the chain groups. We then consider particular examples in dimension 5 and we prove the following theorem:

THEOREM 1.1. *Let V be a simply connected manifold that is a prequantization of a monotone toric symplectic 4-manifold. Let V' be a contact blow up of V that is also a prequantization of a monotone toric symplectic 4-manifold. If γ_1 and γ_2 are the orbits corresponding to the edges of the cone of V' obtained by chopping the edge of the cone of V that corresponds to orbit γ then the Conley–Zehnder index of γ_i, $i = 1, 2$ is the same as the Conley–Zehnder index of γ, counted with multiplicities. Thus, the cylindrical contact homology of V' is completely determined by the cylindrical contact homology of V in the following way $HC_k(V', \xi') = HC_k(V, \xi) + \mathbb{Q}$ when there is a multiply of γ with Conley–Zehnder index equal to k and $HC_k(V', \xi') = HC_k(V, \xi)$ otherwise.*

2. Contact blow up of toric contact manifolds of Reeb type

In this section we recall the construction of symplectic and contact blow up of toric symplectic and toric contact manifolds, respectively. Then we show some properties of the cone corresponding to the contact blow up, that will be used to prove Theorem 1.1.

2.1. Toric symplectic manifolds. A toric symplectic manifold is a symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of the torus $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. To any toric symplectic manifold one can associate a T^n-invariant map, called a moment map, $\mu = (\mu_1, \dots, \mu_n)\colon M \to \mathbb{R}^n$ such that

$$\mu_k(p) = \omega(X_k(p), \cdot) \text{ where } X_k(p) = \frac{d}{dt}\Big|_{t=0} (\exp\ te_k) * p,$$

$k = 1, \dots, n$; $e_k = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$.

• $\big(\mathbb{C}^n, \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j\big)$ with the standard T^n-action $(t_1, \dots, t_n) * (z_1, \dots, z_n) \mapsto (t_1 z_1, \dots, t_n z_n)$ is a toric symplectic manifold with a moment map $\mu(z_1, \dots, z_n) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2) + \text{const}$. The top left picture in Figure 1 represents the moment map image for toric $\mathbb{C}^2$.

If a toric symplectic manifold M is compact, then the image of μ is a compact convex polytope (Delzant polytope) $\Delta = \bigcap_{i=1}^d \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \geqslant \lambda_i\}$, where $v_i \in \mathbb{Z}^n$, $i = 1, \dots, d$ are primitive inward-pointing normal vectors to the facets of the polytope and $\lambda_i \in \mathbb{R}$, $i = 1, \dots, d$. That is, Δ is the convex hull of the images of the fixed points of the action (Atiyah [**2**], Guillemin, Sternberg [**12**]).

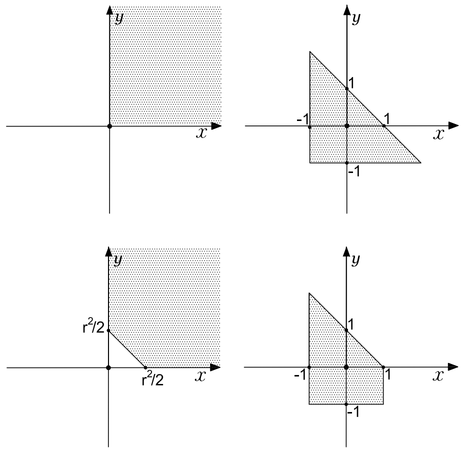

FIGURE 1. The moment map images: up left $(\mathbb{C}^2, \frac{i}{2}\sum_{j=1}^2 z_j \wedge d\bar{z}_j)$, up right $(\mathbb{CP}^2, \omega_{FS})$, bottom left $\mathbb{C}^2\sharp\overline{\mathbb{CP}}^2$, bottom right $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$

Moreover, Delzant in [**8**] proved that compact toric symplectic manifolds, up to equivariant symplectomorphism are classified by their moment map polytopes, up to translation in $\mathbb{R}^n$.

• The complex projective space $\mathbb{CP}^n$ with the Fubini-Study symplectic form ω_{FS} and with the standard toric T^n-action $(t_1, \ldots, t_n) * [z_1 : \cdots : z_{n+1}] \mapsto [t_1 z_1 : \cdots : t_n z_n : z_{n+1}]$ is a compact toric symplectic manifold. The up right picture in Figure 1 represents the polytope corresponding to $\mathbb{CP}^2$.

2.2. Symplectic blow up. By cutting a Darboux neighbourhood of a point in a symplectic manifold (M^{2n}, ω) and replacing it by a neighbourhood of the exceptional divisor (isomorphic to $\mathbb{C}P^{n-1}$) in $\mathbb{C}^n$, one obtains a new symplectic manifold. This process is called a symplectic blow up. For the explicit construction of the symplectic form we refer to [**17**, Section 7.1].

Further, blowing up a toric symplectic manifold in a fixed point one obtains a new toric symplectic manifold (see [**17**, Exercise 3.17]). Before we explain this in term of polytopes, here is the basic example.

• The symplectic blow up of $(\mathbb{C}^n, \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j)$ of size $r^2/2$, at the origin, is obtained by removing the open ball $B^{2n}(r)$ of radius r about the origin and collapsing its boundary along the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. Since the moment map image of the open ball $B^{2n}(r)$ is the corner given as the intersection of $\mathbb{R}^n_+$ and an open half-space $\{x \in \mathbb{R}^n \mid \langle x, (-1, \ldots, -1)\rangle > -r^2/2\}$, it follows that the symplectic blow up of toric $\mathbb{C}^n$ corresponds to the chopping the corner of $\mathbb{R}^n_+$ (see the left down picture in Figure 1).

Let (M^{2n}, ω) be a compact toric symplectic manifold with the moment polytope Δ. Since a Hamiltonian function on a compact manifold has critical points it follows that (M, ω) has fixed points and they are exactly the pre-images under the moment map of the vertices of Δ. By the equivariant Darboux–Weinstein theorem, every fixed point in M has a neighborhood U which is equivariantly symplectomorphic to a neighbourhood of the origin in $\mathbb{C}^n$. Thus, if V_P is the vertex of Δ whose pre-image under the moment map is a fixed point P then symplectic blow up of M at the point P corresponds to the chopping the corner of Δ that contains the vertex V_P. In this way, from Δ is obtained a new polytope Δ' by adding a new facet with the inward normal $v_1 + \cdots + v_n$, where $v_i, i = 1, \ldots, n$ are primitive inward normal vectors to the facets of Δ that meet at the vertex V_P. That is,

$$\Delta' = \Delta \cap \{x \in \mathbb{R}^n \mid \langle x, v_1 + \cdots + v_n \rangle \geqslant \lambda\},$$

where λ depends on the size of a blow up. When $v_1 + \cdots + v_n$ is a primitive vector, then the polytope Δ' is also a Delzant polytope and it corresponds to a toric symplectic manifold M' that is a symplectic blow up of M.

• A symplectic blow up of $(\mathbb{CP}^2, \omega_{FS})$ at one point, $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$, is given by the down right polytope in Figure 1.

We remark that it is also possible to do a blow up of a toric symplectic orbifold and obtain a new toric symplectic orbifold (see [**10**]). Due to Lerman and Tolman, compact toric symplectic orbifolds are classified by their moment polytopes (not Delzant polytopes) together with labels attached to each facet (see [**16**]). The blow up of a smooth point or an isolated singular point, in a symplectic orbifold is the same as a blow up of a point in a symplectic manifold. Moreover, if the orbifold is toric, a chopping of a corner goes in the same way as described above.

2.3. Toric contact manifolds. A toric contact manifold is a contact manifold (V^{2n+1}, ξ) equipped with an effective action of the torus T^{n+1} that preserves the contact structure. To any toric contact manifold one can associate a T^{n+1}-invariant map, called a moment map, $\mu_\alpha = (\mu_1, \ldots, \mu_{n+1})\colon V \to \mathbb{R}^{n+1}$, with respect to an invariant contact form α, such that

$$\mu_k(p) = \alpha_p(X_k(p)) \text{ where } X_k(p) = \frac{d}{dt}\Big|_{t=0} (\exp\ te_k) * p,$$

$k = 1, \ldots, n+1$; $e_k = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{n+1}$. Since a moment map depends on the choice of an invariant contact form, we introduce a moment cone. First, the toric action on V lifts to the toric action on SV, the symplectization of V. Thus, SV is a toric symplectic manifold and it has a corresponding moment map. *A moment cone* of a toric contact manifold (V, ξ) is the union of the origin and a moment map image of the symplectization SV. A moment cone can equivalently be difened as a cone over $\mu_\alpha(V)$.

A moment cone of a toric contact manifold of the Reeb type is always strictly convex cone, which means that it does not contain any linear subspace of a positive dimension. According to Lerman, toric contact manifolds of the Reeb type are classified by their moment cones [**13**, Theorem 2.18]). Thus, a contact blow up can be described in term of these cones.

2.4. Prequantization of a toric symplectic manifold. Let (M^{2n}, ω) be an integral compact toric symplectic manifold and $\Delta = \bigcap_{i=1}^{d}\{x \in \mathbb{R}^n | \langle x, v_i\rangle \geqslant \lambda_i\}$ corresponding Delzant polytope. *The prequantization* V of M (or, the Boothby–Wang manifold of M) is the total space of the S^1-bundle over M with the first Chern class equal to $[\omega] \in H^2(M, \mathbb{Z})$. The connection 1-form of this bundle is a contact form on V whose Reeb vector field generates this S^1-action. Moreover, the toric T^n-action on M lifts to the contact torus T^n-action on V that commutes with that S^1-action. This gives a toric T^{n+1}-action on V making it a toric contact manifold of Reeb type. The moment cone of V is the cone over Δ at the level $x_{n+1} = 1$, that is $C(\Delta) = \{z(x, 1) \in \mathbb{R}^n \times \mathbb{R} | \; x \in \Delta, z \geqslant 0\}$.

LEMMA 2.1. *The cone $C(\Delta)$ can be presented as*

$$C(\Delta) = \bigcap_{i=1}^{d}\{x \in \mathbb{R}^{n+1} \mid \langle x, (v_i, -\lambda_i)\rangle \geqslant 0\}.$$

PROOF. To the facet $f_j = \{x \in \Delta \mid \langle x, v_j\rangle = \lambda_j\}$ of the polytope Δ corresponds the facet $F_j \subset \mathbb{R}^{n+1}$ of the cone $C(\Delta)$ such that $f_j \times \{1\} \subset F_j$ and $(0, \ldots, 0) \in F_j$. Let us check that F_j is contained in the hyperplane $\langle (x_1, \ldots, x_{n+1}), (v_j, -\lambda_j)\rangle = 0$. Obviously the origin $(0, \ldots, 0)$ belongs to this hyperplaneplane. Further, for any point $(x, 1) \in f_j \times \{1\}$ it follows that $\langle (x, 1), (v_j, -\lambda_j)\rangle_{\mathbb{R}^{n+1}} = \langle x, v_j\rangle_{\mathbb{R}^n} - \lambda_j = 0$. Thus, F_j is contained in the hyperplane with the normal $(v_j, -\lambda_j)$. Let us check that this is an inward-pointing normal to the cone $C(\Delta)$. For every point $p = z(x, 1) \in C(\Delta)$, it holds $\langle (zx, z), (v_j, -\lambda_j)\rangle_{\mathbb{R}^{n+1}} = z\langle x, v_j\rangle_{\mathbb{R}^n} - z\lambda_j \geqslant 0$. Thus, $(v_j, -\lambda_j)$, $j = 1, \ldots, d$ are inward normals. □

Any co-oriented contact manifold (V, ξ) has well defined Chern classes (see for instance [**1**, Remark 2.4]) and they are canonically isomorphic to the Chern classes of the tangent bundle of the symplectization SV. We now show how $c_1(V, \xi)$, the first chern class of V, depends on the combinatorial type of the polytope Δ, when V is of the Reeb type with the cone over Δ.

LEMMA 2.2. *If Δ is monotone, that is $\lambda_j = \lambda$, $j = 1, \ldots, d$, then $c_1(V, \xi) = 0$.*

PROOF. Let $\beta \colon \mathbb{R}^d \to \mathbb{R}^{n+1}$ be the map $\beta(e_j) = (v_j, -\lambda)$, $j = 1, \ldots, d$ where $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^d$. Since β is a surjective map, its kernel is a $(d - n - 1)$-dimensional subspace of $\mathbb{R}^d$ (it may not be connected). Denote by $k_j \in \mathbb{R}^d$, $j = 1, \ldots, d-n-1$ the generators of $\operatorname{Ker}\beta$. It follows that $0 = \beta(k_j) = \sum_{i=1}^{d} k_{ji}\tilde{v}_i$, where $\tilde{v}_i = (v_i, -\lambda_i)$ and $k_j = (k_{j1}, \ldots, k_{jd}) = \sum_{i=1}^{d} e_i k_{ji}$. Let π_{n+1} denote the projection to the last coordinate. Thus $0 = \pi_{n+1}(\sum_{i=1}^{d} k_{ji}\tilde{v}_i) = \sum_{i=1}^{d} k_{ji}\pi_{n+1}(\tilde{v}_i) = -\sum_{i=1}^{d} \lambda_i k_{ji}$. In particular, if Δ is monotone, i.e., $\lambda_i = \lambda$, then $\sum_{i=1}^{d} k_{ji} = 0$, for all $j = 1, \ldots, d - n - 1$.

According to Abreu and Macarini [**1**, Remark 2.17], if $\sum_{i=1}^{d} k_{ji} = 0$ for all $j = 1, \ldots, d - n - 1$, then the first Chern class of the tangent bundle of the symplectization SV vanishes, i.e., $c_1(TSV) = 0$. Since $c_1(V, \xi) = c_1(TSV)$ [**1**, Remark 2.4], it follows $c_1(V, \xi) = 0$. □

The following lemma will be also needed to prove Theorem 1.1.

LEMMA 2.3. *If Δ is monotone and V simply connected, then $\lambda = -1$, that is, Δ is reflexive.*

PROOF. According to [**15**, Theorem 1.1] by Lerman, the fundamental group of V is equal to the finite group $\mathbb{Z}^{n+1}\backslash\mathfrak{L}$, where $\mathfrak{L}$ is the sublattice generated by the inward normals to the facets of the cone C. If $\pi_1(V) = 0$ then the normals $(v_j, -\lambda)$, $j = 1, \dots, d$ generate the whole space $\mathbb{Z}^{n+1}$. It follows that the vector $(0, \dots, 0, 1)$ is a $\mathbb{Z}$-linear combination of vectors $(v_j, -\lambda)$, $j = 1, \dots, d$. In particular, 1 is a Z-linear combination of λ's, that is, 1 is an integer multiple of λ. This is possible only if $\lambda \in \{\pm 1\}$. Since $\lambda < 0$ it follows that $\lambda = -1$. $\square$

REMARK 2.1. We remark that every toric contact manifold of the Reeb type is a prequantization of some compact toric symplectic orbifold ([**5**, Theorem 2.7]; see also [**14**, Lemma 3.7]). Here is how we can read the labeled polytope corresponding to that toric symplectic orbifold from the moment cone. Let (V, ξ) be a toric contact manifold of the Reeb type and $C = \bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1} \mid \langle x, u_i \rangle \geqslant 0\}$ the corresponding moment cone. For any vector $R = \sum_{i=1}^{d} a_i u_i$ with $a_1, \dots, a_d \in \mathbb{R}_{>0}$ there exists a T^{n+1}-invariant contact form α such that $R_\alpha(p) = \frac{d}{dt}|_{t=0}(\exp tR) * p$, $p \in V$, is the Reeb vector field corresponding to α (see [**1**, Proposition 2.19]). Here $*$ denotes the action of T^{n+1} on V. Take R in the lattice of the Lie algebra of T^{n+1}, i.e., $R = \sum_{i=1}^{d} a_i u_i \in \mathbb{Z}^{n+1}$, $a_1, \dots, a_d \in \mathbb{R}_{>0}$. Then, the corresponding Reeb flow generates S^1-action on V and we can lift this action to the S^1-action on the symplectization SV. A symplectic reduction of SV with respect to that S^1-action is a symplectic orbifold M and V is a prequantization of M. The polytope corresponding to M is the intersection of the cone C with a hyperplane perpendicular to R. A different choice of that hyperplane corresponds to a rescaling the symplectic form on M. The facets of the polytope of M are intersections of the facets of C with that hyperplane. The label of a facet corresponding to the facet of C with inward normal u_j is the index of the lattice generated by u_j and R inside $\mathrm{span}_{\mathbb{R}}(u_j, R) \cap \mathbb{Z}^d \cong \mathbb{Z}^2$, where $\mathbb{Z}^d$ is the lattice of $\mathbb{R}^d = \mathfrak{t}^d$. If the collection $\{u_j, R\}$ can be completed to a $\mathbb{Z}$ basis of $\mathbb{Z}^d$, the label on the corresponding facet is 1. If, in addition, for any edge of the cone, the inward normals to the facets meeting at that edge together with the vector R form a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$, then V is a prequantization of a toric symplectic manifold.

2.5. Contact blow up of a prequantization space. Let M be an integral toric symplectic manifold with moment polytope Δ and V a prequantization of M. Let M' be a symplectic blow up of M of appropriate size such that a symplectic form on M' is also integral. Then we can take a prequantization of M'. We obtain a new toric contact manifold of Reeb type V' with the moment cone $C(\Delta')$ that is a cone over Δ', a moment polytope for M'. Equivalently, $C(\Delta')$ is obtained from $C(\Delta)$, the cone of V, by chopping one edge, that is, by adding a new facet. More precisely, if $\Delta' = \Delta \cap \{x \in \mathbb{R}^n \mid \langle x, v_1 + \dots + v_n \rangle \geqslant \lambda\}$ then, according to Lemma 2.1

$$C(\Delta') = C(\Delta) \cap \{x \in \mathbb{R}^{n+1} \mid \langle x, (v_1 + \dots + v_n, -\lambda) \rangle \geqslant 0\}.$$

LEMMA 2.4. *If V is simply connected then V' is also simply connected.*

PROOF. According to Lerman [**15**, Theorem 1], V is simply connected if and only if primitive vectors normal to all facets of $C(V)$ span the whole $\mathbb{Z}^{n+1}$. If we add one vector to them, then this new collection will also span the whole $\mathbb{Z}^{n+1}$. □

REMARK 2.2. This procedure of contact blow up can be extended to any toric contact manifold of the Reeb type, since they are all prequantizations of compact toric symplectic orbifolds (see Remark 2.1). More precisely, when a toric contact manifold of the Reeb type is a prequantization only of a toric symplectic orbifold (and not of a manifold), we do the symplectic blow up of that orbifold as explained in [**10**] and obtain another toric symplectic orbifold. The prequantization of the new orbifold is a new toric contact manifold of the Reeb type and that is a contact blow up of a toric contact manifold of the Reeb type we started with.

3. Cylindrical contact homology of a toric contact manifold of Reeb type

A cylindrical contact homology of a contact manifold $(V^{2n+!}, \xi)$ is an invariant of contact structure introduced by Eliashberg, Givental and Hofer in [**9**]. Here is the basic idea, explained in [**4**].

Consider a non-degenerate contact form on V. That is, all periodic orbits of the corresponding Reeb vector field are non-degenerate [**4**, Definition 3]. A periodic orbit of the Reeb vector field is called bad if it is an even multiple of a periodic orbit whose parities of the Conley–Zehnder index of odd and even iterates disagree. An orbit that is not bad is called good. The degree of a good periodic orbit is the Conley–Zehnder index (see [**4**, Definition 4]) of the orbit plus $n-2$. A cylindrical contact homology $HC_*(V, \xi)$ is a homology of the chain complex $C_*(V, \alpha)$, that is given by the graded group with coefficients in $\mathbb{Q}$ and generated by good periodic orbits graded by their degrees. The boundary operator is given by counting rigid holomorphic cylinders in the symplectization SV that connect orbits whose degrees differ by 1.

Toric contact manifolds of Reeb type are completely determined by their moment cones [**13**]. Thus, their cylindrical contact homology should be computed using only the information incoded in the cones. This computation is done by Abreu and Macarini in [**1**] when $c_1(V, \xi) = 0$. We now sketch the algorithm.

Let (V, ξ) be a toric contact manifold of Reeb type and

$$C(\Delta) = \bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1} \mid \langle x, \tilde{v}_i \rangle \geqslant 0\}$$

be the corresponding cone. Consider a contact form α for ξ whose Reeb vector field R_α generates a 1-parametar subgroup that is dense in T^{n+1}. To any edge of the cone C corresponds one simple closed orbit of the Reeb vector field R_α. Since C is a good cone (see [**13**, Definition 2.17]), each edge is the intersection of exactly n facets $F_1, \ldots, F_n$, whose set of normals $\tilde{v}_1, \ldots, \tilde{v}_n$ can be completed to an integral base of $\mathbb{Z}^{n+1}$. Hence, for any edge, we can choose an integral vector $\eta \in \mathbb{Z}^{n+1}$ such that $\tilde{v}_1, \dots, \tilde{v}_n, \eta$ is an integral bases of $\mathbb{Z}^{n+1}$.

Define $\beta : \mathbb{R}^d \to \mathbb{R}^{n+1}$ by $\beta(e_j) = \tilde{v}_j$, $j = 1, \dots, d$ where $e_j = (0, \dots, 1, \dots, 0) \in \mathbb{R}^d$. Since β is surjective and integral, that is $\beta(\mathbb{Z}^d) \subset \mathbb{Z}^{n+1}$, there is the smallest natural number N_0 and an integral vector $\tilde{\eta}$ such that $\beta(\tilde{\eta}) = N_0\eta$.

Note that if $\pi_1(V) = 0$ then $N_0 = 1$. Indeed, if V is simply connected then the vectors $\{\tilde{v}_1, \dots, \tilde{v}_d\}$ span $\mathbb{Z}^{n+1}$ ([**15**, Theorem 1.1.]). Thus η can be written as an integral linear combination of $\tilde{v}_i, i = 1, \dots, d$, for instance $\eta = \sum c_i\tilde{v}_i, c_i \in \mathbb{Z}$. Then $\tilde{\eta} = \sum c_i e_i$ and $N_0 = 1$.

There is a vector $R \in \mathbb{R}^{n+1}$, given as a positive linear combination of vectors $\tilde{v}_i$, such that $R_\alpha(p) = \frac{d}{dt}|_{t=0}(\exp tR) * p$, $p \in V$ (see [**1**, Proposition 2.19]). The vector R can be uniquely written as $R = \sum_{i=1}^n b_i\tilde{v}_i + b_0N_0\tilde{\eta}$, for some constants $b_i, i = 0, 1, \dots, n$.

If γ denotes the simple closed Reeb orbit corresponding to the edge of C and γ^N the N-th multiple of γ, for any $N \in \mathbb{N}$ then *the Conley–Zehnder index* of γ^N can be computed as

$$\mu_{CZ}(\gamma^N) = 2\left(\sum_{i=1}^{n} \left\lfloor N\frac{b_i}{b_0}\right\rfloor + N\sum_{j=1}^{d}\tilde{\eta}_j\right) + n,$$

where $\tilde{\eta} = \sum_{j=1}^d \tilde{\eta}_j e_j$ and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

Since the degree of γ is $\mu_{CZ}(\gamma^N) + n - 2$ it follows that the degree is always an even number. Thus, $C_{2k+1}(V, \alpha) = 0$, $k \in \mathbb{Z}$ and the boundary operator $\partial_k \colon C_k(V, \alpha) \to C_{k-1}(V, \alpha)$, $k \in \mathbb{Z}$ is a zero map (since a domain or a codomain of the boundary map is a zero set). It follows that $HC_k(V, \xi) = C_k(V, \alpha)$, $k \in \mathbb{Z}$.

4. Proof of Theorem 1.1

Let V be a simply connected manifold that is a prequantization of a monotone toric symplectic 4-manifold given by a Delzant polytope Δ. Let V' be a contact blow up of V described in Section 2.5, that is, V' is a prequantization of a monotone toric symplectic manifold given by Δ'. Then, both V and V' are toric contact manifolds of Reeb type. Since V is simply connected then V' is also simply connected (see Lemma 2.4) and according to Lemma 2.2, the corresponding first chern classes are zero, $c_1(V, \xi) = c_1(V', \xi') = 0$. Thus, we can apply the results of Abreu and Macarini described in Section 3.

According to Lemma 2.3, both Δ and Δ' are reflexive polytopes. As explained in Section 2.2, Δ' is obtained from Δ by chopping a vertex V_1, that is, by adding a new facet. Δ' contains 2 new vertices V_{11} and V_{12} instead of V_1. If $v_1, v_2 \in \mathbb{Z}^2$ are primitive inward normals to the facets of Δ meeting at V_1, then $v_1 + v_2$ is a normal to the facet of Δ' that is not a facet of Δ. Let E_1 be the edge of the cone $C(\Delta)$ of V that corresponds to the vertex V_1. According to Lemma 2.1, the vectors $(v_j, 1)$, $j = 1, 2$ are primitive inward normals to the facets of $C(\Delta)$ meeting at E_1. By chopping the edge E_1 of $C(\Delta)$, that is, by taking the cone over Δ', we obtain a new cone $C(\Delta')$, that contains 2 new edges E_{11} and E_{12} instead of E_1 and one new facet F_{11} with the inward normal vector $(v_1 + v_2, 1)$.

Note that the vectors $(v_j, 1)$, $j = 1, 2$ and $(v_1 + v_2, 1)$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^3$. Indeed,

$$\det((v_1 + v_2, 1), (v_1, 1), (v_2, 1)) = \det((0, 0, -1), (v_1, 1), (v_2, 1)) = -\det(v_1, v_2).$$

Since Δ is a Delzant polytope, it is in particular smooth, meaning primitive inward normals to the facets meeting in one vertex form a $\mathbb{Z}$-basis. Thus $\det(v_1, v_2) \in \{1, -1\}$.

We now consider vectors $(v_1, 1), (v_2, 1)$ and $\eta = -(v_1 + v_2, 1)$. Since they form a $\mathbb{Z}$-basis of $\mathbb{Z}^3$, for any vector $R \in \mathbb{R}^3$ there are $b_0, b_1, b_2 \in \mathbb{R}$ such that

$$R = b_1\tilde{v}_1 + b_2\tilde{v}_2 + b_0\eta,$$

where $\tilde{v}_j = (v_j, 1)$, $j = 1, 2$. Assume that R is a positive linear combination of inward normals to the facets of $C(\Delta)$ and consider the Reeb vector field given by the vector R. Let γ_1 be the simple closed Reeb orbit corresponding to the edge E_1 and denote by γ_1^N the N-th multiple of γ_1, for any $N \in \mathbb{N}$. The inward normals of the facets F_1 and F_2, meeting at E_1 are $\tilde{v}_1$ and $\tilde{v}_2$. We complete the $\mathbb{Z}$-basis of $\mathbb{Z}^3$ with the vector η. The Conley–Zehnder index of γ_1^N can be computed as

$$\mu_{CZ}(\gamma_1^N) = 2\bigg(\Big\lfloor N\frac{b_1}{b_0}\Big\rfloor + \Big\lfloor N\frac{b_2}{b_0}\Big\rfloor + N\sum_{i=1}^{d}\tilde{\eta}_i\bigg) + 2 = 2\Big(\Big\lfloor N\frac{b_1}{b_0}\Big\rfloor + \Big\lfloor N\frac{b_2}{b_0}\Big\rfloor - N\Big) + 2.$$

Here we used $\sum_{i=1}^{d}\tilde{\eta}_i = -1$ what can be seen as follows. Since $C(\Delta) = \bigcap_{i=1}^{d}\{x \in \mathbb{R}^3 \mid \langle x, \tilde{v}_i\rangle \geqslant 0\}$ where $\tilde{v}_j = (v_j, 1)$, $j = 1, \ldots, d$ then $\beta(\tilde{\eta}) = \sum_{i=1}^{d}\tilde{\eta}_i\beta(e_i) = \sum_{i=1}^{d}\tilde{\eta}_i\tilde{v}_i$. On the other hand, $\beta(\tilde{\eta}) = \eta$, since V is simply connected (thus $N_0 = 1$). It follows that $\eta = \sum_{i=1}^{d}\tilde{\eta}_i\tilde{v}_i$ and also $\pi_3(\eta) = \pi_3(\sum_{i=1}^{d}\tilde{\eta}_i\tilde{v}_i)$ where $\pi_3 : (\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}) \to \mathbb{R}$ is the projection. From $\pi_3(\eta) = -1$ and $\pi_3(\tilde{v}_i) = 1$, for all $i = 1, \ldots, d$ we conclude $\sum_{i=1}^{d}\tilde{\eta}_i = -1$.

Let γ_{11} and γ_{12} be the simple closed Reeb orbits corresponding to the edges E_{11} and E_{12} respectively and denote by γ_{1j}^N the N-th multiple of γ_{1j}, for any $N \in \mathbb{N}$. The inward normals of the facets F_1 and F_{11}, meeting at E_{11} are $\tilde{v}_1$ and $-\eta$. We complete the $\mathbb{Z}$-basis of $\mathbb{Z}^3$ with the vector $\tilde{v}_2$. Since $R = b_1\tilde{v}_1 + (-b_0)(-\eta) + b_2\tilde{v}_2$, the Conley–Zehnder index of γ_{11}^N can be computed as

$$\mu_{CZ}(\gamma_{11}^N) = 2\bigg(\Big\lfloor N\frac{b_1}{b_2}\Big\rfloor + \Big\lfloor -N\frac{b_0}{b_2}\Big\rfloor + N\sum_{i=1}^{3}(e_2)_i\bigg) + 2 = 2\Big(\Big\lfloor N\frac{b_1}{b_2}\Big\rfloor + \Big\lfloor -N\frac{b_0}{b_2}\Big\rfloor + N\Big) + 2.$$

The inward normals of the facets F_2 and F_{11}, meeting at E_{12} are $\tilde{v}_2$ and $-\eta$. We complete the $\mathbb{Z}$-basis of $\mathbb{Z}^3$ with the vector $\tilde{v}_1$. Since $R = b_2\tilde{v}_2 + (-b_0)(-\eta) + b_1\tilde{v}_1$, the Conley–Zehnder index of γ_{12}^N can be computed as

$$\mu_{CZ}(\gamma_{12}^N) = 2\bigg(\Big\lfloor N\frac{b_2}{b_1}\Big\rfloor + \Big\lfloor -N\frac{b_0}{b_1}\Big\rfloor + N\sum_{i=1}^{3}(e_1)_i\bigg) + 2 = 2\Big(\Big\lfloor N\frac{b_2}{b_1}\Big\rfloor + \Big\lfloor -N\frac{b_0}{b_1}\Big\rfloor + N\Big) + 2.$$

We now consider a vector $R = b_0\eta + b_1\tilde{v}_1 + b_2\tilde{v}_2$ such that $0 < b_1 < b_0 < b_2 < 1$ and that R is a positive linear combination of inward normals to the facets of $C(\Delta)$. This vector corresponds to the Reeb vector field of some contact form α. Then

$$\left\lfloor N\frac{b_1}{b_0}\right\rfloor = \left\lfloor N\frac{b_1}{b_2}\right\rfloor = N-1, \quad \left\lfloor N\frac{b_2}{b_0}\right\rfloor = \left\lfloor N\frac{b_2}{b_1}\right\rfloor = N$$
$$\left\lfloor -N\frac{b_0}{b_1}\right\rfloor = -N-1, \quad \left\lfloor -N\frac{b_0}{b_2}\right\rfloor = -N.$$

It follows that $\mu_{CZ}(\gamma_1^N) = \mu_{CZ}(\gamma_{11}^N) = \mu_{CZ}(\gamma_{12}^N) = 2N$. Since $HC_k(V,\xi) = C_k(V,\alpha)$ and $C_k(V,\alpha)$ is generated by periodic orbits of degree k follows the proof of Theorem 1.1.

5. Application of Theorem 1.1

The requirements in Theorem 1.1 are actually very strict. It turns out that the only 4-dimensional toric symplectic manifolds satisfying these conditions are $(\mathbb{CP}^2, \omega_{FS})$ with the polytope given up right in Figure 1 and its blow ups $\mathbb{CP}^2\sharp m\overline{\mathbb{CP}}^2$, $m = 1,2,3$, given by the polytopes in Figure 2. The prequantization of $(\mathbb{CP}^2, \omega_{FS})$ is the sphere S^5 with the standard contact structure

$$\xi_{\text{st}} = \operatorname{Ker}\left(\frac{i}{4}\sum_{j=1}^{d}(z_j d\bar{z}_j - \bar{z}_j dz_j)\right).$$

The prequantization of $\mathbb{CP}^2\sharp m\overline{\mathbb{CP}}^2$ is the contact blow up of the sphere S^5, blown up in m orbits, $m = 1,2,3$ respectively.

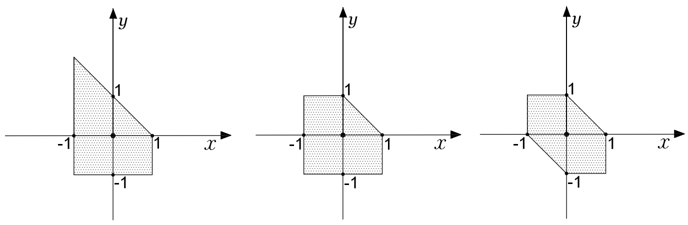

FIGURE 2. The moment map images for $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ on the left, for $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$ in the middle, for $\mathbb{CP}^2\sharp 3\overline{\mathbb{CP}}^2$ on the right.

The chain groups for the contact homology of the sphere S^5 are generated by three orbits γ_j, $j = 1,2,3$, corresponding to the three edges of the moment cone. According to the algorithm by Abreu and Macarini, we compute $\mu_{CZ}(\gamma_1^N) = 6N-2$, $\mu_{CZ}(\gamma_2^N) = 6N$ and $\mu_{CZ}(\gamma_3^N) = 6N+2$, $N \in \mathbb{N}$. Thus, $HC_k(S^5, \xi_{\text{st}}) = \mathbb{Q}$ if $k \in 2\mathbb{N}+2$ and $HC_k(S^5,\xi_{\text{st}}) = 0$ otherwise. From Theorem 1.1 we conclude that the cylindrical contact homology of S^5 blown at γ_j is $\mathbb{Q}^2$ in any even degree equal to $\mu_{CZ}(\gamma_j^N)$ for some $N \in \mathbb{N}$, while in other degrees it is equal to the contact homology of S^5. Similarly, the contact homology of S^5 blown in two orbits γ_j and γ_l, $j,l \in \{1,2,3\}$ is $\mathbb{Q}^2$ in any even degree equal to $\mu_{CZ}(\gamma_j^N)$ or $\mu_{CZ}(\gamma_l^N)$ for some $N \in \mathbb{N}$, while in other degrees it is equal to the contact homology of S^5. Finally, the cylindrical contact homology of S^5 blown up in all three orbits is $\mathbb{Q}^2$ in any even degree greater than 2 and zero otherwise.

Remark 5.1. It would be interesting to extend the result of Theorem 1.1 to more examples. The straightforward generalization to higher dimensions is not possible since the vectors $\tilde{v}_1, \dots, \tilde{v}_n, \eta = -(v_1 + \dots + v_n, 1)$ form only an $\mathbb{R}$-basis of $\mathbb{R}^{n+1}$ when $n > 2$. Indeed, since $\eta = -\tilde{v}_1 - \dots - \tilde{v}_n + (0, \dots, 0, n-1)$, it follows

$$\begin{aligned}\det(\eta, \tilde{v}_1, \dots, \tilde{v}_n) &= \det((0, \dots, 0, n-1), \tilde{v}_1, \dots, \tilde{v}_n)\\ &= (-1)^n (n-1) \det(v_1, \dots, v_n) = (-1)^n (n-1) \det(v_1, \dots, v_n).\end{aligned}$$

The vectors $v_1, \dots, v_n$ are normals of the facets of a Delzant polytope that meet in one vertex. Thus, these normals form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and $\det(v_1, \dots, v_n) \in \{1, -1\}$.

Acknowledgments. I thank Professor Miguel Abreu and Milena Pabiniak for useful discussions.

References

1. M. Abreu, L. Macarini, *Contact homology of good toric contact manifolds*, Compos. Math. **148** (2012), 304–334.
2. M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. Lond. Math. Soc. **14** (1982), 1–15.
3. W. M. Boothby, H. C. Wang, *On contact manifolds*, Annals of Math. **68** (1958), 721–734.
4. F. Bourgeois, *Introduction to contact homology*, lecture notes of mini-course at Summer School in Berder on Holomorphic Curves and Contact Topology, (2003).
5. C. P. Boyer, *Maximal tori in contactomorphism groups*, Differ. Geom. Appl. **31** (2013), 190–216.
6. C. P. Boyer, K. Galicki, *A note on toric contact geometry*, J. Geom. Phys. **35** (2000), 288–298.
7. R. Casals, D. Pancholi, F. Presas, *Contact blow up*, Expo. Math. **33** (2015), 78–100.
8. T. Delzant, *Hamiltoniens périodiques et image convexe de l application moment*, Bull. Soc. Math. Fr. **116** (1988), 315–339.
9. Y. Eliashberg, A. Givental, H. Hofer, *Introduction to Symplectic Field Theory*, Alon, N. (ed.) et al., GAFA 2000. Visions in mathematics – Towards 2000, Proc., Tel Aviv, Israel, August 25–September 3, 1999. Part II. Basel: Birkhäuser, 560–673 (2000).
10. L. Godinho, *Blowing up symplectic orbifolds*, Ann. Global Anal. Geom. **20** (2001), 117–162.
11. ______, *Partial Differential Relations*, Ergeb. Math. Grenzgeb. **9** (1986), Springer-Verlag, Berlin.
12. V. Guillemin, S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
13. E. Lerman, *Contact toric manifolds*, J. Symplectic Geom. **1** (2003), 785–828.
14. ______, *Maximal tori in the contactomorphism groups of circle bundles over Hirzebruch surfaces*, Math. Res. Lett. **10** (2003), 133–144.
15. ______, *Homotopy groups of K-contact toric manifolds*, Trans. Am. Math. Soc. **356** (2004), 4075–4083.
16. E. Lerman, S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Am. Math. Soc. **349** (1997), 4201–4230.
17. D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Math. Monographs, Oxford University Press, New York, 1995.

Center for Mathematical Analysis, Geometry and Dynamical Systems
Mathematics Department
Instituto Superior Técnico
Universidade de Lisboa
Lisboa, Portugal
aleksperisic@yahoo.com

(Received 07 08 2015)