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A NEW APPROACH TO FILTERS IN TRIANGLE ALGEBRAS

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Abstract. We develop the filter theory in triangle algebras. We define several interval valued residuated lattice-filters (IVRL-filters for short) in triangle algebras. We investigate the relationships among these types of IVRL-filters. Also, some special triangle algebras are introduced and studied in details.

1. Introduction

Formal fuzzy logics are generalizations of classical logic that allow us to reason gradually. Indeed, in the scope of these logics, formulas can be assigned not only 0 and 1 as truth values, but also elements of [0,1], or, more generally, of a bounded lattice L. The partial ordering of L then serves to compare the truth values of formulas which can be true to some extent. The best-known examples of formal fuzzy logics are probably monoidal t-norm based logic, basic logic, Gödel logic and Łukasiewicz logic [**3**–**5**, **7**].

The filter theory for logical algebras plays an important role in studying these algebras and the completeness of the corresponding logics. Filters are also particularly interesting because they are closely related to congruence relations, which are used to construct quotient algebras. The filter theory of residuated lattices, BL-algebras and MTL-algebras has been widely studied, and some important results have been published [**5**, **6**, **8**, **9**]. Among these logical algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of the residuated lattices.

Van Gass et al. introduced triangle algebras, a variety of residuated lattices equipped with approximation operators, and a third angular point u, different from 0,1. They proved that there is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL [**11**].

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Triangle algebras are used to cast the essence of using closed intervals of $\mathcal{L}$ as truth values into a set of appropriate logical axioms. Based on the definition and properties of triangle algebras, they also defined triangle logic (TL) and showed that this logic is sound and complete with respect to the variety of triangle algebras [**11**]. The same authors defined filters in triangle algebras. They suggested two different ways to define specific kinds of filters (as Boolean filters and prime filters) in triangle algebras and examine their mutual dependencies and connections. Finally, they obtained some interesting results [**10**].

In this paper, to obtain more properties of triangle algebras, we generalize the concept of filters and introduce some types of filters. We state and prove some theorems that determine relationships among these filters. By studying these special filters, we can define specific triangle algebras such as BL-triangle algebra, G-triangle algebra, MV-triangle algebra and semi G-triangle algebra. Definition of a filter in triangle algebras is different from that of filter in other algebraic structures such as residuated lattices and BL-algebras. Since ν and μ in this structure are important and are used in the definition of a filter, these types of filters play a basic role and thus the extended filters behave differently. Based on these facts, we give a classification for triangle algebras. Finally, we give a diagram in Figure 1, that determines the relations among all IVRL-filters in triangle algebras.

2. Preliminaries

Definition 2.1. [**5**] A residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ with four binary operations and two constant 0,1 such that:

- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leqslant z$ if and only if $x \leqslant (y \to z)$, for all x, y and z in L (residuation principle).

The ordering $\leqslant$ and negation $\neg$ in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ are defined as follows, for all x and y in L: $x \leqslant y$ if and only if $x \wedge y = x$ (or equivalently, if and only if $x \vee y = y$; or, also equivalently, if and only if $x \to y = 1$) and $\neg x = x \to 0$.

Lemma 2.1. [**8, 10**] *Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice. Then the following properties are valid, for all x, y and z in L:*

(1) $x \vee y \leqslant (x \to y) \to y$*(in particular* $x \leqslant \neg\neg x$*),*
(2) $x \to y = ((x \to y) \to y) \to y$,
(3) $\neg\neg\neg x = \neg x$,
(4) $(x \to y) * (y \to z) \leqslant (x \to z)$,
(5) *If* $x \leqslant y$, *then* $x * z \leqslant y * z, z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$,
(6) $(y \to z) \leqslant (x \to y) \to (x \to z)$,
(7) $x \to (y \to z) = y \to (x \to z)$,
(8) $((x \to y) \to y) \to y = x \to y$,
(9) $x \to y \leqslant (y \to z) \to (x \to z)$,
(10) $\neg\neg x * \neg\neg y \leqslant \neg\neg(x * y)$.

Definition 2.2. [**8**, **9**] A nonempty subset D of a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ is called a deductive system if:

(i) $1 \in D$,
(ii) If $x, x \rightarrow y \in D$, then $y \in D$.

An equivalent definition for deductive system is:

(i) If $x \in D$, $y \in L$ and $x \leqslant y$, then $y \in D$,
(ii) If $x, y \in D$, then $x * y \in D$.

Definition 2.3. [**8**] A prime filter of a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ is a filter such that $x \rightarrow y \in F$ or $y \rightarrow x \in F$ (or both), for all $x, y \in L$.

Definition 2.4. [**11**] Given a lattice $\mathcal{A} = (A, \vee, \wedge)$, its triangularization $\mathbb{T}(\mathcal{A})$ is the structure $\mathbb{T}(\mathcal{A}) = (\mathrm{Int}(\mathcal{A}), \vee, \wedge)$ defined by

- $\mathrm{Int}(\mathcal{A}) = \{[x_1, x_2] : (x_1, x_2) \in A^2 \text{ and } x_1 \leqslant x_2\}$,
- $[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,
- $[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_{\mathcal{A}} = \{[x, x] : x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

Definition 2.5. [**11**] An interval-valued residuated lattice (IVRL) is a residuated lattice $(\mathrm{Int}(\mathcal{A}), \vee, \wedge, \odot, \rightarrow_{\odot}, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{A})$ of a bounded lattice $\mathcal{A}$, in which the diagonal $D_{\mathcal{A}}$ is closed under $\odot$ and $\rightarrow_{\odot}$, i.e., $[x, x] \odot [y, y] \in D_{\mathcal{A}}$ and $[x, x] \rightarrow_{\odot} [y, y] \in D_{\mathcal{A}}$, for all x, y in A.

In triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$, operator ν (necessity) and μ (possibility) are modal operators, and u (uncertainty, $u \neq 0$, $u \neq 1$) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

Theorem 2.1. [**11**] *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

Definition 2.6. [**11**] A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the following conditions:

(T.1) $\nu x \leqslant x$,	(T.1′) $x \leqslant \mu x$,
(T.2) $\nu x \leqslant \nu\nu x$,	(T.2′) $\mu\mu x \leqslant \mu x$,
(T.3) $\nu(x \wedge y) = \nu x \wedge \nu y$,	(T.3′) $\mu(x \wedge y) = \mu x \wedge \mu y$,
(T.4) $\nu(x \vee y) = \nu x \vee \nu y$,	(T.4′) $\mu(x \vee y) = \mu x \vee \mu y$,
(T.5) $\nu u = 0$,	(T.5′) $\mu u = 1$,
(T.6) $\nu\mu x = \mu x$,	(T.6′) $\mu\nu x = \nu x$,
(T.7) $\nu(x \rightarrow y) \leqslant \nu x \rightarrow \nu y$,	
(T.8) $(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leqslant (x \leftrightarrow y)$,	
(T.9) $\nu x \rightarrow \nu y \leqslant \nu(\nu x \rightarrow \nu y)$.	

Definition 2.7. [**10**] Let $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An element x in A is called exact if $\nu x = x$. The set of exact elements of $\mathcal{A}$ is denoted by $E(\mathcal{A})$.

$E(\mathcal{A})$ is closed under all the defined operations on $\mathcal{A}$ [**11**]. We denote the subalgebra $(E(\mathcal{A}), \vee, \wedge, *, \rightarrow, 0, 1)$ by $\mathcal{E}(\mathcal{A})$ which is a residuated lattice.

THEOREM 2.2. [**10**] *In a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$, the implication $\rightarrow$ and the product $*$ are completely determined by their action on $E(\mathcal{A})$ and the value of $\mu(u * u)$. More specifically:*

- $\nu(x \rightarrow y) = (\nu x \rightarrow \nu y) \wedge (\mu x \rightarrow \mu y)$,
- $\mu(x \rightarrow y) = (\mu x \rightarrow (\mu(u * u) \rightarrow \mu y)) \wedge (\nu x \rightarrow \mu y)$,
- $\nu(x * y) = \nu x * \nu y$,
- $\mu(x * y) = (\nu x * \mu y) \vee (\mu x * \nu y) \vee (\mu x * \mu y * \mu(u * u))$.

DEFINITION 2.8. [**10**] Let $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An IVRL-filter (F) of $\mathcal{A}$ is a non-empty subset F of A satisfying:

(F.1) If $x \in F, y \in A$ and $x \leqslant y$, then $y \in F$,
(F.2) If $x, y \in F$, then $x * y \in F$,
(F.3) If $x \in F$, then $\nu x \in F$.

For all $x, y \in A$, we write $x \sim_F y$ if and only if $x \rightarrow y$ and $y \rightarrow x$ are both in F.

The relation $\sim_F$ is always a congruence [**10**]. Note that (F.3) is a necessary condition for this statement. Indeed, if $\sim_F$ is a congruence relation on a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ and $x \in F$, than $x \sim_F 1$ and therefore $\nu x \sim_F \nu 1 = 1$, which is equivalent with $\nu x \in F$.

PROPOSITION 2.1. [**10**] *Let $\mathcal{A}$ be a triangle algebra, $\mathcal{E}(A) = (E(A), \vee, \wedge, *, \rightarrow, 0, 1)$ be its subalgebra of exact elements and $F \subseteq A$. Then F is a filter of the triangle algebra $\mathcal{A}$ if and only if* (F.3$'$) *holds and $F \bigcap E(A)$ is a filter of the residuated lattice $\mathcal{E}(A)$.*

Proposition 2.1 suggests two different ways to define specific kinds of IVRL-filters of triangle algebras. The first is to impose a property on a filter of the subalgebra of exact elements and extend this filter to the whole triangle algebra, using (F.3$'$). We call these IVRL-extended filters. For example, an IVRL-extended prime filter of triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ is a subset F of A such that $F \cap E(\mathcal{A})$ is an prime filter of $\mathcal{E}(\mathcal{A})$ and $x \in F$ if and only if $\nu x \in F \cap \mathcal{E}(\mathcal{A})$.

The second way is to impose a property on the whole IVRL-filter. For example, a prime IVRL-filter of a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ is an IVRL-filter of A such that F is a prime filter of $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ [**10**].

3. Implicative filters in triangle algebras

From now on $\mathcal{A} = (A, \vee, \wedge, \rightarrow, *, \nu, \mu, 0, u, 1)$ or simply A is a triangle algebra unless otherwise specified.

Now, we can define two types of implicative filters in triangle algebras as follows:

DEFINITION 3.1. F is an IVRL-extended implicative filter (EIF) if for $x, y, z \in A$, $(\nu x \rightarrow (\nu y \rightarrow \nu z)), (\nu x \rightarrow \nu y) \in F$, implies $\nu x \rightarrow \nu z \in F$.

DEFINITION 3.2. F is an implicative IVRL-filter (IF) if for $x, y, z \in A$, $\nu(x \rightarrow (y \rightarrow z)), \nu(x \rightarrow y) \in F$, implies $\nu(x \rightarrow z) \in F$.

It is clear that every implicative IVRL-filter is an IVRL-extended implicative filter, but the converse is not true.

EXAMPLE 3.1. Let $A = \{0, u, 1\}$ be a chain. We define operations $\nu, \mu, *, \to$ as follows:

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$*$	0	u	1
0	0	0	0
u	0	0	u
1	0	u	1

$\to$	0	u	1
0	1	1	1
u	u	1	1
1	0	u	1

$\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a triangle algebra. It is clear that, $F = \{1\}$ is an IVRL-extended implicative filter of $\mathcal{A}$. Let $x = y = u$, $z = 0$. Then $\nu(x \to (y \to z)) = \nu(u \to (u \to 0)) = 1 \in F$, $\nu(x \to y) = \nu(u \to u) = 1 \in F$, but $\nu(x \to z) = \nu(u \to 0) = 0 \notin F$. Thus F is not an implicative IVRL-filter of $\mathcal{A}$.

THEOREM 3.1. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an IVRL-extended implicative filter of A,*
(ii) *For any $a \in A$, $F_a = \{x \in A : \nu a \to \nu x \in F\}$ is an IVRL-filter of A,*
(iii) *$\nu x \to (\nu x \to \nu y) \in F$ implies $\nu x \to \nu y \in F$, for all $x, y \in A$,*
(iv) *$\nu x \to (\nu y \to \nu z) \in F$ implies $(\nu x \to \nu y) \to (\nu x \to \nu z) \in F$, for all $x, y, z \in A$.*

PROOF. (i $\Rightarrow$ ii) For all $a \in A$, we have $\nu a \to \nu 1 \in F$, $1 \in F_a$. If $x, x \to y \in F_a$, then $\nu a \to \nu x, \nu a \to \nu(x \to y) \in F$. By (T.7), $\nu a \to \nu(x \to y) \leqslant \nu a \to (\nu x \to \nu y)$. Then $\nu a \to \nu y \in F$ and $y \in F_a$. Let $x \in F_a$. So $\nu a \to \nu x \in F$. Since $\nu\nu x = \nu x$, $\nu a \to \nu\nu x \in F$, $\nu x \in F_a$. Hence F_a is an IVRL-filter of A.

(ii $\Rightarrow$ iii) If $x, y \in A$ and $\nu x \to (\nu x \to \nu y) \in F$, then $\nu(\nu x \to (\nu x \to \nu y)) \in F$. By (T.7), we have $\nu\nu x \to \nu(\nu x \to \nu y) \in F$, so $\nu x \to \nu y \in F_x$. Since $\nu x, \nu x \to \nu y \in F_x$, $\nu y \in F_x$, that is, $\nu x \to \nu y \in F$.

(iii $\Rightarrow$ iv) Let $x, y, z \in A$ be such that $\nu x \to (\nu y \to \nu z) \in F$. We have $\nu x \to (\nu y \to \nu z) \leqslant \nu x \to ((\nu x \to \nu y) \to (\nu x \to \nu z))$ and so $\nu x \to (\nu x \to ((\nu x \to \nu y) \to \nu z) \in F$. By hypothesis, we deduce that $\nu x \to ((\nu x \to \nu y) \to \nu z) \in F$, hence $(\nu x \to \nu y) \to (\nu x \to \nu z) \in F$.

(iv $\Rightarrow$ i) Let $x, y, z \in A$ be such that $\nu x \to (\nu y \to \nu z), \nu x \to \nu y \in F$. We have $\nu x \to (\nu y \to \nu z) \leqslant (\nu x \to \nu y) \to (\nu x \to (\nu x \to \nu z))$. Then $\nu x \to (\nu x \to \nu z) \in F$ and so $\nu x \to \nu z \in F$, that is F is an IVRL-extended implicative filter. $\square$

Similarly we have:

THEOREM 3.2. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an implicative IVRL-filter of A,*
(ii) *For any $a \in A$, $F_a = \{\nu x \in A : a \to x \in F\}$ is an IVRL-filter of A,*
(iii) *$\nu(x \to (x \to y)) \in F$ implies $\nu(x \to y) \in F$, for all $x, y \in A$,*
(iv) *$\nu(x \to (y \to z)) \in F$ implies $\nu((x \to y) \to (x \to z)) \in F$, for all $x, y, z \in A$.*

LEMMA 3.1. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an IVRL-extended implicative filter of A,*
(ii) *$\nu x \to \nu x^2 \in F$, for all $x \in A$.*

PROOF. (i) $\Rightarrow$ (ii) We have $\nu x \to (\nu x \to \nu x^2) = \nu x^2 \to \nu x^2 = 1 \in F$, hence $\nu x \to \nu x^2 \in F$.
(ii) $\Rightarrow$ (i) Consider $x, y \in A$ such that $\nu x \to (\nu x \to \nu y) \in F$, hence $\nu x^2 \to \nu y \in F$. Since $\nu x \to \nu x^2 \in F$, then $\nu x \to \nu y \in F$. Thus F is an IVRL-extended implicative filter of A. $\square$

Similarly, we prove that F is an implicative IVRL-filter of A if and only if $\nu(x \to x^2) \in F$, for all $x \in A$.

THEOREM 3.3. *If F, G are IVRL-filters of A, $F \subseteq G$ and F is an IVRL-extended implicative filter(implicative IVRL-filter) of A, then so is G.*

PROOF. Let $x, y, z \in A$ be such that $a = \nu z \to (\nu x \to \nu y) \in G$. Since $\nu z \to (\nu x \to (a \to \nu y)) = a \to (\nu z \to (\nu x \to \nu y)) = a \to a = 1 \in F$, we conclude that $(\nu z \to \nu x) \to (\nu z \to (a \to \nu y)) \in F$ so $(\nu z \to \nu x) \to (\nu z \to (a \to \nu y)) \in G$. Also, $(\nu z \to \nu x) \to (a \to (\nu z \to \nu y)) = a \to ((\nu z \to \nu x) \to (\nu z \to \nu y)) \in G$. Hence $(\nu z \to \nu x) \to (\nu z \to \nu y) \in G$. Therefore, G is an IVRL-extended implicative filter of A. $\square$

DEFINITION 3.3. A triangle algebra A is called a BL-triangle algebra if it satisfies the following identities, for all $x, y \in A$:
$(x \to y) \vee (y \to x) = 1$ (prelinearity), $\quad x \wedge y = x * (x \to y)$ (divisibility).
A BL-triangle algebra A is called an MV-triangle algebra if and only if $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in A$.
A triangle algebra A is called a Gödel-triangle algebra (G-triangle algebra) if $x^2 = x$, for all $x \in A$.
We say that A is a semi-G-triangle algebra if $\neg(x^2) = \neg x$, for all $x \in A$; indeed, every G-triangle algebra is a semi-G-triangle algebra.

EXAMPLE 3.2. (a) In Example 3.1, clearly $\mathcal{A}$ is an MV-triangle algebra.
(b) Let $A = \{0, u, 1\}$ be a chain. We define operations $\nu, \mu, \odot, \Rightarrow$ as follows:

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$\odot$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\Rightarrow$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

Then $\mathcal{A} = (A, \vee, \wedge, \odot, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra and it is clear that, $\mathcal{A}$ is a G-triangle algebra.

We recall the following corollary form [**12**].

COROLLARY 3.1. *Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice and $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in L$. Then we have $(x \to y) \to y = x \vee y$.*

Proposition 3.1. *Let $\mathcal{A} = (A, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice. Then the following conditions are equivalent, for all $x, y \in A$:*

(i) $(x \to y) \to y = (y \to x) \to x$,
(ii) $((x \to y) \to y) \to x = y \to x$.

Proposition 3.2. *By Corollary 3.1, it is clear that if A is an MV-triangle algebra, then $x \vee y = (x \to y) \to y$. Also, A is an MV-triangle algebra if and only if $((x \to y) \to y) \to x = y \to x$, for all $x, y \in A$.*

Lemma 3.2. *The following conditions are equivalent:*

(i) *A is a G-triangle algebra,*
(ii) *Every IVRL-filter of A is an implicative IVRL-filter of A,*
(iii) *$\{1\}$ is an implicative IVRL-filter of A.*

Proof. (i) $\Rightarrow$ (ii) Let $x, y \in A$ be such that $\nu(x \to (x \to y)) \in F$. Since A is a G-triangle algebra, $\nu(x \to (x \to y)) = \nu(x^2 \to y) = \nu(x \to y)$. Hence $\nu(x \to y) \in F$, that is, F is an implicative IVRL-filter of A.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) Let $x \in A$. Since $\nu(x \to (x \to x^2)) \in \{1\}$ and $\{1\}$ is an implicative IVRL-filter of A then $\nu(x \to x^2) \in \{1\}$. Hence $x \leqslant x^2$ and so $x = x^2$, A is a G-triangle algebra. □

Corollary 3.2. *If A is a G-triangle algebra, then every IVRL-filter of A is an IVRL-extended implicative filter of A.*

In the following example, we show that the converse of above corollary is not true.

Example 3.3. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended implicative filters of A. But $u * u = 0 \neq u$, so it is not a G-triangle algebra.

Lemma 3.3. *Let F be an IVRL-filter of A. A/F is a G-triangle algebra if and only if F is an implicative IVRL-filter of A.*

Proof. Consider $x, y \in A$ such that $\nu(x \to (x \to y)) \in F$; then $x \to (x \to y) \in F$ so $x/F \to (x/F \to y/F) = 1/F$ and we obtain successively $(x/F)^2 \to y/F = 1/F$, $x/F \to y/F = 1/F$. Hence $x \to y \in F$ so $\nu(x \to y) \in F$. Thus F is an implicative IVRL-filter of A.

The converse is clear by Lemma 3.2 and Theorem 3.1 (iii). □

Remark 3.1. By the above lemma it is clear that if A/F is a G-triangle algebra, then F is an IVRL-extended implicative filter of A.

Example 3.4. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended implicative filter of A, but A/F is not a G-triangle algebra.

4. Positive implicative filters in triangle algebras

Definition 4.1. F is an IVRL-extended positive implicative filter (EPIF) if for $x, y \in A$, $(\nu x \to \nu y) \to \nu x \in F$, implies $\nu x \in F$.

Definition 4.2. F is a positive implicative IVRL-filter (PIF) if for $x, y \in A$, $\nu((x \to y) \to x) \in F$, implies $\nu x \in F$.

It is clear that every positive implicative IVRL-filter of A is an IVRL-extended positive implicative filter of A, but the converse is not true.

Example 4.1. In Example 3.2(b), it is clear that $F = \{1\}$ is an IVRL-extended positive implicative filter of A. Let $x = u, y = 0$. Then $\nu((u \Rightarrow 0) \Rightarrow u) = 1 \in F$, but $\nu u = 0 \notin F$. Thus F is not a positive implicative IVRL-filter of A.

Theorem 4.1. *Every IVRL-extended positive implicative filter (positive implicative IVRL-filter) of A is an IVRL-extended implicative filter (implicative IVRL-filter) of A.*

Proof. Let $x, y \in A$ be such that $\nu x \to (\nu x \to \nu y) \in F$. We must prove that $\nu x \to \nu y \in F$. We have $\nu x \to (\nu x \to \nu y) \leqslant ((\nu x \to \nu y) \to \nu y) \to (\nu x \to \nu y)$. Thus F is an IVRL-extended implicative filter of A. □

In the next two examples we show that the converse of the above theorem is not true.

Example 4.2. [**11**] Let $\mathcal{L}^I = (L^I, \vee, \wedge, *, \to, 0, 1)$, that we define $*, \to$ as follows:

$x * y = \min(x, y)$ and $x \to y = \left\{\begin{smallmatrix} 1 & x \leqslant y \\ y & y < x \end{smallmatrix}\right.$, is a residuated lattice and define

$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, x_2 * y_2]$,

$[x_1, x_2] \Rightarrow [y_1, y_2] = [(x_1 \to y_1) \wedge (x_2 \to y_2), x_2 \to y_2]$

The structure $\mathcal{L}^I = (L^I, \vee, \wedge, \odot, \Rightarrow, [0, 0], [1, 1])$ is a residuated lattice too. If we define

$$\nu[x_1, x_2] = [x_1, x_1], \mu[x_1, x_2] = [x_2, x_2], u = [0, 1]$$

then $(\mathrm{Int}(\mathcal{L}), \vee, \wedge, \odot, \Rightarrow, \nu, \mu, [0, 0], u, [1, 1])$ is a triangle algebra. Consider $F = \{[1, 1]\}$. It is clear that F is an IVRL-extended implicative filter. Let $x = [0.7, 0.8]$, $y = [0.6, 0.9]$. Then $\nu x = [0.7, 0.7]$, $\nu y = [0.6, 0.6]$. We have $(\nu x \Rightarrow \nu y) \Rightarrow \nu x \in F$. But $\nu x \notin F$, so F is not an IVRL-extended positive implicative filter.

Example 4.3. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended implicative filter. Let $x = u, y = 0$. So $\nu((x \to y) \to x) \in F$, but $\nu x = \nu u = 0 \notin F$. Thus F is not a positive implicative IVRL-filter.

Theorem 4.2. *Let F be an IVRL-filter of A. Consider the following assertions:*

(i) *F is an IVRL-extended positive implicative filter of A.*
(ii) *If $x \in A$ and $\neg \nu x \to \nu x \in F$, then $\nu x \in F$.*
(iii) *If $x, y \in A$ and $(\nu x \to \nu y) \to \nu y \in F$, then $(\nu y \to \nu x) \to \nu x \in F$.*

Then:

a) (i) $\Leftrightarrow$ (ii).
b) (i) $\Rightarrow$ (iii).
c) *If F is an IVRL-extended implicative filter of A, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

PROOF. a) It is clear that (i) $\Rightarrow$ (ii). Conversely, let $y, z \in A$ be such that $(\nu y \to \nu z) \to \nu y \in F$. We have $(\nu y \to \nu z) \to \nu y \leqslant \neg\nu y \to \nu y$, so $\neg\nu y \to \nu y \in F$ and $\nu y \in F$, that is, F is an IVRL-extended positive implicative filter of A.

b) Let $x, y \in A$ and denote $a = (\nu x \to \nu y) \to \nu y$, $b = (\nu y \to \nu x) \to \nu x$. Then we must show that

$$(*) \qquad a \leqslant (b \to \nu y) \to b$$

Since $\nu x \leqslant b$ we deduce that $(\nu x \to \nu y) \to b \leqslant (b \to \nu y) \to b$, wherefrom we have

$$\begin{aligned} a = (\nu x \to \nu y) \to \nu y &\leqslant (\nu y \to \nu x) \to ((\nu x \to \nu y) \to \nu x) \\ &= (\nu x \to \nu y) \to b \leqslant (b \to \nu y) \to b, \end{aligned}$$

that is, (*) is true. If $a \in F$, by (*) we deduce that $(b \to \nu y) \to b \in F$, so $b \in F$.

c) Let $x \in A$ be such that $\neg\nu x \to \nu x \in F$. We deduce that $\neg\nu x \to \nu x \leqslant \neg\nu x \to \neg\neg\nu x$. Since $\neg\nu x \to \neg\neg\nu x = \neg\nu x \to (\neg\nu x \to 0)$, F is an IVRL-extended implicative filter of A and by Theorem 3.1 (iii), $\neg\nu x \to 0 \in F$, that is $\neg\neg\nu x \in F$. But $\neg\neg\nu x = (\nu x \to 0) \to 0$ and by hypothesis $\nu x = (0 \to \nu x) \to \nu x \in F$. So (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i), hence, (i) $\Leftrightarrow$ (iii). $\square$

THEOREM 4.3. *If F, G are two IVRL-filters of A, $F \subseteq G$ and F is an IVRL-extended positive implicative filter(positive implicative IVRL-filter) of A, then G is an IVRL-extended positive implicative filter(positive implicative IVRL-filter) of A.*

PROOF. By Theorem 4.1, F is an IVRL-extended implicative filter of A. Since $F \subseteq G$, G is an IVRL-extended implicative filter of A. By Theorem 4.2 it is suffice to prove that if $x, y \in A$ and $a = (\nu x \to \nu y) \to \nu y \in G$, then $(\nu y \to \nu x) \to \nu x \in G$. We have $a \to ((\nu x \to \nu y) \to \nu y) \in F$. By Theorem 3.1(iv), $(a \to (\nu x \to \nu y)) \to (a \to \nu y) \in F$. Hence $(\nu x \to (a \to \nu y)) \to (a \to \nu y) \in F$. Then $((a \to \nu y) \to \nu x) \to \nu x \in F$. So $((a \to \nu y) \to \nu x) \to \nu x \in G$. Since

$$\begin{aligned} a &= (\nu x \to \nu y) \to \nu y \\ &\leqslant (((\nu x \to \nu y) \to \nu y) \to \nu y) \to \nu y = (a \to \nu y) \to \nu y \\ &\leqslant (\nu y \to \nu x) \to ((a \to \nu y) \to \nu x) \\ &\leqslant [((a \to \nu y) \to \nu x) \to \nu x] \to [(\nu y \to \nu x) \to \nu x] \end{aligned}$$

and $a \in G$, $[((a \to \nu y) \to \nu x) \to \nu x] \to [(\nu y \to \nu x) \to \nu x] \in G$. Since $((a \to \nu y) \to \nu x) \to \nu x \in G$, $(\nu y \to \nu x) \to \nu x \in G$. So G is an IVRL-extended positive implicative filter of A. $\square$

DEFINITION 4.3. A triangle algebra A is called Boolean triangle algebra if $x \vee \neg x = 1$, for all $x \in A$.

Clearly we have:

PROPOSITION 4.1. *The following assertions are equivalent, for all $x, y \in A$:*

(i) $(x \to y) \to x = x$,
(ii) $\neg x \to x = x$,
(iii) *A is a Boolean triangle algebra.*

Definition 4.4. For a nonempty subset $S \subseteq A$, the smallest IVRL-filter of A which contains S, i.e., $\bigcap\{F : S \subseteq F\}$, is said to be the IVRL-filter of A generated by S and will be denoted by $[S)$. If $S = \{a\}$, with $a \in A$, we denote by $[a)$ the IVRL-filter generated by $\{a\}$ ($[a)$ is called principal).

Proposition 4.2. *Let $S \subseteq A$, a nonempty subset of A, $a \in A$. Then $[S) = \{x \in A : s_1 * \cdots * s_n \leqslant \nu x$, for some $n \geqslant 1$ and $s_1, \ldots, s_n \in S\}$. In particular, $[a) = \{x \in A : a^n \leqslant \nu x$, for some $n \geqslant 1\}$.*

Proof. Let M$=\{x \in A : s_1 * \cdots * s_n \leqslant \nu x$, for some $n \geqslant 1$ and $s_1, \ldots, s_n \in S\}$. Then M is an IVRL-filter which contains the set S, hence $[S) \subseteq M$. Let F be an IVRL-filter such that $S \subseteq F$ and $x \in M$. Then there exist $s_1, s_2, \ldots, s_n \in S$ such that $s_1 * \cdots * s_n \leqslant \nu x$. Since $s_1, s_2, \ldots, s_n \in F$, $s_1 * \cdots * s_n \in F, \nu x \in F$, $x \in F$. Hence $M \subseteq F$, therefore $M \subseteq F = [S)$, that is, $[S) = M$. □

Lemma 4.1. *The following conditions are equivalent:*

(i) *$\{1\}$ is an IVRL-extended positive implicative filter of A,*
(ii) *For every $a \in A$, $[a)$ is an IVRL-extended positive implicative filter of A,*
(iii) *$(\nu x \to \nu y) \to \nu x = \nu x$, for all $x, y \in A$,*

Proof. (i $\Rightarrow$ ii) $[a)$ is an IVRL-filter so by Theorem 4.3 and (i), $[a)$ is an IVRL-extended positive implicative filter of A.

(ii $\Rightarrow$ iii) Let $a = (\nu x \to \nu y) \to \nu x$. Then $(\nu x \to \nu y) \to \nu x \in [a)$ and $[a)$ is an IVRL-extended positive implicative filter of A, $\nu x \in [a)$, that is $a^n \leqslant \nu x$. Hence $(\nu x \to \nu y) \to \nu x = \nu x$.

(iii $\Rightarrow$ i) It is clear by Theorem 4.2. □

Lemma 4.2. *The following conditions are equivalent:*

(i) *$\{1\}$ is a positive implicative IVRL-filter of A,*
(ii) *For every $a \in A$, $[a)$ is a positive implicative IVRL-filter of A,*
(iii) *$\nu((x \to y) \to x) = \nu x$, for all $x, y \in A$,*
(iv) *A is a Boolean-triangle algebra.*

Proposition 4.3. *Let F be an IVRL-filter of A. A/F is a Boolean triangle algebra if and only if F is a positive implicative IVRL-filter of A.*

Proof. Let $x, y \in A$ be such that, $\nu((x \to y) \to x) \in F$. So $(x \to y) \to x \in F$. Then $(x/F \to y/F) \to x/F = 1/F \in \{[1]\}$. Since A/F is a Boolean triangle algebra, thus $\{1\}$ is a positive implicative IVRL-filter of A. Then $x/F \in \{[1]\}$, hence $x/F = 1/F$, that is, $x \in F$ so $\nu x \in F$. So F is a positive implicative IVRL-filter of A. □

Corollary 4.1. *Let A/F be a Boolean triangle algebra. Then F is an IVRL-extended positive implicative filter of A.*

In the following example we show that the converse of the above corollary is not true.

Example 4.4. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended positive implicative filter of A. But since $\neg u \vee u = u$, $A/\{1\}$ is not a Boolean triangle algebra.

Definition 4.5. An IVRL-filter F of A will be called IVRL-extended MV-filter if $((\nu x \to \nu y) \to \nu y) \to ((\nu y \to \nu x) \to \nu x) \in F$, and will be called an MV-IVRL-filter if $\nu(((x \to y) \to y) \to ((y \to x) \to x)) \in F$, for all $x, y \in A$.

Corollary 4.2. *Let F be an IVRL-extended MV-filter (MV-IVRL-filter) of A. Then $\neg\neg\nu x \to \nu x \in F$ ($\nu(\neg\neg x \to x) \in F$), for all $x \in A$.*

Proof. Indeed, $[(\nu x \to \nu 0) \to \nu 0] \to [(\nu 0 \to \nu x) \to \nu x] = \neg\neg\nu x \to \nu x$, hence $\neg\neg\nu x \to \nu x \in F$, for all $x \in A$. □

Theorem 4.4. *F is an MV-IVRL-filter of A if and only if A/F is an MV-triangle algebra.*

Proof. By Proposition 3.2, A/F is an MV-triangle algebra if and only if $(x/F \to y/F) \to y/F = (y/F \to x/F) \to x/F$ if and only if $((x \to y) \to y) \to ((y \to x) \to x) \in F$ if and only if $\nu(((x \to y) \to y) \to ((y \to x) \to x)) \in F$ if and only if F is an MV-IVRL-filter of A. □

5. Fantastic filters in triangle algebras

Definition 5.1. F is an IVRL-extended fantastic filter (EFF) if for all $x, y \in A$, $(\nu x \to \nu y) \in F$, implies $((\nu y \to \nu x) \to \nu x) \to \nu y \in F$.

Definition 5.2. F is a fantastic IVRL-filter (FF) if for all $x, y \in A$, $\nu(x \to y) \in F$, implies $\nu((y \to x) \to x) \to y \in F$.

It is clear that every fantastic IVRL-filter of A is an IVRL-extended fantastic filter of A, but the converse is not true.

Example 5.1. In Example 3.2 (b), it is clear that $F = \{1\}$ is an IVRL-extended fantastic filter of A. Let $x = 0, y = u$. Then $\nu(0 \Rightarrow u) \in F$, but $\nu(((u \Rightarrow 0) \Rightarrow 0) \Rightarrow u) = \nu u = 0 \notin F$. Thus F is not an fantastic IVRL-filter of A.

Theorem 5.1. *If F, G are two filters of A, $F \subseteq G$ and F is an IVRL-extended fantastic filter (fantastic IVRL-filter) of A, then G is an IVRL-extended fantastic filter (fantastic IVRL-filter) of A.*

Proof. Consider $x, y \in A$ such that $\nu x \to \nu y \in G$. Clearly $\nu x \to ((\nu x \to \nu y) \to \nu y) = 1 \in F$. Since F is an IVRL-extended fantastic filter of A, $((((\nu x \to \nu y) \to \nu y) \to \nu x) \to \nu x) \to ((\nu x \to \nu y) \to \nu y) \in F$. Then $(\nu x \to \nu y) \to (((((\nu x \to \nu y) \to \nu y) \to \nu x) \to \nu x) \to \nu y) \in F \subseteq G$. Since $\nu x \to \nu y \in G$ and G is an IVRL-filter of A, $((((\nu x \to \nu y) \to \nu y) \to \nu x) \to \nu x) \to \nu y \in G$. Since $\nu y \leqslant (\nu x \to \nu y) \to \nu y$, $((((\nu x \to \nu y) \to \nu y) \to \nu x) \to \nu x) \to \nu y \leqslant ((\nu y \to \nu x) \to \nu x) \to \nu y$, $((\nu y \to \nu x) \to \nu x) \to \nu y \in G$. Thus G is an IVRL-extended fantastic filter of A. □

Theorem 5.2. *Every IVRL-extended positive implicative filter (positive implicative IVRL-filter) of A is an IVRL-extended fantastic filter (fantastic IVRL-filter) of A.*

Proof. Let $x, y \in A$ be such that $\nu x \to \nu y \in F$. Since $\nu y \leqslant ((\nu y \to \nu x) \to \nu x) \to \nu y$, then

$$(**) \qquad ((\nu y \to \nu x) \to \nu x) \to \nu y) \to \nu x \leqslant \nu y \to \nu x$$

We have

$$\nu x \to \nu y \leqslant [(\nu y \to \nu x) \to \nu x] \to [(\nu y \to \nu x) \to \nu y],$$
$$\nu x \to \nu y \leqslant (\nu y \to \nu x) \to [((\nu y \to \nu x) \to \nu x) \to \nu y].$$

By (**) we have

$$\nu x \to \nu y \leqslant ((((\nu y \to \nu x) \to \nu x) \to \nu y) \to \nu x) \to (((\nu y \to \nu x) \to \nu x) \to \nu y).$$

Then $((((\nu y \to \nu x) \to \nu x) \to \nu y) \to \nu x) \to (((\nu y \to \nu x) \to \nu x) \to \nu y) \in F$. Since F is an IVRL-extended positive implicative filter of A, $((\nu y \to \nu x) \to \nu x) \to \nu y \in F$. Hence F is an IVRL-extended fantastic filter of A. $\square$

The following example shows that the converse of the above theorem is not true.

Example 5.2. $\mathcal{L}^I = (L^I, \vee, \wedge, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra if:
for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in L^I,

$$\mathcal{T}_{T,\alpha}(x, y) = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))],$$

induces a residuated lattice on $\mathcal{L}^I$, with the residual implicator

$$\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x, y) = [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, \alpha), y_2), I_T(x_1, y_2))].$$

Also $\nu x = [x_1, x_1]$ and $\mu x = [x_2, x_2]$, for all $\alpha \in I$ and $x = [x_1, x_2] \in L^I$ [**11**]. Let $T(x, y) = \min(x, y)$ and $I_T(x, y) = \left\{\begin{smallmatrix} 1 & x \leqslant y \\ y & y < x \end{smallmatrix}\right.$, $\alpha = 1$, $F = \{[1, 1]\}$. Clearly F is an IVRL-extended fantastic filter.

For $x = [0.8, 0.9] \in L^I$, we have $\neg\nu x = \mathcal{I}_{\mathcal{T}_{T,\alpha}}(\nu x, 0) = [0, 0]$, so $\mathcal{I}_{\mathcal{T}_{T,\alpha}}(\neg\nu x, \nu x) = [1, 1] \in F$, but $\nu x = [0.8, 0.8] \notin F$. Thus F is not an IVRL-extended positive implicative filter and so it is not a positive implicative IVRL-filter.

Example 5.3, shows that there is an implicative IVRL-filter which is not an IVRL-extended fantastic filter.

Example 5.3. In Example 4.2, $F = \{[1, 1]\}$ is an implicative IVRL-filter. Let $x = [0.4, 0.5], y = [0.7, 0.8]$. Then $\nu x = [0.4, 0.4], \nu y = [0.7, 0.7]$, we have $\nu x \Rightarrow \nu y \in F$, but $((\nu y \Rightarrow \nu x) \Rightarrow \nu x) \Rightarrow \nu y = [0.7, 0.7] \notin F$. Thus F is not an IVRL-extended fantastic filter.

Corollary 5.1. *$\{1\}$ is an IVRL-extended fantastic filter of A if and only if every IVRL-filter of A is an IVRL-extended fantastic filter.*

Proposition 5.1. *The following conditions are equivalent:*

(i) *$\{1\}$ is a fantastic IVRL-filter of A,*

(ii) $\nu(((x \to y) \to y) \to x) = \nu(y \to x)$, *for all* $x, y \in A$.

PROOF. Let $x, y \in A$. We have $x \to ((x \to y) \to y) \in \{1\}$. Since F is an IVRL-extended fantastic filter of A, $\nu(((((x \to y) \to y) \to x) \to x) \to ((x \to y) \to y)) = 1$. Thus $(((x \to y) \to y) \to x) \to x = (x \to y) \to y$. We have $(y \to x) \to x \leqslant (((x \to y) \to y) \to x) \to x = (x \to y) \to y$. So $(y \to x) \to x \leqslant (x \to y) \to y$ and similarly $(x \to y) \to y \leqslant (y \to x) \to x$. By Proposition 3.2, $\nu(((x \to y) \to y) \to x) = \nu(y \to x)$. $\square$

It is clear that if A is an MV-triangle algebra, then $\nu((x \to y) \to y) \to x) = \nu(y \to x)$, for all $x, y \in A$. Thus every IVRL-filter of MV-triangle algebra A, is a fantastic IVRL-filter and so, is an IVRL-extended fantastic filter of A.

THEOREM 5.3. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is a fantastic IVRL-filter of A,*

(ii) *A/F is an MV-triangle algebra.*

PROOF. (i $\Rightarrow$ ii) It is suffice to prove that $\{1\}$ is a fantastic IVRL-filter of A. Consider $x, y \in A$ such that, $x/F \to y/F = 1/F$. Then $x \to y \in F$, so $\nu(x \to y) \in F$. Since F is a fantastic IVRL-filter of A, then $\nu(((y \to x) \to x) \to y) \in F$. So $((y \to x) \to x) \to y \in F$, hence $((y/F \to x/F) \to x/F) \to y/F) = 1/F$. Therefore A/F is an MV-triangle algebra.

(ii $\Rightarrow$ i) It is clear by Proposition 3.2. $\square$

COROLLARY 5.2. *Let A/F be an MV-triangle algebra. Then F is an IVRL-extended fantastic filter of A.*

In the following example we show that the converse of the above corollary is not true.

EXAMPLE 5.4. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended fantastic filter of A. Since $\neg\neg u = 1 \neq u$, A/F is not an MV-triangle algebra.

6. Easy filters in triangle algebras

DEFINITION 6.1. F is an IVRL-extended easy filter (EEF) if for $x, y, z \in A$, $\neg\neg\nu x \to (\nu y \to \nu z), \neg\neg\nu x \to \nu y \in F$, implies $\neg\neg\nu x \to \nu z \in F$.

DEFINITION 6.2. F is an easy IVRL-filter (EF) if for $x, y, z \in A$, $\nu(\neg\neg x \to (y \to z)), \nu(\neg\neg x \to y) \in F$, implies $\nu(\neg\neg x \to z) \in F$.

It is clear that every easy IVRL-filter is an IVRL-extended easy filter, but the converse is not true.

EXAMPLE 6.1. In Example 3.1, it is clear that $F = \{1\}$ is an IVRL-extended easy filter of A. Let $x = y = u, z = 0$. Then $\nu(\neg\neg u \to (u \to 0)) \in F$, $\nu(\neg\neg u \to u) \in F$, but $\nu(\neg\neg u \to 0) \notin F$. So F is not an easy IVRL-filter of A.

PROPOSITION 6.1. *Every IVRL-extended implicative filter (implicative IVRL-filter) of A is an IVRL-extended easy filter (easy IVRL-filter) of A.*

PROOF. Let $x, y, z \in A$ be such that $\neg\neg\nu x \to (\nu y \to \nu z), \neg\neg\nu x \to \nu y \in F$. By Theorem 3.1 (iv), we conclude that $(\neg\neg\nu x \to \nu y) \to (\neg\neg\nu x \to \nu z) \in F$. Since $\neg\neg\nu x \to \nu y \in F$, $\neg\neg\nu x \to \nu z \in F$. Thus F is an IVRL-extended easy filter of A. $\square$

The following example shows that every easy IVRL-filter is not an IVRL-extended implicative filter

EXAMPLE 6.2. In Example 4.2, define $x * y = x \cdot y$, $x \to y = \left\{ \begin{smallmatrix} 1 & x \leqslant y \\ y/x & y<x \end{smallmatrix} \right.$. Then $F = \{[1, 1]\}$ is an IVRL-filter on $\mathcal{L}^I$. It is clear that F is an easy IVRL-filter and so is an IVRL-extended easy filter. Let $x = [0.5, 0.7]$, so $\nu x = [0.5, 0.5]$. We have $(\nu x)^2 = [0.5, 0.5] \odot [0.5, 0.5]$, $\nu x \Rightarrow (\nu x)^2 = [0.25/0.5, 0.25/0.5] \notin F$. Thus F is not an IVRL-extended implicative filter.

Example 6.3, shows that there is an implicative IVRL-filter which is not an IVRL-extended fantastic filter.

EXAMPLE 6.3. In Example 4.2, define

$$x * y = \max(0, x + y - 1), \quad x \to y = \min(1, 1 - x + y)$$

Then $F = \{[1, 1]\}$ is an IVRL-filter. Let $x = [0.6, 0.8]$, so $\nu x = [0.6, 0.6]$ and $\neg\nu x = [0.6, 0.6] \Rightarrow [0, 0] = [0.4, 0.4]$. So $\neg\neg\nu x = [0.6, 0.6]$ and $(\neg\neg\nu x)^2 = [0.2, 0.2]$. $\neg\neg\nu x \Rightarrow (\neg\neg\nu x)^2 = [0.6, 0.6] \notin F$, thus F is not an IVRL-extended easy filter. Clearly F is an fantastic IVRL-filter.

PROPOSITION 6.2. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an IVRL-extended easy filter of A,*
(ii) *If $\neg\neg\nu x \to (\nu y \to \nu z) \in F$, then $(\neg\neg\nu x \to \nu y) \to (\neg\neg\nu x \to \nu z) \in F$, for all $x, y, z \in A$,*
(iii) *If $\neg\neg\nu x \to (\neg\neg\nu x \to \nu y) \in F$, then $\neg\neg\nu x \to \nu y \in F$, for all $x, y \in A$.*

PROOF. (i) $\Rightarrow$ (ii) Let $x, y, z \in A$ be such that $\neg\neg\nu x \to (\nu y \to \nu z) \in F$. Then

$$\begin{aligned} \neg\neg\nu x \to [\neg\neg\nu x \to ((\neg\neg\nu x \to \nu y) \to \nu z)] &= \neg\neg\nu x \to [(\neg\neg\nu x \to \nu y) \to (\neg\neg\nu x \to \nu z)] \\ &\geqslant \neg\neg\nu x \to (\neg\neg\nu x \to (\nu y \to \nu z)) \\ &\geqslant \neg\neg\nu x \to (\nu y \to \nu z) \end{aligned}$$

Since $\neg\neg\nu x \to \neg\neg\nu x \in F$, we have $\neg\neg\nu x \to [(\neg\neg\nu x \to \nu y) \to \nu z] \in F$. We also have $\neg\neg\nu x \to [(\neg\neg\nu x \to \nu y) \to \nu z] = (\neg\neg\nu x \to \nu y) \to (\neg\neg\nu x \to \nu z)$, and then $(\neg\neg\nu x \to \nu y) \to (\neg\neg\nu x \to \nu z) \in F$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Let $\neg\neg\nu x \to (\nu y \to \nu z), (\neg\neg\nu x \to \nu y) \in F$. Since

$$\begin{aligned} \neg\neg\nu x \to (\nu y \to \nu z) &= \nu y \to (\neg\neg\nu x \to \nu z) \\ &\leqslant (\neg\neg\nu x \to \nu y) \to [\neg\neg\nu x \to (\neg\neg\nu x \to \nu z)], \end{aligned}$$

we have $\neg\neg\nu x \to (\neg\neg\nu x \to \nu z) \in F$. By hypothesis, $\neg\neg\nu x \to \nu z \in F$. Thus F is an IVRL-extended easy filter of A. $\square$

PROPOSITION 6.3. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an easy IVRL-filter of A,*
(ii) *If $\nu(\neg\neg x \to (y \to z)) \in F$, then $\nu((\neg\neg x \to y) \to (\neg\neg x \to z)) \in F$, for all $x, y, z \in A$,*
(iii) *If $\nu(\neg\neg x \to (\neg\neg x \to y)) \in F$, then $\nu(\neg\neg x \to y) \in F$, for all $x, y \in A$.*

THEOREM 6.1. *If F, G are IVRL-filters of A, $F \subseteq G$ and F is an IVRL-extended easy filter (easy IVRL-filter) of A, then G is an IVRL-extended easy filter (easy IVRL-filter) of A.*

PROOF. Let $a = \neg\neg\nu x \to (\neg\neg\nu x \to \nu y) \in G$. Since

$$\begin{aligned}\neg\neg\nu x \to (\neg\neg\nu x \to (a \to \nu y)) &= \neg\neg\nu x \to (a \to (\neg\neg\nu x \to \nu y)) \\ &= a \to (\neg\neg\nu x \to (\neg\neg\nu x \to \nu y) \in F,\end{aligned}$$

we have $\neg\neg\nu x \to (\neg\neg\nu x \to (a \to \nu y)) \in F$. By Proposition 6.2(iii), we have $\neg\neg\nu x \to (a \to \nu y) \in F \subseteq G$. Then $\neg\neg\nu x \to \nu y \in G$. Thus G is an IVRL-extended easy filter of A. $\square$

REMARK 6.1. By Theorem 6.1, $\{1\}$ is an IVRL-extended easy filter(easy IVRL-filter) of A if and only if every IVRL-filter of A is an IVRL-extended easy filter (easy IVRL-filter).

THEOREM 6.2. *Let F be an IVRL-filter of A. Then F is an IVRL-extended easy filter if and only if $\neg\neg\nu x \to (\neg\neg\nu x)^2 \in F$, for all $x \in A$.*

PROOF. Let F be an IVRL-extended easy filter of A. Then for $x \in A$,

$$\neg\neg\nu x \to (\neg\neg\nu x \to (\neg\neg\nu x)^2) = (\neg\neg\nu x)^2 \to (\neg\neg\nu x)^2 = 1 \in F.$$

By Proposition 6.2(iii), $\neg\neg\nu x \to (\neg\neg\nu x)^2 \in F$. Conversely, let $x, y \in A$ be such that $\neg\neg\nu x \to (\neg\neg\nu x \to \nu y) \in F$. Then $(\neg\neg\nu x)^2 \to \nu y \in F$. Since $\neg\neg\nu x \to (\neg\neg\nu x)^2 \in F$, we have $\neg\neg\nu x \to \nu y \in F$. Then F is an IVRL-extended easy filter of A. $\square$

THEOREM 6.3. *Let F be an IVRL-filter of A. F is an easy IVRL-filter if and only if $\nu(\neg\neg x \to (\neg\neg x)^2) \in F$, for all $x \in A$.*

PROPOSITION 6.4. *Let A be a triangle algebra and $\neg\neg x = (\neg\neg x)^2$. Then $\neg(x^2) = \neg x$, for all $x \in A$.*

PROOF. By Lemma 2.1 (10), we have $(\neg\neg x)^2 \leqslant \neg\neg(x^2)$. By hypothesis we have $\neg\neg x \leqslant \neg\neg(x^2)$, $\neg(x^2) \leqslant \neg x$. Since $\neg x \leqslant \neg(x^2)$, we have $\neg(x^2) = \neg x$. $\square$

REMARK 6.2. Let A be a BL-triangle algebra. Then the converse of Proposition 6.4 holds. Indeed, if $\neg(x^2) = \neg x$, then $\neg\neg(x * x) = \neg\neg x$, so $\neg\neg x * \neg\neg x = \neg\neg x$, for all $x, y \in A$.

COROLLARY 6.1. *Let F be an IVRL-extended easy filter (easy IVRL-filter) of A. Then A/F is a semi-G-triangle algebra.*

PROOF. If $x \in A$, then $\neg\neg(x/F) \leqslant (\neg\neg(x/F))^2 \leqslant \neg\neg(x/F)$. Thus $\neg\neg(x/F) = (\neg\neg(x/F))^2$, so by Proposition 6.4, $\neg[(x/F)^2] = \neg(x/F)$, that is, A/F is a semi-G-triangle algebra. □

By Theorem 6.2, Remark 6.2 and Corollary 6.1, we have:

COROLLARY 6.2. *Let F be a BL-triangle algebra. Then F is an IVRL-extended easy filter of A if and only if A/F is a semi-G-triangle algebra.*

THEOREM 6.4. *F is an IVRL-extended easy filter and IVRL-extended MV-filter (easy IVRL-filter and MV-IVRL-filter) of A if and only if F is an IVRL-extended implicative filter and IVRL-extended MV-filter (implicative IVRL-filter and MV-IVRL-filter) of A.*

PROOF. If F is an IVRL-extended easy filter and IVRL-extended MV-filter, then F is an IVRL-extended easy filter and IVRL-extended MV-filter. Now, let $x, y \in A$ be such that $\nu x \to (\nu x \to \nu y) \in F$. We have to prove that $\nu x \to \nu y \in F$. Since F is an IVRL-extended MV-filter of A, we have $\neg\neg\nu x \to \nu x \in F$. From $\nu x \to (\nu x \to \nu y)$, $\neg\neg\nu x \to \nu x \in F$ and Lemma 2.1, we get $\neg\neg\nu x \to (\nu x \to \nu y) \in F$. Since $\neg\neg\nu x \to \nu x \in F$ and F is an IVRL-extended easy filter of A, we have $\neg\neg\nu x \to \nu y \in F$. Since $\neg\neg\nu x \to \nu y \leqslant \nu x \to \nu y$, we have $\nu x \to \nu y \in F$, hence F is an IVRL-extended implicative filter of A. □

PROPOSITION 6.5. *Let F be an IVRL-extended easy filter of A and $x \in A$ be such that $\neg[(\neg\neg\nu x)^n] \in F$, for some $n \geqslant 1$. Then $\neg\nu x \in F$.*

PROOF. We have $\neg[(\neg\neg\nu x)^n] = \underbrace{\neg\neg\nu x \to (\neg\neg\nu x \to \cdots \to (\neg\neg\nu x}_{n-times} \to 0)\dots)$ therefore $\neg\neg\nu x \to 0 = \neg\neg\neg\nu x = \neg\nu x \in F$. □

7. Conclusion and future work

The notions of triangle algebra and interval valued residuated lattices are defined by Van Gass et al. [**11**]. They showed that the definitions of the different kinds of filters of residuated lattices can be extended to triangle algebras in two different ways. They examined the relationships between the obtained concepts [**10**].

In this paper, we developed filter theory in triangle algebras. Mainly, we introduced different kinds of filters in triangle algebras, such as implicative, positive implicative, fantastic, easy IVRL-filters and IVRL-extended implicative, positive implicative, fantastic and easy filters. We have given some characterizations and several examples. Figure 1 gives a schematic summary of relations among all IVRL-filters that we considered. For example the diagram below shows that every positive implicative IVRL-filter (PIF) is an implicative IVRL-filter (IF) but the converse is not true.

The investigation of other such generalizations can be an interesting object for further work.

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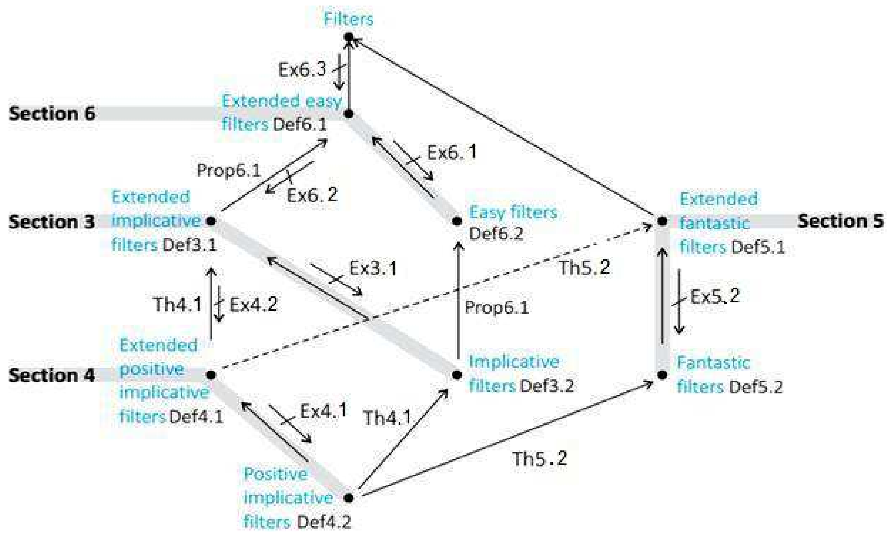

FIGURE 1. The relationship between filters in triangle algebra

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