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COMPARISON OF ITERATES OF A CLASS OF DIFFERENTIAL OPERATORS IN ROUMIEU SPACES

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Abstract. Considering a class of differential operators with constant coefficients including the hypoelliptic operators, we show that the comparison of the operators implies the inclusion between their spaces of Roumieu vectors.

1. Introduction

The iterate property of linear differential operators has been generalized to the comparison in the sense of inclusion between the spaces of Gevrey vectors of those operators for the first time by Newberger and Zielezny [**17**], they have given necessary and sufficient conditions to get this comparison considering hypoelliptic differential operators with constant coefficients. This result has been generalized and extended by Bouzar–Chaili [**4**], to a class of systems of differential operators with constant coefficients including the class of hypoelliptic differential operators. In [**11**] Juan–Huguet has extended the theorem of Newberger–Zielezny in the spaces of ultradifferentiable functions of type Roumieu.

The aim of this work is to refine these results considering a class of differential operators as in [**4**] and Roumieu spaces of type M_p. For more details on the results concerning the iterate problem see [**1**, **3**, **5**–**7**, **10**, **12**, **14**, **16**, **20**–**22**].

Let (M_p) be a sequence of positive real numbers satisfying the following conditions logarithmic convexity:

$$M_p^2 \leqslant M_{p-1}M_{p+1}, \quad \forall p \in \mathbb{N}^*, \tag{1.1}$$

non-quasi-analyticity:

$$\sum_{p=0}^{\infty} \frac{M_{p-1}}{M_p} < \infty, \tag{1.2}$$

stability under derivation and multiplication:

$$\exists A > 0, \exists H > 0, \exists c > 0 : cC_p^j M_{p-j}M_j \leqslant M_p \leqslant AH^p M_{p-j}M_j, \ \forall p \in \mathbb{N}, \ j \leqslant p \tag{1.3}$$

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Example 1.1. The sequence $M_p = p!^s$, $s \geqslant 1$, called Gevrey sequence of order s, satisfies conditions (1.1)–(1.3).

Let Ω be an open subset of $\mathbb{R}^n$, P a linear differential operator with constant coefficients of order m.

Definition 1.1. We call Roumieu vector (or vector of type M_p) of the operator P in Ω, any function $u \in C^\infty(\Omega)$ such that

$$\forall H \text{ compact of } \Omega, \exists C > 0, \forall l \in \mathbb{Z}_+ : \|P^l u\|_{L^2(H)} \leqslant C^{l+1} M_{lm}$$

The space of Roumieu vectors of P in Ω is denoted $R_M(\Omega, P)$.

Definition 1.2. We call Roumieu space in Ω, and we denote $R_M(\Omega)$, the space of functions $u \in C^\infty(\Omega)$ such that

$$\forall H \text{ compact } \Omega, \exists C > 0, \forall \alpha \in \mathbb{Z}_+^n : \|D^\alpha u\|_{L^2(H)} \leqslant C^{|\alpha|+1} M_{|\alpha|}$$

Example 1.2. If $M_p = p!^s$, $s \geqslant 1$, then $R_M(\Omega)$ is the Gevrey space of order s in Ω, and it is denoted $G^s(\Omega)$.

Similarly $R_M(\Omega, P)$ is denoted $G^s(\Omega, P)$.

Definition 1.3. We denote $\mathcal{H}$ the set of linear differential operators with constant coefficients P satisfying the following condition

$$\exists C > 0, \exists \gamma \geqslant \deg P, \forall \alpha \in \mathbb{Z}_+^n, \forall \xi \in \mathbb{R}^n : |P^{(\alpha)}(\xi)| \leqslant C(1 + |P(\xi)|)^{1-\frac{|\alpha|}{\gamma}}, \tag{1.4}$$

where $P^{(\alpha)}(\xi) = \partial_\xi^\alpha P(\xi)$.

Example 1.3. If P is an hypoelliptic operator, it satisfies condition (1.4), and so $P \in \mathcal{H}$ (see [**8**]). In particular, if P is elliptic, then $P \in \mathcal{H}$ and (1.4) is fulfilled for $\gamma = \deg P$.

2. Basic estimates

In this section we recall the notion of comparison of differential operators, see e.g. [**9**, **19**] and we give the basic estimate which is essential for the main result of this work.

Definition 2.1. Let P and Q be two linear differential operators with constant coefficients on $\mathbb{R}^n$, we say that P is weaker than Q and we denote $P \prec Q$ if for any relatively compact open subset Ω of $\mathbb{R}^n$, there exists a constant $C = C(P, Q, \Omega) > 0$, such that

$$\|Pv\|_{L^2(\Omega)} \leqslant C\|Qv\|_{L^2(\Omega)}, \quad \forall v \in \mathbb{C}_0^\infty(\Omega).$$

The operators P and Q are said to be equally strong if $P \prec Q \prec P$.

Remark 2.1. If P and Q are equally strong and $P \in \mathcal{H}$, then $Q \in \mathcal{H}$, further more if P satisfies condition (1.4) for some γ, then Q satisfies also condition (1.4) for the same constant γ, see [**8**].

For any open subset ω of $\mathbb{R}^n$ and $\delta > 0$ we set $\omega_\delta = \{x \in \omega,\ d(x, C\omega) > \delta\}$. If $f \in L^2_{loc}(\omega)$, $\mu > 0$ and $t > 0$, we define

$$N_{\omega,\mu,t}(f) = \sup_{0<\delta\leqslant t} \delta^\mu \|f\|_{L^2(\omega_\delta)}.$$

Without loss of generality, we suppose that ω is a bounded open subset of diameter < 1, and for simplify we denote $N_{\omega,\mu,t}(f) = N_\mu(f)$.

We need the following proposition given in Hörmander [**8**].

Proposition 2.1. *Let ω be a bounded open subset of $\mathbb{R}^n$ and let $P \in \mathcal{H}$; then there exists $C > 0$ such that*

$$\sum_\alpha N_{\gamma-|\alpha|}(P^{(\alpha)}u) \leqslant C(N_\gamma(Pu) + \|u\|_{L^2(\omega)}), \quad \forall u \in \mathbb{C}^\infty(\omega). \tag{2.1}$$

The following result is analogous to that of [**8**, Proposition 4.2].

Proposition 2.2. *Let ω be a bounded open subset of $\mathbb{R}^n$ and let Q , $P \in \mathcal{H}$. If $Q \prec P$, then there exists $C > 0$ such that*

$$N_\gamma(Qu) \leqslant C(N_\gamma(Pu) + \|u\|_{L^2(\omega)}), \quad \forall u \in \mathbb{C}^\infty(\omega),$$

where γ is the constant for which P satisfies condition (1.4).

Proof. Suppose that $Q \prec P$, so there exists $C > 0$ such that

$$\|Qv\|_{L^2(\omega)} \leqslant C\|Pv\|_{L^2(\omega)}, \quad \forall v \in \mathbb{C}_0^\infty(\omega) \tag{2.2}$$

Let $\varphi \in \mathbb{C}_0^\infty(\omega_{\delta/2})$ such that $\varphi(x) = 1$ in ω_δ, $0 \leqslant \varphi(x) \leqslant 1$ and

$$|D^\alpha \varphi(x)| \leqslant C_\alpha (\delta/2)^{-|\alpha|},$$

where C_α depends only on n and α. From (2.2), we have for every $u \in \mathbb{C}^\infty(\omega)$,

$$\|Qu\|_{L^2(\omega_\delta)} \leqslant \|Q(\varphi u)\|_{L^2(\omega_{\frac{\delta}{2}})} \leqslant C\|P(\varphi u)\|_{L^2(\omega_{\frac{\delta}{2}})}$$

By the Leibniz formula $P(\varphi u) = \sum_\alpha \frac{1}{\alpha!} D^\alpha \varphi P^{(\alpha)} u$, we obtain

$$\begin{aligned}\|Qu\|_{L^2(\omega_\delta)} &\leqslant C\sum_\alpha \frac{C_\alpha}{\alpha!}\Big(\frac{\delta}{2}\Big)^{-|\alpha|}\|P^{(\alpha)}u\|_{L^2(\omega_{\frac{\delta}{2}})} \\ &\leqslant \Big(\frac{\delta}{2}\Big)^{-\gamma}\sum_\alpha C'_\alpha N_{\gamma-|\alpha|}(P^{(\alpha)}u) \leqslant \delta^{-\gamma}\sum_\alpha C''_\alpha N_{\gamma-|\alpha|}(P^{(\alpha)}u)\end{aligned}$$

Multiplying both sides of this inequality by δ^γ, we get

$$N_\gamma(Qu) \leqslant \sum_\alpha C''_\alpha N_{\gamma-|\alpha|}(P^{(\alpha)}u) \leqslant \Big(\max_{|\alpha|\leqslant \deg P} C''_\alpha\Big)\sum_\alpha N_{\gamma-|\alpha|}(P^{(\alpha)}u),$$

which gives with (2.1)

$$N_\gamma Q(u) \leqslant \widetilde{C}(N_\gamma(Pu) + \|u\|_{L^2(\omega)}),$$

with another constant $\widetilde{C} > 0$. □

Definition 2.2. We denote $M_p \subset N_p$ if

$$\exists L > 0, \exists C > 0 : M_p \leqslant CL^p N_p, \quad \forall p \in \mathbb{N}$$

EXAMPLE 2.1. We have $p! \subset M_p$, for all sequences M_p satisfying conditions (1.1)–(1.3). In fact we have a more stronger estimate, see [**13**]

$$\forall L > 0, \exists C > 0 : p! \leqslant CL^p M_p, \quad \forall p \in \mathbb{N}$$

3. Comparison of Roumieu vectors

The main result of this paper is the following theorem.

THEOREM 3.1. *Let* (M_p) *be a sequence satisfying conditions* (1.1)–(1.3)*, and let* $Q, P \in \mathcal{H}$ *such that* $Q \prec P$*. If in addition the sequence* (M_p) *satisfies*

$$(p!)^d \subset M_p \text{ where } d = \frac{\gamma(P)}{\deg(P)}, \tag{3.1}$$

then $R_M(\Omega, P) \subset R_M(\Omega, Q)$.

PROOF. Let ω be a bounded open subset. From Proposition 2.2 we have $\exists C(Q, P, \text{diam}\Omega) > 0$, $\forall t > 0$, $\forall \rho \geqslant 0$, $\forall \upsilon \in C^\infty(\omega_\rho)$,

$$\sup_{0 \leqslant \tau \leqslant t} \tau^\gamma \|(Q\upsilon)\|_{L^2(\omega_{\rho+\tau})} \leqslant C\Big(\sup_{0 \leqslant \tau \leqslant t} \tau^\gamma \|P\upsilon\|_{L^2(\omega_{\rho+\tau})} + \|\upsilon\|_{L^2(\omega_\rho)}\Big),$$

hence $t^\gamma \|(Q\upsilon)\|_{L^2(\omega_{\rho+t})} \leqslant C\big(t^\gamma \|P\upsilon\|_{L^2(\omega_\rho)} + \|\upsilon\|_{L^2(\omega_\rho)}\big)$, and so

$$\|Q\upsilon\|_{L^2(\omega_{\rho+t})} \leqslant C\big(\|P\upsilon\|_{L^2(\omega_\rho)} + t^{-\gamma}\|\upsilon\|_{L^2(\omega_\rho)}\big), \quad \upsilon \in \mathbb{C}^\infty(\omega_\rho). \tag{3.2}$$

Let us show by recurrence that $\exists C > 0$, $\forall k \geqslant 0$, $\forall \delta > 0$,

$$\|Q^k u\|_{L^2(\omega_\delta)} \leqslant C^k \sum_{i=0}^{k} \tbinom{k}{i} \Big(\frac{k}{\delta}\Big)^{i\gamma} \|P^{(k-i)} u\|_{L^2(\omega)}, \quad \forall u \in \mathbb{C}^\infty(\omega) \tag{3.3}$$

For $k = 1$, (3.3) is fulfilled from (3.2), it suffices to take $\rho = 0$ and $t = \delta$. Suppose that estimate (3.3) is true until the order k and let us prove it at the order $k + 1$. Replacing in (3.2) t with $\frac{\delta}{k+1}$, ρ with $\frac{k\delta}{k+1}$ and υ with $Q^k(D)u$, then we obtain

$$\begin{aligned}
\|Q^{(k+1)} u\|_{L^2(\omega_\delta)} &\leqslant C\big(\|P(Q^k u)\|_{L^2(\omega_\rho)} + t^{-\gamma}\|Q^k u\|_{L^2(\omega_\rho)}\big) \\
&\leqslant C^{k+1} \sum_{i=0}^{k} \tbinom{k}{i} \Big(\frac{k+1}{\delta}\Big)^{i\gamma} \|P^{(k-i)} P u\|_{L^2(\omega)} \\
&\quad + C^{k+1} \sum_{i=0}^{k} \tbinom{k}{i} \Big(\frac{k+1}{\delta}\Big)^{(i+1)\gamma} \|P^{(k-i)} u\|_{L^2(\omega)} \\
&\leqslant C^{k+1} \sum_{i=0}^{k} \tbinom{k}{i} \Big(\frac{k+1}{\delta}\Big)^{i\gamma} \|P^{(k+1-i)} u\|_{L^2(\omega)} \\
&\quad + C^{k+1} \sum_{i=1}^{k+1} \tbinom{k}{i-1} \Big(\frac{k+1}{\delta}\Big)^{i\gamma} \|P^{(k+1-i)} u\|_{L^2(\omega)}
\end{aligned}$$

But $\binom{k}{i}+\binom{k}{i-1}=\binom{k+1}{i}$, then

$$\|Q^{(k+1)}u\|_{L^2(\omega_\delta)} \leqslant C^{k+1}\sum_{i=1}^{k+1}\binom{k+1}{i}\Big(\frac{k+1}{\delta}\Big)^{i\gamma}\|P^{(k+1-i)}u\|_{L^2(\omega)}$$

Suppose now that $u \in R_M(\Omega, P)$, so for any compact subset H of Ω there exist a bounded open subset ω and $\delta > 0$ such that $H \subset \omega_\delta \subset \omega \subset \Omega$, and therefore there exists $B > 0$ such that $\|P^i u\|_{L^2(\omega)} \leqslant B^{i+1} M_{im}$, $i = 0, 1, \ldots$ Taking into account the relation $(km)^{km} \leqslant (km)!e^{km}$, we obtain for all $i \leqslant k$,

$$\begin{aligned}
k^{i\gamma}\|P^{(k-i)}u\|_{L^2(\omega)} &\leqslant B^{k-i+1}(km)^{i\gamma}M_{(k-i)m}\\
&\leqslant B^{k+1}\Big((km)^{kmd}\frac{M_{km}}{M_{km}}\Big)^{\frac{im}{km}}M_{(k-i)m}\\
&\leqslant B^{k+1}\Big(\frac{(km)!^d e^{kmd}}{M_{km}}\Big)^{\frac{im}{km}}M_{(k-i)m}(M_{km})^{\frac{im}{km}},
\end{aligned}$$

which gives from condition (3.1),

$$k^{i\gamma}\|P^{(k-i)}u\|_{L^2(\omega)} \leqslant B_1^{k+1}M_{(k-i)m}(M_{km})^{\frac{im}{km}} \tag{3.4}$$

On the other hand we can show from (1.1) that $(M_p)^{1/p}$ is an increasing sequence. In fact we will prove by induction the equivalent condition

$$\frac{p+1}{p}\log M_p \leqslant \log M_{p+1}, \quad \forall p \in \mathbb{N}^* \tag{3.5}$$

It is trivial for $p = 1$. Suppose that (3.5) is true for p, then from (1.1) we get

$$2\log M_{p+1} \leqslant \log M_p + \log M_{p+2} \leqslant \log M_{p+2} + \frac{p}{p+1}\log M_{p+1},$$

hence

$$2\log M_{p+1} - \frac{p}{p+1}\log M_{p+1} = \frac{p+2}{p+1}\log M_{p+1} \leqslant \log M_{p+2}$$

In particular we have, $\forall h \leqslant p$, $(M_h)^p \leqslant (M_p)^h$. Applying for $h = km - im$ and $p = km$, we obtain $M_{(k-i)m} \leqslant (M_{km})^{\frac{km-im}{km}}$, which implies with (3.4)

$$k^{i\gamma}\|P^{(k-i)}(D)u\|_{L^2(\omega)} \leqslant B_1^{k+1}M_{km}.$$

Substituting in (3.3) we get

$$\|Q^k u\|_{L^2(H)} \leqslant \|Q^k u\|_{L^2(\omega_\delta)} \leqslant C^k\sum_{i=0}^{k}\binom{k}{i}\Big(\frac{1}{\delta}\Big)^{kmd}B_1^{k+1}M_{km} \leqslant B_2^{k+1}M_{km}$$

Thus $u \in R_M(\Omega, Q)$. □

Corollary 3.1. *Let P and Q be differential operators belonging to $\mathcal{H}$ and equally strong. If $(p!)^d \leqslant M_p$ with $d = \frac{\gamma(P)}{\deg(P)}$, then $R_M(\Omega, P) = R_M(\Omega, Q)$.*

Proof. From Remark 2.1, we have $\deg(P) = \deg(Q)$ and $\gamma(P) = \gamma(Q)$, which implies the result. □

COROLLARY 3.2. *If* $M_p = p!^s$*, Theorem* 3.1 *coincides with Theorem* 1 *of Newberger–Zielezny* [**17**] *in the class of hypoelliptic operators.*

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