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PELL NUMBERS WHOSE EULER FUNCTION IS A PELL NUMBER

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Abstract. We show that the only Pell numbers whose Euler function is also a Pell number are 1 and 2.

1. Introduction

Let $\phi(n)$ be the Euler function of the positive integer n. Recall that if n has the prime factorization

$$n = p_1^{a_1} \cdots p_k^{a_k}$$

with distinct primes $p_1, \dots, p_k$ and positive integers $a_1, \dots, a_k$, then

$$\phi(n) = p_1^{a_1-1}(p_1-1)\cdots p_k^{a_k-1}(p_k-1).$$

There are many papers in the literature dealing with diophantine equations involving the Euler function in members of a binary recurrent sequence. For example, in [**11**], it is shown that 1, 2, and 3 are the only Fibonacci numbers whose Euler function is also a Fibonacci number, while in [**4**] it is shown that the Diophantine equation $\phi(5^n-1) = 5^m-1$ has no positive integer solutions (m, n). Furthermore, the divisibility relation $\phi(n) \mid n-1$ when n is a Fibonacci number, or a Lucas number, or a Cullen number (that is, a number of the form $n2^n+1$ for some positive integer n), or a rep-digit $(g^m-1)/(g-1)$ in some integer base $g \in [2, 1000]$ have been investigated in [**10, 5, 7, 3**], respectively.

Here we look for a similar equation with members of the *Pell sequence*. The Pell sequence $(P_n)_{n\geqslant 0}$ is given by $P_0 = 0$, $P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for all $n \geqslant 0$. Its first terms are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, \dots$$

We have the following result.

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THEOREM 1.1. *The only solutions in positive integers* (n, m) *of the equation*

$$\phi(P_n) = P_m \tag{1.1}$$

are $(n, m) = (1, 1), (2, 1)$.

For the proof, we begin by following the method from [**11**], but we add to it some ingredients from [**10**].

2. Preliminary results

Let $(\alpha, \beta) = (1 + \sqrt{2}, 1 - \sqrt{2})$ be the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell sequence $\{P_n\}_{n \geqslant 0}$. The Binet formula for P_n is

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \geqslant 0.$$

This implies easily that the inequalities

$$\alpha^{n-2} \leqslant P_n \leqslant \alpha^{n-1} \tag{2.1}$$

hold for all positive integers n.

We let $\{Q_n\}_{n \geqslant 0}$ be the companion Lucas sequence of the Pell sequence given by $Q_0 = 2$, $Q_1 = 2$ and $Q_{n+2} = 2Q_{n+1} + Q_n$ for all $n \geqslant 0$. Its first few terms are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, 551614, \ldots$$

The Binet formula for Q_n is

$$Q_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geqslant 0. \tag{2.2}$$

We use the well-known result.

LEMMA 2.1. *The relations* (i) $P_{2n} = P_n Q_n$ *and* (ii) $Q_n^2 - 8P_n^2 = 4(-1)^n$ *hold for all* $n \geqslant 0$.

For a prime p and a nonzero integer m let $\nu_p(m)$ be the exponent with which p appears in the prime factorization of m. The following result is well known and easy to prove.

LEMMA 2.2. *The relations* (i) $\nu_2(Q_n) = 1$ *and* (ii) $\nu_2(P_n) = \nu_2(n)$ *hold for all positive integers* n.

The following divisibility relations among the Pell numbers are well known.

LEMMA 2.3. *Let* m *and* n *be positive integers. We have:*

(i) *If* $m \mid n$, *then* $P_m \mid P_n$, (ii) $\gcd(P_m, P_n) = P_{\gcd(m,n)}$.

For each positive integer n, let $z(n)$ be the smallest positive integer k such that $n \mid P_k$. It is known that this exists and $n \mid P_m$ if and only if $z(n) \mid m$. This number is referred to as *the order of appearance of* n in the Pell sequence. Clearly, $z(2) = 2$. Further, putting for an odd prime p, $e_p = \left(\frac{2}{p}\right)$, where the above notation stands for the Legendre symbol of 2 with respect to p, we have that $z(p) \mid p - e_p$. A prime factor p of P_n such that $z(p) = n$ is called *primitive for* P_n. It is known that P_n has a primitive divisor for all $n \geqslant 2$ (see [**2**] or [**1**]). Write $P_{z(p)} = p^{e_p} m_p$, where

m_p is coprime to p. It is known that if $p^k \mid P_n$ for some $k > e_p$, then $pz(p) \mid n$. In particular,

$$\nu_p(P_n) \leqslant e_p \quad \text{whenever} \quad p \nmid n. \tag{2.3}$$

We need a bound on e_p. We have the following result.

LEMMA 2.4. *The inequality*

$$e_p \leqslant \frac{(p+1)\log\alpha}{2\log p}. \tag{2.4}$$

holds for all primes p.

PROOF. Since $e_2 = 1$, the inequality holds for the prime 2. Assume that p is odd. Then $z(p) \mid p+\varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Furthermore, by Lemmas 2.1 and 2.3, we have $p^{e_p} \mid P_{z(p)} \mid P_{p+\varepsilon} = P_{(p+\varepsilon)/2}Q_{(p+\varepsilon)/2}$. By Lemma 2.1, it follows easily that p cannot divide both P_n and Q_n for $n = (p+\varepsilon)/2$ since otherwise p will also divide

$$Q_n^2 - 8P_n^2 = \pm 4,$$

a contradiction since p is odd. Hence, p^{e_p} divides one of $P_{(p+\varepsilon)/2}$ or $Q_{(p+\varepsilon)/2}$. If p^{e_p} divides $P_{(p+\varepsilon)/2}$, we have, by (2.1), that $p^{e_p} \leqslant P_{(p+\varepsilon)/2} \leqslant P_{(p+1)/2} < \alpha^{(p+1)/2}$, which leads to the desired inequality (2.4) upon taking logarithms of both sides. In case p^{e_p} divides $Q_{(p+\varepsilon)/2}$, we use the fact that $Q_{(p+\varepsilon)/2}$ is even by Lemma 2.2 (i). Hence, p^{e_p} divides $Q_{(p+\varepsilon)/2}/2$, therefore, by formula (2.2), we have

$$p^{e_p} \leqslant \frac{Q_{(p+\varepsilon)/2}}{2} \leqslant \frac{Q_{(p+1)/2}}{2} < \frac{\alpha^{(p+1)/2}+1}{2} < \alpha^{(p+1)/2},$$

which leads again to the desired conclusion by taking logarithms of both sides. □

For a positive real number x we use $\log x$ for the natural logarithm of x. We need some inequalities from the prime number theory. For a positive integer n we write $\omega(n)$ for the number of distinct prime factors of n. The following inequalities (i), (ii) and (iii) are inequalities (3.13), (3.29) and (3.41) in [**15**], while (iv) is Théoréme 13 from [**6**].

LEMMA 2.5. *Let $p_1 < p_2 < \cdots$ be the sequence of all prime numbers. We have:*

(i) *The inequality $p_n < n(\log n + \log\log n)$ holds for all $n \geqslant 6$.*

(ii) *The inequality*

$$\prod_{p \leqslant x}\Big(1 + \frac{1}{p-1}\Big) < 1.79\log x\Big(1 + \frac{1}{2(\log x)^2}\Big)$$

holds for all $x \geqslant 286$.

(iii) *The inequality*

$$\phi(n) > \frac{n}{1.79\log\log n + 2.5/\log\log n}$$

holds for all $n \geqslant 3$.

(iv) *The inequality*

$$\omega(n) < \frac{\log n}{\log\log n - 1.1714}$$

holds for all $n \geqslant 26$.

For a positive integer n, we put $\mathcal{P}_n = \{p : z(p) = n\}$. We need the following result.

LEMMA 2.6. *Put* $S_n := \sum_{p\in\mathcal{P}_n} \frac{1}{p-1}$. *For* $n > 2$, *we have*

$$S_n < \min\Big\{\frac{2\log n}{n}, \frac{4+4\log\log n}{\phi(n)}\Big\}. \tag{2.5}$$

PROOF. Since $n > 2$, it follows that every prime factor $p \in \mathcal{P}_n$ is odd and satisfies the congruence $p \equiv \pm 1 \pmod n$. Further, putting $\ell_n := \#\mathcal{P}_n$, we have

$$(n-1)^{\ell_n} \leqslant \prod_{p\in\mathcal{P}_n} p \leqslant P_n < \alpha^{n-1}$$

(by inequality (2.1)), giving

$$\ell_n \leqslant \frac{(n-1)\log\alpha}{\log(n-1)}. \tag{2.6}$$

Thus, the inequality

$$\ell_n < \frac{n\log\alpha}{\log n} \tag{2.7}$$

holds for all $n \geqslant 3$, since it follows from (2.6) for $n \geqslant 4$ via the fact that the function $x \mapsto x/\log x$ is increasing for $x \geqslant 3$, while for $n = 3$ it can be checked directly. To prove the first bound, we use (2.7) to deduce that

$$\begin{aligned} S_n &\leqslant \sum_{1\leqslant\ell\leqslant\ell_n}\Big(\frac{1}{n\ell-2}+\frac{1}{n\ell}\Big) \leqslant \frac{2}{n}\sum_{1\leqslant\ell\leqslant\ell_n}\frac{1}{\ell} + \sum_{m\geqslant n}\Big(\frac{1}{m-2}-\frac{1}{m}\Big) \\ &\leqslant \frac{2}{n}\Big(\int_1^{\ell_n}\frac{dt}{t}+1\Big) + \frac{1}{n-2} + \frac{1}{n-1} \leqslant \frac{2}{n}\Big(\log\ell_n + 1 + \frac{n}{n-2}\Big) \\ &\leqslant \frac{2}{n}\log\Big(n\Big(\frac{(\log\alpha)e^{2+2/(n-2)}}{\log n}\Big)\Big). \end{aligned} \tag{2.8}$$

Since the inequality $\log n > (\log\alpha)e^{2+2/(n-2)}$ holds for all $n \geqslant 800$, (2.8) implies that $S_n < \frac{2}{n}\log n$ for $n \geqslant 800$. The remaining range for n can be checked on an individual basis. For the second bound on S_n, we follow the argument from [**10**] and split the primes in $\mathcal{P}_n$ in three groups:

$$\text{(i) } p < 3n; \quad \text{(ii) } p \in (3n, n^2); \quad \text{(iii) } p > n^2;$$

We have

$$T_1 = \sum_{\substack{p\in\mathcal{P}_n\\ p<3n}}\frac{1}{p-1} \leqslant \begin{cases} \frac{1}{n-2}+\frac{1}{n}+\frac{1}{2n-2}+\frac{1}{2n}+\frac{1}{3n-2} & < \frac{10.1}{3n}, \quad n \text{ even},\\ \frac{1}{2n-2}+\frac{1}{2n} & < \frac{7.1}{3n}, \quad n \text{ odd}, \end{cases} \tag{2.9}$$

where the last inequalities above hold for all $n \geqslant 84$. For the remaining primes in $\mathcal{P}_n$, we have

$$\sum_{\substack{p\in\mathcal{P}_n\\ p>3n}} \frac{1}{p-1} < \sum_{\substack{p\in\mathcal{P}_n\\ p>3n}} \frac{1}{p} + \sum_{m\geqslant 3n+1} \left(\frac{1}{m-1} - \frac{1}{m}\right) = T_2 + T_3 + \frac{1}{3n}, \tag{2.10}$$

where T_2 and T_3 denote the sums of the reciprocals of the primes in $\mathcal{P}_n$ satisfying (ii) and (iii), respectively. The sum T_2 was estimated in [**10**] using the large sieve inequality of Montgomery and Vaughan [**13**] (see also page 397 in [**11**]), and the bound on it is

$$T_2 = \sum_{3n<p<n^2} \frac{1}{p} < \frac{4}{\phi(n)\log n} + \frac{4\log\log n}{\phi(n)} < \frac{1}{\phi(n)} + \frac{4\log\log n}{\phi(n)}, \tag{2.11}$$

where the last inequality holds for $n \geqslant 55$. Finally, for T_3, we use estimate (2.7) on ℓ_n to deduce that

$$T_3 < \frac{\ell_n}{n^2} < \frac{\log\alpha}{n\log n} < \frac{0.9}{3n}, \tag{2.12}$$

where the last bound holds for all $n \geqslant 19$. To summarize, for $n \geqslant 84$, we have, by (2.9), (2.10), (2.11) and (2.12),

$$S_n < \frac{10.1}{3n} + \frac{1}{3n} + \frac{0.9}{3n} + \frac{1}{\phi(n)} + \frac{4\log\log n}{\phi(n)} = \frac{4}{n} + \frac{1}{\phi(n)} + \frac{4\log\log n}{\phi(n)} \leqslant \frac{3+4\log\log n}{\phi(n)}$$

for n even, which is stronger that the desired inequality. Here, we used that $\phi(n) \leqslant n/2$ for even n. For odd n, we use the same argument except that the first fraction $10.1/(3n)$ on the right-hand side above gets replaced by $7.1/(3n)$ (by (2.9)), and we only have $\phi(n) \leqslant n$ for odd n. This was for $n \geqslant 84$. For $n \in [3, 83]$, the desired inequality can be checked on an individual basis. □

The next lemma from [**9**] gives an upper bound on the sum appearing in the right-hand side of (2.5).

LEMMA 2.7. *We have*

$$\sum_{d\mid n} \frac{\log d}{d} < \left(\sum_{p\mid n} \frac{\log p}{p-1}\right) \frac{n}{\phi(n)}.$$

Throughout the rest of this paper we use $p,\ q,\ r$ with or without subscripts to denote prime numbers.

3. Proof of the Theorem

3.1. A bird'e eye view of the proof of the Theorem. In this section, we explain the plan of attack for the proof of the Theorem. We assume $n > 2$. We put k for the number of distinct prime factors of P_n and $\ell = n - m$. We first show that $2^k \mid m$ and that any putative solution must be large. This only uses the fact that $p - 1 \mid \phi(P_n) = P_m$ for all prime factors p of P_n, and all such primes with at most one exception are odd. We show that $k \geqslant 416$ and $n > m \geqslant 2^{416}$. This is Lemma 3.1. We next bound ℓ in terms of n by showing that $\ell < \log\log\log n/\log\alpha + 1.1$

(Lemma 3.2). Next we show that k is large, by proving that $3^k > n/6$ (Lemma 3.3). When n is odd, then every prime factor of P_n is congruent to 1 modulo 4. This implies that $4^k \mid m$. Thus, $3^k > n/6$ and $n > m \geqslant 4^k$, a contradiction in our range for n. This is done in Subsection 3.5. When n is even, we write $n = 2^s n_1$ with an odd integer n_1 and bound s and the smallest prime factor r_1 of n_1. We first show that $s \leqslant 3$, that if n_1 and m have a common divisor larger than 1, then $r_1 \in \{3, 5, 7\}$ (Lemma 3.4). A lot of effort is spend into finding a small bound on r_1. As we saw, $r_1 \leqslant 7$ if n_1 and m are not coprime. When n_1 and m are coprime, we succeed in proving that $r_1 < 10^6$. Putting e_r for the exponent of r in the factorization of $P_{z(r)}$, it turns out that our argument works well when $e_r = 1$ and we get a contradiction, but when $e_r = 2$, then we need some additional information about the prime factors of Q_r. It is always the case that $e_r = 1$ for all primes $r < 10^6$, except for $r \in \{13, 31\}$ for which $e_r = 2$, but, lucky for us, both Q_{13} and Q_{31} have two suitable prime factors each which allows us to obtain a contradiction. Our efforts in obtaining $r_1 < 10^6$ involve quite a complicated argument (roughly the entire argument after Lemma 3.4 until the end), which we believe it is justified by the existence of the mighty prime $r_1 = 1546463$, for which $e_{r_1} = 2$. Should we have only obtained say $r_1 < 1.6 \times 10^6$, we would have had to say something nontrivial about the prime factors of $Q_{15467463}$, a nuisance which we succeeded in avoiding simply by proving that r_1 cannot get that large!

3.2. Some lower bounds on m and $\omega(P_n)$. We start with a computation showing that there are no other solutions than $n = 1,\ 2$ when $n \leqslant 100$. So, from now on $n > 100$. We write $P_n = q_1^{\alpha_1} \ldots q_k^{\alpha_k}$, where $q_1 < \cdots < q_k$ are primes and $\alpha_1, \ldots, \alpha_k$ are positive integers. Clearly, $m < n$.

McDaniel [**12**], proved that P_n has a prime factor $q \equiv 1 \pmod 4$ for all $n > 14$. Thus, McDaniel's result applies for us showing that $4 \mid q - 1 \mid \phi(P_n) \mid P_m$, so $4 \mid m$ by Lemma 2.2. Further, it follows from a the result of the second author [**5**], that $\phi(P_n) \geqslant P_{\phi(n)}$. Hence, $m \geqslant \phi(n)$. Thus,

$$m \geqslant \phi(n) \geqslant \frac{n}{1.79 \log\log n + 2.5/\log\log n}, \tag{3.1}$$

by Lemma 2.5 (iii). The function

$$x \mapsto \frac{x}{1.79 \log\log x + 2.5/\log\log x}$$

is increasing for $x \geqslant 100$. Since $n \geqslant 100$, inequality (3.1) together with the fact that $4 \mid m$, show that $m \geqslant 24$.

Put $\ell = n - m$. Since m is even, we have $\beta^m > 0$, therefore

$$\frac{P_n}{P_m} = \frac{\alpha^n - \beta^n}{\alpha^m - \beta^m} > \frac{\alpha^n - \beta^n}{\alpha^m} \geqslant \alpha^\ell - \frac{1}{\alpha^{m+n}} > \alpha^\ell - 10^{-40}, \tag{3.2}$$

where we used the fact that

$$\frac{1}{\alpha^{m+n}} \leqslant \frac{1}{\alpha^{124}} < 10^{-40}.$$

We now are ready to provide a large lower bound on n. We distinguish the following cases.

Case 1: *n is odd.* Here, we have $\ell \geqslant 1$. So, $P_n/P_m > \alpha - 10^{-40} > 2.4142$. Since n is odd, it follows that P_n is divisible only by primes q such that $z(q)$ is odd. Among the first 10000 primes, there are precisely 2907 of them with this property. They are

$$\mathcal{F}_1 = \{5, 13, 29, 37, 53, 61, 101, 109, \ldots, 104597, 104677, 104693, 104701, 104717\}.$$

Since

$$\prod_{p\in\mathcal{F}_1}\left(1-\frac{1}{p}\right)^{-1} < 1.963 < 2.4142 < \frac{P_n}{P_m} = \prod_{i=1}^{k}\left(1-\frac{1}{q_i}\right)^{-1},$$

we get that $k > 2907$. Since $2^k \mid \phi(P_n) \mid P_m$, we get, by Lemma 2.2, that

$$n > m > 2^{2907}. \tag{3.3}$$

Case 2: $n \equiv 2 \pmod 4$. Since both m and n are even, we get $\ell \geqslant 2$. Thus,

$$\frac{P_n}{P_m} > \alpha^2 - 10^{-40} > 5.8284. \tag{3.4}$$

If q is a prime factor of P_n, as in Case 1, we have that $z(q)$ is not divisible by 4. Among the first 10000 primes, there are precisely 5815 of them with this property. They are

$$\mathcal{F}_2 = \{2, 5, 7, 13, 23, 29, 31, 37, 41, 47, 53, 61, \ldots, 104693, 104701, 104711, 104717\}.$$

Writing p_j as the j^{th} prime number in $\mathcal{F}_2$, we check with Mathematica that

$$\prod_{i=1}^{415}\left(1-\frac{1}{p_i}\right)^{-1} = 5.82753\ldots \quad \prod_{i=1}^{416}\left(1-\frac{1}{p_i}\right)^{-1} = 5.82861\ldots,$$

which via inequality (3.4) shows that $k \geqslant 416$. Of the k prime factors of P_n, we have that only $k-1$ of them are odd ($q_1 = 2$ because n is even), but one of those is congruent to 1 modulo 4 by McDaniel's result. Hence, $2^k \mid \phi(P_n) \mid P_m$, which shows, via Lemma 2.2, that

$$n > m \geqslant 2^{416}. \tag{3.5}$$

Case 3: $4 \mid n$. In this case, since both m and n are multiples of 4, we get that $\ell \geqslant 4$. Therefore, $P_n/P_m > \alpha^4 - 10^{-40} > 33.97$. Letting $p_1 < p_2 < \cdots$ be the sequence of all primes, we have that

$$\prod_{i=1}^{2000}\left(1-\frac{1}{p_i}\right)^{-1} < 17.41\ldots < 33.97 < \frac{P_n}{P_m} = \prod_{i=1}^{k}\left(1-\frac{1}{q_i}\right),$$

showing that $k > 2000$. Since $2^k \mid \phi(P_n) = P_m$, we get

$$n > m \geqslant 2^{2000}. \tag{3.6}$$

To summarize, from (3.3), (3.5) and (3.6), we get the following results.

Lemma 3.1. *If $n > 2$, then* (i) $2^k \mid m$; (ii) $k \geqslant 416$; (iii) $n > m \geqslant 2^{416}$.

3.3. Bounding ℓ in term of n. We saw in the preceding section that $k \geqslant 416$. Since $n > m \geqslant 2^k$, we have

$$k < k(n) := \frac{\log n}{\log 2}. \tag{3.7}$$

Let p_j be the j^{th} prime number. Lemma 2.5 shows that

$$p_k \leqslant p_{\lfloor k(n) \rfloor} \leqslant k(n)(\log k(n) + \log\log k(n)) := q(n).$$

We then have, using Lemma 2.5 (ii), that

$$\frac{P_m}{P_n} = \prod_{i=1}^{k}\Big(1 - \frac{1}{q_i}\Big) \geqslant \prod_{2 \leqslant p \leqslant q(n)}\Big(1 - \frac{1}{p}\Big) > \frac{1}{1.79 \log q(n)(1 + 1/(2(\log q(n))^2))}.$$

Inequality (ii) of Lemma 2.5 requires that $x \geqslant 286$, which holds for us with $x = q(n)$ because $k(n) \geqslant 416$. Hence, we get

$$1.79 \log q(n)\Big(1 + \frac{1}{(2(\log q(n))^2)}\Big) > \frac{P_n}{P_m} > \alpha^{\ell} - 10^{-40} > \alpha^{\ell}\Big(1 - \frac{1}{10^{40}}\Big).$$

Since $k \geqslant 416$, we have $q(n) > 3256$. Hence, we get

$$\log q(n)\Big(1.79\Big(1 - \frac{1}{10^{40}}\Big)^{-1}\Big(1 + \frac{1}{2(\log(3256))^2}\Big)\Big) > \alpha^{\ell},$$

which yields, after taking logarithms, to

$$\ell \leqslant \frac{\log\log q(n)}{\log \alpha} + 0.67. \tag{3.8}$$

The inequality

$$q(n) < (\log n)^{1.45} \tag{3.9}$$

holds in our range for n (in fact, it holds for all $n > 10^{83}$, which is our case since for us $n > 2^{416} > 10^{125}$). Inserting inequality (3.9) into (3.8), we get

$$\ell < \frac{\log\log(\log n)^{1.45}}{\log \alpha} + 0.67 < \frac{\log\log\log n}{\log \alpha} + 1.1.$$

Thus, we proved the following result.

Lemma 3.2. *If $n > 2$, then*

$$\ell < \frac{\log\log\log n}{\log \alpha} + 1.1.$$

3.4. Bounding the primes q_i for $i = 1, \ldots, k$. Write $P_n = q_1 \cdots q_k B$, where $B = q_1^{\alpha_1 - 1} \cdots q_k^{\alpha_k - 1}$. Clearly, $B \mid \phi(P_n)$, therefore $B \mid P_m$. Since also $B \mid P_n$, we have, by Lemma 2.3, that $B \mid \gcd(P_n, P_m) = P_{\gcd(n,m)} \mid P_\ell$ where the last relation follows again by Lemma 2.3 because $\gcd(n, m) \mid \ell$. Using inequality (2.1) and Lemma 3.2, we get

$$B \leqslant P_{n-m} \leqslant \alpha^{n-m-1} \leqslant \alpha^{0.1} \log\log n. \tag{3.10}$$

To bound the primes q_i for all $i = 1, \dots, k$, we use the inductive argument from Section 3.3 in [**11**]. We write

$$\prod_{i=1}^{k}\Big(1-\frac{1}{q_i}\Big)=\frac{\phi(P_n)}{P_n}=\frac{P_m}{P_n}.$$

Therefore,

$$1-\prod_{i=1}^{k}\Big(1-\frac{1}{q_i}\Big)=1-\frac{P_m}{P_n}=\frac{P_n-P_m}{P_n}\geqslant\frac{P_n-P_{n-1}}{P_n}>\frac{P_{n-1}}{P_n}.$$

Using the inequality

$$1-(1-x_1)\cdots(1-x_s)\leqslant x_1+\cdots+x_s \quad \text{valid for all} \quad x_i\in[0,1] \ \text{ for } \ i=1,\dots,s,$$

we get,

$$\frac{P_{n-1}}{P_n}<1-\prod_{i=1}^{k}\Big(1-\frac{1}{q_i}\Big)\leqslant\sum_{i=1}^{k}\frac{1}{q_i}<\frac{k}{q_1},$$

therefore, $q_1 < k(P_n/P_{n-1}) < 3k$. Using the method of the proof of inequality (13) in [**11**], one proves by induction on the index $i \in \{1, \dots, k\}$ that if we put $u_i := \prod_{j=1}^{i} q_j$, then $u_i < (2\alpha^{2.1}k\log\log n)^{(3^i-1)/2}$. In particular,

$$q_1\cdots q_k=u_k<(2\alpha^{2.1}k\log\log n)^{(3^k-1)/2},$$

which together with formula (3.8) and (3.10) gives

$$P_n=q_1\cdots q_kB<(2\alpha^{2.1}k\log\log n)^{1+(3^k-1)/2}=(2\alpha^{2.1}k\log\log n)^{(3^k+1)/2}.$$

Since $P_n > \alpha^{n-2}$ by inequality (2.1), we get

$$(n-2)\log\alpha<\frac{(3^k+1)}{2}\log(2\alpha^{2.1}k\log\log n).$$

Since $k < \log n/\log 2$ (see (3.7)), we get

$$3^k>(n-2)\Big(\frac{2\log\alpha}{\log(2\alpha^{2.1}(\log n)(\log\log n)(\log 2)^{-1})}\Big)-1>0.17(n-2)-1>\frac{n}{6},$$

where the last two inequalities above hold because $n > 2^{416}$.

So, we proved the following result.

Lemma 3.3. *If $n > 2$, then $3^k > n/6$.*

3.5. The case when n is odd. Assume that $n > 2$ is odd and let q be any prime factor of P_n. Reducing relation $Q_n^2-8P_n^2 = 4(-1)^n$ of Lemma 2.1 (ii) modulo q, we get $Q_n^2 \equiv -4 \pmod q$. Since q is odd, (because n is odd), we get that $q \equiv 1 \pmod 4$. This is true for all prime factors q of P_n. Hence,

$$4^k\mid\prod_{i=1}^{k}(q_i-1)\mid\phi(P_n)\mid P_m,$$

which, by Lemma 2.2 (ii), gives $4^k \mid m$. Thus, $n > m \geqslant 4^k$, inequality which together with Lemma 3.3 gives $n > (3^k)^{\log 4/\log 3} > \left(\frac{n}{6}\right)^{\log 4/\log 3}$, so

$$n < 6^{\log 4/\log(4/3)} < 5621,$$

in contradiction with Lemma 3.1.

3.6. Bounding n. From now on, $n > 2$ is even. We write it as

$$n = 2^s r_1^{\lambda_1} \cdots r_t^{\lambda_t} =: 2^s n_1,$$

where $s \geqslant 1$, $t \geqslant 0$ and $3 \leqslant r_1 < \cdots < r_t$ are odd primes. Thus, by inequality (3.2), we have

$$\alpha^\ell \Big(1 - \frac{1}{10^{40}}\Big) < \alpha^\ell - \frac{1}{10^{40}} < \frac{P_n}{\phi(P_n)} = \prod_{p \mid P_n} \Big(1 + \frac{1}{p-1}\Big) = 2 \prod_{\substack{d \geqslant 3 \\ d \mid n}} \prod_{p \in \mathcal{P}_d} \Big(1 + \frac{1}{p-1}\Big),$$

and taking logarithms we get

$$\begin{aligned} \ell \log \alpha - \frac{1}{10^{39}} &< \log \Big(\alpha^\ell \Big(1 - \frac{1}{10^{40}}\Big)\Big) \\ &< \log 2 + \sum_{\substack{d \geqslant 3 \\ d \mid n}} \sum_{p \in \mathcal{P}_d} \log \Big(1 + \frac{1}{p-1}\Big) < \log 2 + \sum_{\substack{d \geqslant 3 \\ d \mid n}} S_d. \end{aligned} \tag{3.11}$$

In the above, we used the inequality $\log(1 - x) > -10x$ valid for all $x \in (0, 1/2)$ with $x = 1/10^{40}$ and the inequality $\log(1 + x) \leqslant x$ valid for all real numbers x with $x = p$ for all $p \in \mathcal{P}_d$ and all divisors $d \mid n$ with $d \geqslant 3$.

Let us deduce that the case $t = 0$ is impossible. Indeed, if this were so, then n is a power of 2 and so, by Lemma 3.1, both m and n are divisible by 2^{416}. Thus, $\ell \geqslant 2^{416}$. Inserting this into (3.11), and using Lemma 2.6, we get

$$2^{416} \log \alpha - \frac{1}{10^{39}} < \sum_{a \geqslant 1} \frac{2 \log(2^a)}{2^a} = 4 \log 2,$$

a contradiction.

Thus, $t \geqslant 1$ so $n_1 > 1$. We now put $\mathcal{I} := \{i : r_i \mid m\}$ and $\mathcal{J} = \{1, \ldots, t\} \smallsetminus \mathcal{I}$. We put $M = \prod_{i \in \mathcal{I}} r_i$. We also let j be minimal in $\mathcal{J}$. We split the sum appearing in (3.11) in two parts:

$$\sum_{d \mid n} S_d = L_1 + L_2,$$

where

$$L_1 := \sum_{\substack{d \mid n \\ r \mid d \Rightarrow r \mid 2M}} S_d \quad \text{and} \quad L_2 := \sum_{\substack{d \mid n \\ r_u \mid d \text{ for some } u \in \mathcal{J}}} S_d.$$

To bound L_1, we note that all divisors involved divide n', where

$$n' = 2^s \prod_{i \in \mathcal{I}} r_i^{\lambda_i}.$$

Using Lemmas 2.6 and 2.7, we get

$$L_1 \leqslant 2\sum_{d|n'} \frac{\log d}{d} < 2\Big(\sum_{r|n'} \frac{\log r}{r-1}\Big)\Big(\frac{n'}{\phi(n')}\Big) = 2\Big(\sum_{r|2M} \frac{\log r}{r-1}\Big)\Big(\frac{2M}{\phi(2M)}\Big). \tag{3.12}$$

We now bound L_2. If $\mathcal{J} = \emptyset$, then $L_2 = 0$ and there is nothing to bound. So, assume that $\mathcal{J} \neq \emptyset$. We argue as follows. Note that since $s \geqslant 1$, by Lemma 2.1 (i), we have $P_n = P_{n_1}Q_{n_1}Q_{2n_1}\cdots Q_{2^{s-1}n_1}$. Let q be any odd prime factor of Q_{n_1}. By reducing relation (ii) of Lemma 2.1 modulo q and using the fact that n_1 and q are both odd, we get $2P_{n_1}^2 \equiv 1 \pmod q$, therefore $\big(\frac{2}{q}\big) = 1$. Hence, $z(q) \mid q-1$ for such primes q. Now let d be any divisor of n_1 which is a multiple of r_j. The number of them is $\tau(n_1/r_j)$, where $\tau(u)$ is the number of divisors of the positive integer u. For each such d, there is a primitive prime factor q_d of $Q_d \mid Q_{n_1}$. Thus, $r_j \mid d \mid q_d - 1$. This shows that

$$\nu_{r_j}(\phi(P_n)) \geqslant \nu_{r_j}(\phi(Q_{n_1})) \geqslant \tau(n_1/r_j) \geqslant \tau(n_1)/2, \tag{3.13}$$

where the last inequality follows from the fact that

$$\frac{\tau(n_1/r_j)}{\tau(n_1)} = \frac{\lambda_j}{\lambda_j+1} \geqslant \frac{1}{2}.$$

Since r_j does not divide m, it follows from (2.3) that

$$\nu_{r_j}(P_m) \leqslant e_{r_j}. \tag{3.14}$$

Hence, (3.13), (3.14) and (1.1) imply that

$$\tau(n_1) \leqslant 2e_{r_j}.$$

Invoking Lemma 2.4, we get

$$\tau(n_1) \leqslant \frac{(r_j+1)\log\alpha}{\log r_j}. \tag{3.15}$$

Now every divisor d participating in L_2 is of the form $d = 2^a d_1$, where $0 \leqslant a \leqslant s$ and d_1 is a divisor of n_1 divisible by r_u for some $u \in \mathcal{J}$. Thus,

$$L_2 \leqslant \tau(n_1)\min\Bigg\{\sum_{\substack{0\leqslant a\leqslant s,\ d_1|n_1\\ r_u|d_1 \text{ for some } u\in\mathcal{J}}} S_{2^a d_1}\Bigg\} := g(n_1, s, r_1).$$

In particular, $d_1 \geqslant 3$ and since the function $x \mapsto \log x/x$ is decreasing for $x \geqslant 3$, we have that

$$g(n_1, s, r_1) \leqslant 2\tau(n_1)\sum_{0\leqslant a\leqslant s} \frac{\log(2^a r_j)}{2^a r_j}. \tag{3.16}$$

Putting also $s_1 := \min\{s, 416\}$, we get, by Lemma 3.1, that $2^{s_1} \mid \ell$. Thus, inserting this as well as (3.12) and (3.16) all into (3.11), we get

$$\ell\log\alpha - \frac{1}{10^{39}} < 2\Big(\sum_{r|2M}\frac{\log r}{r-1}\Big)\Big(\frac{2M}{\phi(2M)}\Big) + g(n_1, s, r_1). \tag{3.17}$$

Since

$$\sum_{0\leqslant a\leqslant s}\frac{\log(2^a r_j)}{2^a r_j}<\frac{4\log 2+2\log r_j}{r_j}, \tag{3.18}$$

inequalities (3.18), (3.15) and (3.16) give us that

$$g(n_1,s,r_1)\leqslant 2\Big(1+\frac{1}{r_j}\Big)\Big(2+\frac{4\log 2}{\log r_j}\Big)\log\alpha := g(r_j).$$

The function $g(x)$ is decreasing for $x\geqslant 3$. Thus, $g(r_j)\leqslant g(3)<10.64$. For a positive integer N put

$$f(N):=N\log\alpha-\frac{1}{10^{39}}-2\Big(\sum_{r\mid N}\frac{\log r}{r-1}\Big)\Big(\frac{N}{\phi(N)}\Big).$$

Then inequality (3.17) implies that both inequalities

$$f(\ell)<g(r_j),\quad (\ell-M)\log\alpha+f(M)<g(r_j) \tag{3.19}$$

hold. Assuming that $\ell\geqslant 26$, we get, by Lemma 2.5, that

$$\ell\log\alpha-\frac{1}{10^{39}}-2(\log 2)\frac{(1.79\log\log\ell+2.5/\log\log\ell)\log\ell}{\log\log\ell-1.1714}\leqslant 10.64.$$

Mathematica confirmed that the above inequality implies $\ell\leqslant 500$. Another calculation with Mathematica showed that the inequality $f(\ell)<10.64$ for even values of $\ell\in[1,500]\cap\mathbb{Z}$ implies that $\ell\in[2,18]$. The minimum of the function $f(2N)$ for $N\in[1,250]\cap\mathbb{Z}$ is at $N=3$ and $f(6)>-2.12$. For the remaining positive integers N, we have $f(2N)>0$. Hence, inequality (3.19) implies

$$(2^{s_1}-2)\log\alpha<10.64\quad\text{and}\quad(2^{s_1}-2)3\log\alpha<10.64+2.12=12.76,$$

according to whether $M\neq 3$ or $M=3$, and either one of the above inequalities implies that $s_1\leqslant 3$. Thus, $s=s_1\in\{1,2,3\}$. Since $2M\mid\ell$, $2M$ is square-free and $\ell\leqslant 18$, we have that $M\in\{1,3,5,7\}$. Assume $M>1$ and let i be such that $M=r_i$. Let us show that $\lambda_i=1$. Indeed, if $\lambda_i\geqslant 2$, then

$$199\mid Q_9\mid P_n,\quad 29201\mid P_{25}\mid P_n,\quad 1471\mid Q_{49}\mid P_n,$$

according to whether $r_i=3,5,7$, respectively, and $3^2\mid 199-1$, $5^2\mid 29201-1$, $7^2\mid 1471-1$. Thus, we get that $3^2,\ 5^2,\ 7^2$ divide $\phi(P_n)=P_m$, showing that $3^2,\ 5^2,\ 7^2$ divide ℓ. Since $\ell\leqslant 18$, only the case $\ell=18$ is possible. In this case, $r_j\geqslant 5$, and inequality (3.19) gives $8.4<f(18)\leqslant g(5)<7.9$, a contradiction. Let us record what we have deduced so far.

Lemma 3.4. *If $n>2$ is even, then $s\in\{1,2,3\}$. Further, if $\mathcal{I}\neq\emptyset$, then $\mathcal{I}=\{i\}$, $r_i\in\{3,5,7\}$ and $\lambda_i=1$.*

We now deal with $\mathcal{J}$. For this, we return to (3.11) and use the better inequality namely

$$2^s M\log\alpha-\frac{1}{10^{39}}\leqslant\ell\log\alpha-\frac{1}{10^{39}}\leqslant\log\Big(\frac{P_n}{\phi(P_n)}\Big)\leqslant\sum_{d\mid 2^s M}\sum_{p\in\mathcal{P}_d}\log\Big(1+\frac{1}{p-1}\Big)+L_2,$$

so

$$L_2 \geqslant 2^s M \log \alpha - \frac{1}{10^{39}} - \sum_{d|2^s M} \sum_{p \in \mathcal{P}_d} \log\Big(1 + \frac{1}{p-1}\Big).$$

In the right-hand side above, $M \in \{1,3,5,7\}$ and $s \in \{1,2,3\}$. The values of the right-hand side above are in fact

$$h(u) := u \log \alpha - \frac{1}{10^{39}} - \log(P_u/\phi(P_u))$$

for $u = 2^s M \in \{2,4,6,8,10,12,14,20,24,28,40,56\}$. Computing we get:

$$h(u) \geqslant H_{s,M}\Big(\frac{M}{\phi(M)}\Big) \quad \text{for} \quad M \in \{1,3,5,7\}, \quad s \in \{1,2,3\},$$

where

$$H_{1,1} > 1.069, \quad H_{1,M} > 2.81 \quad \text{for} \quad M > 1, \quad H_{2,M} > 2.426, \quad H_{3,M} > 5.8917.$$

We now exploit the relation

$$H_{s,M}\Big(\frac{M}{\phi(M)}\Big) < L_2. \tag{3.20}$$

Our goal is to prove that $r_j < 10^6$. Assume this is not so. We use the bound

$$L_2 < \sum_{\substack{d|n \\ r_u|d \text{ for sume } u\in\mathcal{J}}} \frac{4 + 4\log\log d}{\phi(d)}$$

of Lemma 2.6. Each divisor d participating in L_2 is of the form $2^a d_1$, where $a \in [0,s] \cap \mathbb{Z}$ and d_1 is a multiple of a prime at least as large as r_j. Thus,

$$\frac{4 + 4\log\log d}{\phi(d)} \leqslant \frac{4 + 4\log\log 8d_1}{\phi(2^a)\phi(d_1)} \quad \text{for} \quad a \in \{0,1,\ldots,s\},$$

and

$$\frac{d_1}{\phi(d_1)} \leqslant \frac{n_1}{\phi(n_1)} \leqslant \frac{M}{\phi(M)}\Big(1 + \frac{1}{r_j - 1}\Big)^{\omega(n_1)}.$$

Using (3.15), we get

$$2^{\omega(n_1)} \leqslant \tau(n_1) \leqslant \frac{(r_j+1)\log \alpha}{\log r_j} < r_j,$$

where the last inequality holds because r_j is large. Thus,

$$\omega(n_1) < \frac{\log r_j}{\log 2} < 2\log r_j. \tag{3.21}$$

Hence,

$$\begin{aligned} \frac{n_1}{\phi(n_1)} &\leqslant \frac{M}{\phi(M)}\Big(1 + \frac{1}{r_j - 1}\Big)^{\omega(n_1)} < \frac{M}{\phi(M)}\Big(1 + \frac{1}{r_j - 1}\Big)^{2\log r_j} \\ &< \frac{M}{\phi(M)} \exp\Big(\frac{2\log r_j}{r_j - 1}\Big) < \frac{M}{\phi(M)}\Big(1 + \frac{4\log r_j}{r_j - 1}\Big), \end{aligned} \tag{3.22}$$

where we used the inequalities $1+x<e^x$, valid for all real numbers x, as well as $e^x<1+2x$ which is valid for $x\in(0,1/2)$ with $x=2\log r_j/(r_j-1)$ which belongs to $(0,1/2)$ because r_j is large. Thus, the inequality

$$\frac{4+4\log\log d}{\phi(d)}\leqslant\Big(\frac{4+4\log\log 8d_1}{d_1}\Big)\Big(1+\frac{4\log r_j}{r_j-1}\Big)\Big(\frac{1}{\phi(2^a)}\Big)\frac{M}{\phi(M)}$$

holds for $d=2^ad_1$ participating in L_2. The function $x\mapsto(4+4\log\log(8x))/x$ is decreasing for $x\geqslant 3$. Hence,

$$L_2\leqslant\Big(\frac{4+4\log\log(8r_j)}{r_j}\Big)\tau(n_1)\Big(1+\frac{4\log r_j}{r_j-1}\Big)\Big(\sum_{0\leqslant a\leqslant s}\frac{1}{\phi(2^a)}\Big)\Big(\frac{M}{\phi(M)}\Big). \tag{3.23}$$

Inserting inequality (3.15) into (3.23) and using (3.20), we get

$$\log r_j<4\Big(1+\frac{1}{r_j}\Big)\Big(1+\frac{4\log r_j}{r_j-1}\Big)(1+\log\log(8r_j))(\log\alpha)\Big(\frac{G_s}{H_{s,M}}\Big), \tag{3.24}$$

where

$$G_s=\sum_{0\leqslant a\leqslant s}\frac{1}{\phi(2^a)}.$$

For $s=2,\ 3$, inequality (3.24) implies $r_j<900{,}000$ and $r_j<300$, respectively. For $s=1$ and $M>1$, inequality (3.24) implies $r_j<5000$. When $M=1$ and $s=1$, we get $n=2n_1$ and $j=1$. Here, inequality (3.24) implies that $r_1<8\times10^{12}$. This is too big, so we use the bound

$$S_d<\frac{2\log d}{d}$$

of Lemma 2.6 instead for the divisors d of participating in L_2, which in this case are all the divisors of n larger than 2. We deduce that

$$1.06<L_2<2\sum_{\substack{d\mid 2n_1\\ d>2}}\frac{\log d}{d}<4\sum_{d_1\mid n_1}\frac{\log d_1}{d_1}.$$

The last inequality above follows from the fact that all divisors $d>2$ of n are either of the form d_1 or $2d_1$ for some divisor $d_1\geqslant 3$ of n_1, and the function $x\mapsto\log x/x$ is decreasing for $x\geqslant 3$. Using Lemma 2.7 and inequalities (3.21) and (3.22), we get

$$\begin{aligned}1.06&<4\Big(\sum_{r\mid n_1}\frac{\log r}{r-1}\Big)\Big(\frac{n_1}{\phi(n_1)}\Big)<\Big(\frac{4\log r_1}{r_1-1}\Big)\omega(n_1)\Big(1+\frac{4\log r_1}{r_1-1}\Big)\\&<\Big(\frac{4\log r_1}{r_1-1}\Big)\Big(2\log r_1\Big)\Big(1+\frac{4\log r_1}{r_1-1}\Big),\end{aligned}$$

which gives $r_1<159$. So, in all cases, $r_j<10^6$. Here, we checked that $e_r=1$ for all such r except $r\in\{13,31\}$ for which $e_r=2$. If $e_{r_j}=1$, we then get $\tau(n_1/r_j)\leqslant 1$, so $n_1=r_j$. Thus, $n\leqslant 8\cdot10^6$, in contradiction with Lemma 3.1. Assume now that $r_j\in\{13,31\}$. Say $r_j=13$. In this case, 79 and 599 divide Q_{13} which divides P_n, therefore $13^2\mid(79-1)(599-1)\mid\phi(P_n)=P_m$. Thus, if there is some other prime factor r' of $n_1/13$, then $13r'\mid n_1$, and $Q_{13r'}$ has a primitive prime factor $q\equiv 1\pmod{13r'}$. In particular, $13\mid q-1$. Thus, $\nu_{13}(\phi(P_n))\geqslant 3$, showing that

$13^3 \mid P_m$. Hence, $13 \mid m$, therefore $13 \mid M$, a contradiction. A similar contradiction is obtained if $r_j = 31$ since Q_{31} has two primitive prime factors namely 424577 and 865087 so $31 \mid M$.

This finishes the proof.

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